Rewarding Learning

ADVANCED
General Certificate of Education
2025

Centre Number

Candidate Number

Further Mathematics

StemUp Model Solution

Assessment Unit A2 1

assessing

Pure Mathematics

[AFM11]

AFM11

WEDNESDAY 21 MAY, MORNING

TIME

2 hours 15 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer **all eleven** questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages.
Complete in black ink only. **Do not write with a gel pen.**
Questions which require drawing or sketching should be completed using an HB pencil.
Show clearly the full development of your answers. **Answers without working may not gain full credit.**
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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1 Solve the differential equation

$$(x^2 + 4)\frac{dy}{dx} - 2xy = 3x^2 + 12$$

given that $y = 4$ when $x = 0$ [10]

$\frac{dy}{dx} - \frac{2x}{x^2+4}y = 3$ (dividing by $x^2 + 4$)

Integrating factor $= e^{\int -\frac{2x}{x^2+4}\,dx} = e^{-\ln(x^2+4)}$

$= \frac{1}{x^2+4}$

$\frac{1}{x^2+4}\frac{dy}{dx} - \frac{2x}{(x^2+4)^2}y = \frac{3}{x^2+4}$

$\frac{d}{dx}\left(\frac{y}{x^2+4}\right) = \frac{3}{x^2+4}$

$\frac{y}{x^2+4} = \int \frac{3}{x^2+4}\,dx$

$\frac{y}{x^2+4} = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$x = 0, y = 4$: $\frac{4}{4} = \frac{3}{2}\tan^{-1}0 + c \Rightarrow c = 1$

$\frac{y}{x^2+4} = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + 1$

$y = (x^2 + 4)\left(\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + 1\right)$

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2 **(a)** Evaluate the improper integral

$$\int_0^2 \ln x \, dx$$

giving your answer in the form $a \ln b + c$

You may assume that $\lim_{t \to 0} (t \ln t) = 0$ [8]

$\int_0^2 \ln x \, dx = \lim_{t \to 0} \int_t^2 \ln x \, dx$

By parts: $u = \ln x$, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$, $v = x$

$\int_t^2 \ln x \, dx = [x \ln x]_t^2 - \int_t^2 x \times \frac{1}{x} \, dx$

$= [x \ln x - x]_t^2$

$= (2 \ln 2 - 2) - (t \ln t - t)$

As $t \to 0$, $t \ln t \to 0$ and $t \to 0$

$\int_0^2 \ln x \, dx = 2 \ln 2 - 2$

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(b) (i) Express

$$25x^2 + 10x - 8$$

in the form $(px + q)^2 + r$ where p, q and r are integers. [2]

$25x^2 + 10x - 8 = (5x + 1)^2 - 1 - 8$

$= (5x + 1)^2 - 9 \qquad (p = 5,\ q = 1,\ r = -9)$

(ii) Hence show that

$$\int_{\frac{2}{5}}^{2} \frac{1}{\sqrt{25x^2 + 10x - 8}}\,dx = \frac{1}{5}\cosh^{-1}\left(\frac{11}{3}\right) \qquad [5]$$

$\int_{2/5}^{2} \frac{1}{\sqrt{(5x + 1)^2 - 9}}\,dx$

Let $u = 5x + 1$, $\frac{du}{dx} = 5$, $dx = \frac{1}{5}du$

$x = \frac{2}{5} \Rightarrow u = 3; \qquad x = 2 \Rightarrow u = 11$

$= \frac{1}{5}\int_{3}^{11} \frac{1}{\sqrt{u^2 - 3^2}}\,du$

$= \frac{1}{5}\left[\cosh^{-1}\left(\frac{u}{3}\right)\right]_3^{11}$

$= \frac{1}{5}\left(\cosh^{-1}\left(\frac{11}{3}\right) - \cosh^{-1}1\right)$

$= \frac{1}{5}\cosh^{-1}\left(\frac{11}{3}\right)$ since $\cosh^{-1}1 = 0$, as required

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3 **Fig. 1** below shows the design for an enamel badge.

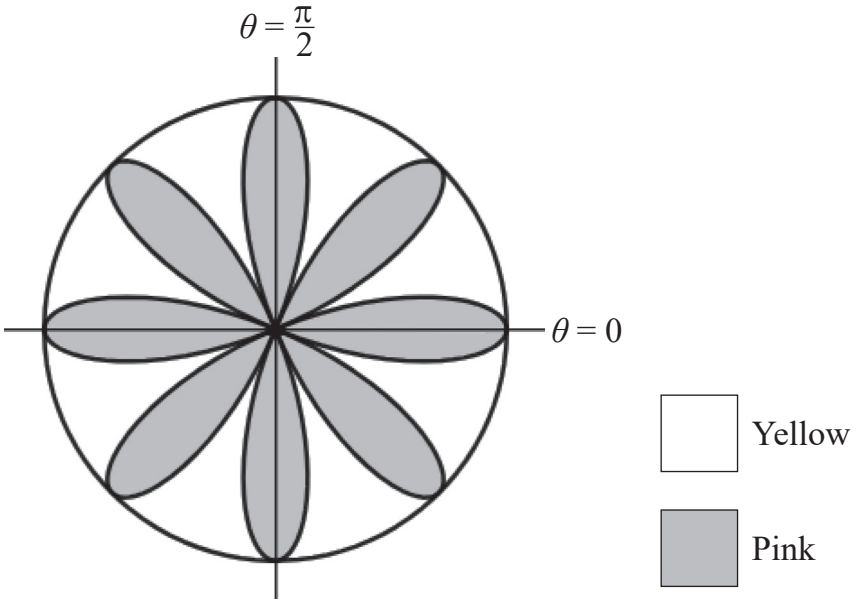


Fig. 1

The design is modelled by a curve enclosed within a circle, such that the curve touches the circle at multiple points, as shown.

The equation of the curve is $r = 6\cos 4\theta$

The shaded region is to be pink and the clear region is to be yellow.

By finding the area of the petal symmetric about the initial line ($\theta = 0$), show that an equal amount of each colour is required. [11]

Petal about $\theta = 0$ ends where $r = 0$: $\cos 4\theta = 0$

$4\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \pm\frac{\pi}{8}$

Area of one petal $A = \frac{1}{2}\int_{-\pi/8}^{\pi/8} (6\cos 4\theta)^2 \, d\theta$

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$A = \int_{-\pi/8}^{\pi/8} 18\cos^2 4\theta \, d\theta$

$= \int_{-\pi/8}^{\pi/8} 9(1 + \cos 8\theta) \, d\theta \qquad (\cos^2 4\theta = \frac{1}{2}(1 + \cos 8\theta))$

$= [9\theta + \frac{9}{8}\sin 8\theta]_{-\pi/8}^{\pi/8}$

$= (\frac{9\pi}{8} + \frac{9}{8}\sin \pi) - (-\frac{9\pi}{8} + \frac{9}{8}\sin(-\pi))$

$= \frac{9\pi}{8} + \frac{9\pi}{8} = \frac{9\pi}{4}$

There are 8 identical petals:

Shaded (pink) area $= 8 \times \frac{9\pi}{4} = 18\pi$

Maximum $r = 6$, so the circle has radius 6

Area of circle $= \pi \times 6^2 = 36\pi$

Clear (yellow) area $= 36\pi - 18\pi = 18\pi$

Pink area = yellow area $= 18\pi$, so an equal amount of each colour is required, as required

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4 The distance x cm of a particle P from a fixed point O, at time t seconds, can be modelled by

$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 12x = 24$$

(i) Solve this differential equation given that $\frac{dx}{dt} = 13$ and $x = 0$ when $t = 0$ [13]

Auxiliary equation: $m^2 + 7m + 12 = 0$

$(m + 3)(m + 4) = 0 \Rightarrow m = -3,\ m = -4$

Complementary function: $x = Ae^{-3t} + Be^{-4t}$

Particular integral: try $x = C$, $\frac{dx}{dt} = 0$, $\frac{d^2x}{dt^2} = 0$

$0 + 0 + 12C = 24 \Rightarrow C = 2$

General solution: $x = Ae^{-3t} + Be^{-4t} + 2$

$t = 0, x = 0$: $0 = A + B + 2 \Rightarrow A + B = -2$

$\frac{dx}{dt} = -3Ae^{-3t} - 4Be^{-4t}$

$t = 0, \frac{dx}{dt} = 13$: $13 = -3A - 4B$

From the first: $3A + 3B = -6$

Adding: $7 = -B \Rightarrow B = -7$

$A = -2 - (-7) = 5$

$x = 5e^{-3t} - 7e^{-4t} + 2$

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(ii) Find the limiting value of x as t increases. [1]

As $t \to \infty$, $e^{-3t} \to 0$ and $e^{-4t} \to 0$

so $x \to 2$. The limiting value of x is 2 cm

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5 **(i)** Using the exponential definitions of $\sinh x$ and $\cosh x$ show that for $|x| < 1$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad [6]$$

Let $y = \tanh^{-1} x$, so $\tanh y = x$

$\frac{\sinh y}{\cosh y} = x$

$\frac{\frac{1}{2}(e^{y} - e^{-y})}{\frac{1}{2}(e^{y} + e^{-y})} = x$

$\frac{e^{2y} - 1}{e^{2y} + 1} = x$ (multiplying top and bottom by e^{y})

$e^{2y} - 1 = x(e^{2y} + 1)$

$e^{2y} - xe^{2y} = 1 + x$

$e^{2y}(1 - x) = 1 + x$

$e^{2y} = \frac{1+x}{1-x}$ (valid as $|x| < 1$, so this is positive)

$2y = \ln\left(\frac{1+x}{1-x}\right)$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, as required

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(ii) Hence, or otherwise, solve the equation

$$x = 4\tanh\left(\ln\sqrt{3x}\right)$$

where $0 < x < 4$ [6]

$\frac{x}{4} = \tanh(\ln\sqrt{3x})$

$\tanh^{-1}\left(\frac{x}{4}\right) = \ln\sqrt{3x} = \frac{1}{2}\ln(3x)$

$\frac{1}{2}\ln\left(\frac{1+x/4}{1-x/4}\right) = \frac{1}{2}\ln(3x)$ (valid as $0 < x < 4$)

$\frac{1}{2}\ln\left(\frac{4+x}{4-x}\right) = \frac{1}{2}\ln(3x)$

$\frac{4+x}{4-x} = 3x$

$4 + x = 12x - 3x^2$

$3x^2 - 11x + 4 = 0$

$x = \frac{11 \pm \sqrt{121-48}}{6} = \frac{11 \pm \sqrt{73}}{6}$

$x = 3.26$ or $x = 0.409$ (both lie in $0 < x < 4$)

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6 **(i)** Show that

$$(r+1)^2 r^2 (2r+1) - r^2 (r-1)^2 (2r-1) \equiv 10r^4 + 2r^2 \qquad [4]$$

LHS = $r^2(r^2 + 2r + 1)(2r + 1) - r^2(r^2 - 2r + 1)(2r - 1)$

$= r^2(2r^3 + 5r^2 + 4r + 1) - r^2(2r^3 - 5r^2 + 4r - 1)$

$= r^2(2r^3 + 5r^2 + 4r + 1 - 2r^3 + 5r^2 - 4r + 1)$

$= r^2(10r^2 + 2)$

$= 10r^4 + 2r^2 \equiv$ RHS, as required

(ii) Hence show that

$$\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2 + 3n - 1) \qquad [8]$$

Let $f(r) = r^2(r-1)^2(2r-1)$, so $f(r+1) = (r+1)^2 r^2 (2r+1)$

$\sum_{r=1}^{n} (10r^4 + 2r^2) = \sum_{r=1}^{n} [f(r+1) - f(r)]$

$= [f(2) - f(1)] + [f(3) - f(2)] + \ldots + [f(n+1) - f(n)]$

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Writing out the terms:

$r = 1$: $2^2(1)^2(3) - 1^2(0)^2(1)$

$r = 2$: $3^2(2)^2(5) - 2^2(1)^2(3)$

$r = 3$: $4^2(3)^2(7) - 3^2(2)^2(5)$

...

$r = n - 1$: $n^2(n-1)^2(2n-1) - (n-1)^2(n-2)^2(2n-3)$

$r = n$: $(n+1)^2n^2(2n+1) - n^2(n-1)^2(2n-1)$

All terms cancel except the first and last, and $f(1) = 0$:

$$10\sum_{r=1}^{n} r^4 + 2\sum_{r=1}^{n} r^2 = n^2(n+1)^2(2n+1)$$

$$10\sum_{r=1}^{n} r^4 = n^2(n+1)^2(2n+1) - 2 \times \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{3}n(n+1)(2n+1)[3n(n+1) - 1]$$

$$= \frac{1}{3}n(n+1)(2n+1)(3n^2 + 3n - 1)$$

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1), \text{ as required}$$

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7 For each non-negative integer n let

$$I_n = \int_0^1 x^n e^{-4x}\,dx$$

(i) Show that for $n \geqslant 1$

$$I_n = -\frac{1}{4}e^{-4} + \frac{1}{4}nI_{n-1} \qquad [7]$$

$I_n = \int_0^1 x^n e^{-4x}\,dx$

By parts: $u = x^n$, $\frac{dv}{dx} = e^{-4x}$

$\frac{du}{dx} = nx^{n-1}$, $v = -\frac{1}{4}e^{-4x}$

$I_n = \left[-\frac{1}{4}x^n e^{-4x}\right]_0^1 + \int_0^1 \frac{1}{4}nx^{n-1}e^{-4x}\,dx$

$= \left(-\frac{1}{4}e^{-4} - 0\right) + \frac{1}{4}n\int_0^1 x^{n-1}e^{-4x}\,dx$ $\quad (n \geq 1,\ \text{so } 0^n = 0)$

$I_n = -\frac{1}{4}e^{-4} + \frac{1}{4}nI_{n-1}$, as required

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(ii) Hence find the exact value of

$$\int_0^1 x^2 e^{-4x}\,dx \qquad [5]$$

$\int_0^1 x^2e^{-4x}\,dx = I_2$

$I_0 = \int_0^1 e^{-4x}\,dx = \left[-\frac{1}{4}e^{-4x}\right]_0^1 = -\frac{1}{4}e^{-4} + \frac{1}{4}$

$I_1 = -\frac{1}{4}e^{-4} + \frac{1}{4}\left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right)$

$= -\frac{5}{16}e^{-4} + \frac{1}{16}$

$I_2 = -\frac{1}{4}e^{-4} + \frac{2}{4}\left(-\frac{5}{16}e^{-4} + \frac{1}{16}\right)$

$= -\frac{1}{4}e^{-4} - \frac{5}{32}e^{-4} + \frac{1}{32}$

$= -\frac{13}{32}e^{-4} + \frac{1}{32}$

$\int_0^1 x^2e^{-4x}\,dx = \frac{1}{32}(1 - 13e^{-4})$

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8 **(a)** Derive the Maclaurin expansion for

$$\cos^{-1} x$$

up to and including the term in x^3 [8]

$f(x) = \cos^{-1} x$

$f'(x) = -\frac{1}{\sqrt{1-x^2}} = -(1-x^2)^{-1/2}$

$f''(x) = \frac{1}{2}(1-x^2)^{-3/2} \times (-2x) = -x(1-x^2)^{-3/2}$

$f'''(x) = -(1-x^2)^{-3/2} - x \times (-\frac{3}{2})(1-x^2)^{-5/2} \times (-2x)$

$= -(1-x^2)^{-3/2} - 3x^2(1-x^2)^{-5/2}$

$= \frac{-(1-x^2) - 3x^2}{(1-x^2)^{5/2}} = \frac{-1-2x^2}{(1-x^2)^{5/2}}$

$f(0) = \frac{\pi}{2}, \quad f'(0) = -1, \quad f''(0) = 0, \quad f'''(0) = -1$

$\cos^{-1} x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$= \frac{\pi}{2} - x + 0x^2 - \frac{1}{3!}x^3 + \ldots$

$\cos^{-1} x \approx \frac{\pi}{2} - x - \frac{1}{6}x^3$

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(b) Using small angle approximations, show that

$$\frac{1 + 2\tan\theta}{13 + 5\sin\theta - 12\cos\theta} \approx 1 - 3\theta + 9\theta^2$$

when θ is small. [6]

For small θ: tan θ ≈ θ, sin θ ≈ θ, cos θ ≈ $1 - \frac{1}{2}\theta^2$

$$\frac{1 + 2\tan\theta}{13 + 5\sin\theta - 12\cos\theta} \approx \frac{1 + 2\theta}{13 + 5\theta - 12\left(1 - \frac{1}{2}\theta^2\right)}$$

$$= \frac{1 + 2\theta}{13 - 12 + 5\theta + 6\theta^2}$$

$$= \frac{1 + 2\theta}{1 + 5\theta + 6\theta^2}$$

$$= \frac{1 + 2\theta}{(1 + 2\theta)(1 + 3\theta)}$$

$$= (1 + 3\theta)^{-1}$$

Binomial expansion:

$$(1 + 3\theta)^{-1} = 1 + (-1)(3\theta) + \frac{(-1)(-2)}{2!}(3\theta)^2 + \ldots$$

$$= 1 - 3\theta + 9\theta^2 + \ldots$$

$$\frac{1 + 2\tan\theta}{13 + 5\sin\theta - 12\cos\theta} \approx 1 - 3\theta + 9\theta^2, \text{ as required}$$

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9 The expression

$$\cos\theta - \sqrt{3}\sin\theta$$

can be written in the form

$$R\cos(\theta+\alpha) \quad \text{where} \quad 0 < \alpha < \frac{\pi}{2} \quad \text{and} \quad R > 0$$

(i) Find the values of R and α. [5]

$R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 1, \quad R\sin\alpha = \sqrt{3}$

$R = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\tan\alpha = \sqrt{3} \quad \Rightarrow \quad \alpha = \frac{\pi}{3}$

$\cos\theta - \sqrt{3}\sin\theta = 2\cos(\theta + \frac{\pi}{3})$

(ii) Hence prove by mathematical induction that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\mathrm{e}^x\cos\left(\sqrt{3}x\right)\right] = 2^n\mathrm{e}^x\cos\left(\sqrt{3}x + \frac{n\pi}{3}\right)$$

for all integers $n \geqslant 1$ [11]

Test $n = 1$:

LHS $= \frac{d}{dx}[e^x\cos(\sqrt{3}x)] = e^x\cos(\sqrt{3}x) - \sqrt{3}e^x\sin(\sqrt{3}x)$

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$= e^x[\cos(\sqrt{3}x) - \sqrt{3}\sin(\sqrt{3}x)] = 2e^x\cos(\sqrt{3}x + \frac{\pi}{3})$ using (i)

RHS $= 2^1 e^x\cos(\sqrt{3}x + \frac{\pi}{3}) =$ LHS, so true for $n = 1$

Assume true for $n = k$:

$\frac{d^k}{dx^k}[e^x\cos(\sqrt{3}x)] = 2^k e^x\cos(\sqrt{3}x + \frac{k\pi}{3})$

Then for $n = k + 1$:

$\frac{d^{k+1}}{dx^{k+1}}[e^x\cos(\sqrt{3}x)] = \frac{d}{dx}[2^k e^x\cos(\sqrt{3}x + \frac{k\pi}{3})]$

$= 2^k e^x\cos(\sqrt{3}x + \frac{k\pi}{3}) - \sqrt{3} \times 2^k e^x\sin(\sqrt{3}x + \frac{k\pi}{3})$

$= 2^k e^x[\cos(\sqrt{3}x + \frac{k\pi}{3}) - \sqrt{3}\sin(\sqrt{3}x + \frac{k\pi}{3})]$

Using (i) with $\theta = \sqrt{3}x + \frac{k\pi}{3}$:

$= 2^k e^x \times 2\cos(\sqrt{3}x + \frac{k\pi}{3} + \frac{\pi}{3})$

$= 2^{k+1} e^x\cos(\sqrt{3}x + \frac{(k+1)\pi}{3})$

which is the result with $n = k + 1$, so true for $n = k + 1$.

Since it is true for $n = 1$, and if true for $n = k$ it is also true for $n = k + 1$, by mathematical induction it is true for all integers $n \geq 1$.

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10 (i) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x \sinh^{-1}(3x)\right] = \sinh^{-1}(3x) + \frac{3x}{\sqrt{1+9x^2}} \qquad [5]$$

Product rule: $\frac{d}{dx}[x \sinh^{-1}(3x)] = 1 \times \sinh^{-1}(3x) + x \times \frac{d}{dx}[\sinh^{-1}(3x)]$

$= \sinh^{-1}(3x) + x \times \frac{1}{\sqrt{1+(3x)^2}} \times 3$

$= \sinh^{-1}(3x) + \frac{3x}{\sqrt{1+9x^2}}$, as required

Fig. 2 below shows part of the graphs of $y = \sinh^{-1}(3x)$, $y = 0.5x$ and $x = 6$

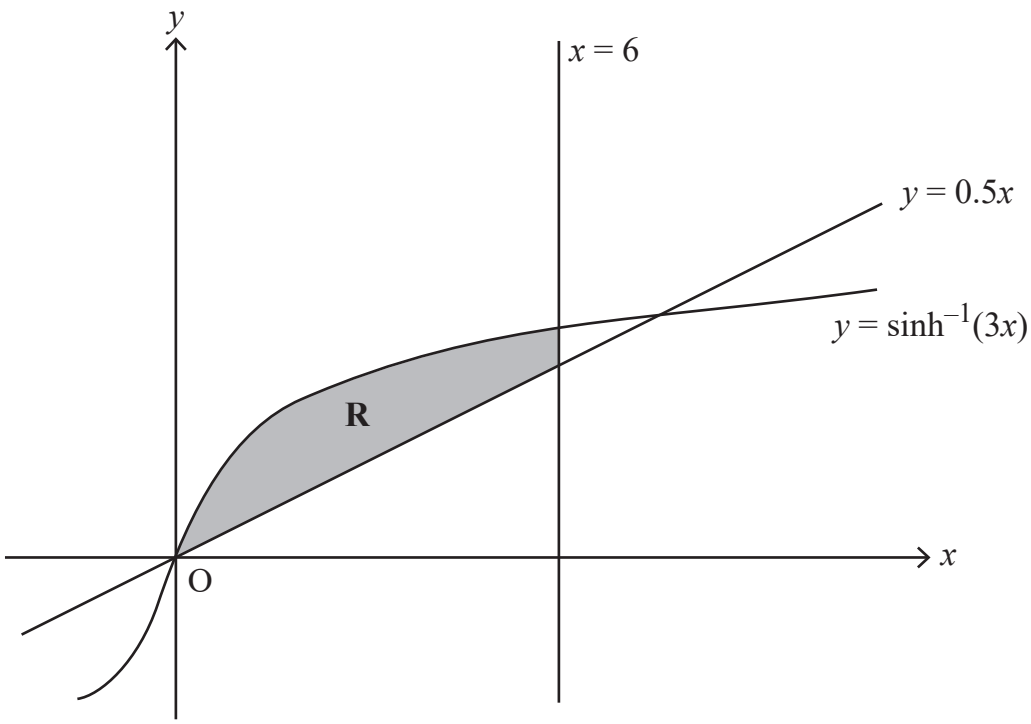


Fig. 2

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(ii) Using part **(i)**, find the exact area of the shaded region R in **Fig. 2**, enclosed between the curve $y = \sinh^{-1}(3x)$ and the lines $y = 0.5x$ and $x = 6$ [9]

From (i): $\sinh^{-1}(3x) = \frac{d}{dx}[x \sinh^{-1}(3x)] - \frac{3x}{\sqrt{1+9x^2}}$

Area under curve $= \int_0^6 \sinh^{-1}(3x)\,dx$

$= [x \sinh^{-1}(3x)]_0^6 - \int_0^6 \frac{3x}{\sqrt{1+9x^2}}\,dx$

$= [x \sinh^{-1}(3x)]_0^6 - [\frac{1}{3}\sqrt{1+9x^2}]_0^6$

$= 6 \sinh^{-1}(18) - (\frac{1}{3}\sqrt{325} - \frac{1}{3})$

$= 6 \sinh^{-1}(18) - \frac{1}{3}(5\sqrt{13} - 1)$

Line $y = 0.5x$ meets $x = 6$ at $y = 3$

Area of triangle $= \frac{1}{2} \times 6 \times 3 = 9$

Area of R $= 6 \sinh^{-1}(18) - \frac{1}{3}(5\sqrt{13} - 1) - 9$

$= 6 \sinh^{-1}(18) - \frac{5\sqrt{13}}{3} + \frac{1}{3} - 9$

$= 6 \sinh^{-1}(18) - \frac{1}{3}(5\sqrt{13} + 26)$ square units

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11 (a) Find, in the form $re^{i\theta}$, the values of the three roots of the equation

$$z^3 = \frac{-5 + 3i}{4 + i}$$ [6]

$\frac{-5 + 3i}{4 + i} = \frac{(-5 + 3i)(4 - i)}{(4 + i)(4 - i)}$

$= \frac{-20 + 5i + 12i - 3i^2}{16 + 1} = \frac{-17 + 17i}{17}$

$z^3 = -1 + i$

$|-1 + i| = \sqrt{1 + 1} = \sqrt{2}$

$\arg(-1 + i) = \frac{3\pi}{4}$ (second quadrant)

$z^3 = \sqrt{2}\, e^{i(\frac{3\pi}{4} + 2n\pi)}$

$z = (\sqrt{2})^{1/3}\, e^{i(\frac{\pi}{4} + \frac{2n\pi}{3})}$ for $n = 0, 1, 2$

$(\sqrt{2})^{1/3} = 2^{1/6}$

$n = 0$: $z = 2^{1/6}\, e^{i\pi/4}$

$n = 1$: $z = 2^{1/6}\, e^{i11\pi/12}$

$n = 2$: $z = 2^{1/6}\, e^{i19\pi/12}$

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(b) (i) By expanding $(\cos\theta + i\sin\theta)^6$ and applying de Moivre's theorem, prove that

$$\frac{\sin 6\theta}{\sin 2\theta} \equiv 2\cos 4\theta + 1 \quad \text{where} \quad 0 < \theta < \frac{\pi}{2} \qquad [10]$$

Let $c = \cos\theta$, $s = \sin\theta$. Binomial expansion:

$(c + is)^6 = c^6 + 6c^5(is) + 15c^4(is)^2 + 20c^3(is)^3$

$+ 15c^2(is)^4 + 6c(is)^5 + (is)^6$

$= c^6 + 6ic^5s - 15c^4s^2 - 20ic^3s^3 + 15c^2s^4 + 6ics^5 - s^6$

By de Moivre's theorem: $(c + is)^6 = \cos 6\theta + i\sin 6\theta$

Equating imaginary parts:

$\sin 6\theta = 6c^5s - 20c^3s^3 + 6cs^5$

$= 2sc(3c^4 - 10c^2s^2 + 3s^4)$

$= \sin 2\theta\,(3c^4 - 10c^2s^2 + 3s^4)$

Since $0 < \theta < \frac{\pi}{2}$, $\sin 2\theta \neq 0$, so

$\frac{\sin 6\theta}{\sin 2\theta} = 3c^4 - 10c^2s^2 + 3s^4$

$= 3c^4 - 10c^2(1 - c^2) + 3(1 - c^2)^2$

$= 3c^4 - 10c^2 + 10c^4 + 3 - 6c^2 + 3c^4$

$= 16c^4 - 16c^2 + 3$



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Now $2\cos 4\theta + 1 = 2(2\cos^2 2\theta - 1) + 1$

$= 4\cos^2 2\theta - 1$

$= 4(2c^2 - 1)^2 - 1$

$= 4(4c^4 - 4c^2 + 1) - 1$

$= 16c^4 - 16c^2 + 3$

Hence $\frac{\sin 6\theta}{\sin 2\theta} \equiv 2\cos 4\theta + 1$, as required

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(ii) Hence solve the equation

$$\sin 6\theta = 2\sin 2\theta \quad \text{for} \quad 0 < \theta < \frac{\pi}{2} \qquad [4]$$

$\sin 6\theta = 2 \sin 2\theta$ and $\sin 2\theta \neq 0$ for $0 < \theta < \frac{\pi}{2}$

$\frac{\sin 6\theta}{\sin 2\theta} = 2$

$2 \cos 4\theta + 1 = 2$

$\cos 4\theta = \frac{1}{2} \qquad (0 < 4\theta < 2\pi)$

$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

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