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ADVANCED SUBSIDIARY (AS)
General Certificate of Education
2024

Centre Number

Candidate Number

Further Mathematics

StemUp Model Solution

Assessment Unit AS 1

assessing

Pure Mathematics

[SFM11]

SFM11

MONDAY 13 MAY, AFTERNOON

TIME

1 hour 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer **all eight** questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages.
Complete in black ink only. **Do not write with a gel pen.**
Questions which require drawing or sketching should be completed using an HB pencil.
Show clearly the full development of your answers. **Answers without working may not gain full credit.**
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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1 Matrices **M** and **I** are defined by

$$\mathbf{M} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(i) Show that $\mathbf{M}^2 = 5\mathbf{M} - 6\mathbf{I}$ [4]

$M^2 = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 4+0 & 0+0 \\ -2-3 & 0+9 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 \\ -5 & 9 \end{pmatrix}$

$5M - 6I = 5\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 10 & 0 \\ -5 & 15 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 \\ -5 & 9 \end{pmatrix}$

$\therefore M^2 = 5M - 6I$ as required

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(ii) Hence, or otherwise, express the matrix $\mathbf{M}^4$ in the form $a\mathbf{M} + b\mathbf{I}$, where a and b are integers. [4]

$M^4 = (M^2)^2 = (5M - 6I)^2$

$= 25M^2 - 30MI - 30IM + 36I^2$

$= 25M^2 - 60M + 36I$ (since $MI = IM = M$, $I^2 = I$)

$= 25(5M - 6I) - 60M + 36I$

$= 125M - 150I - 60M + 36I$

$= 65M - 114I$

$a = 65$, $b = -114$

2 The roots of the quadratic equation

$$3x^2 - 2x + 4 = 0$$

are α and β.

(i) Write down the values of $(\alpha + \beta)$ and $\alpha\beta$. [3]

$\alpha + \beta = -\frac{b}{a} = \frac{2}{3}$

$\alpha\beta = \frac{c}{a} = \frac{4}{3}$

(ii) Find a quadratic equation with integer coefficients whose roots are

$\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ [6]

Sum of new roots $= \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{4}{9} - \frac{8}{3} = -\frac{20}{9}$

$\alpha^2\beta^2 = (\alpha\beta)^2 = \frac{16}{9}$

Sum $= -\frac{20}{9} \div \frac{16}{9} = -\frac{20}{16} = -\frac{5}{4}$

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Product of new roots $= \frac{1}{\alpha^2} \times \frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2}$

$= 1 \div \frac{16}{9} = \frac{9}{16}$

Equation: $x^2 - (\text{sum})x + (\text{product}) = 0$

$x^2 + \frac{5}{4}x + \frac{9}{16} = 0$

$16x^2 + 20x + 9 = 0$

[Turn over

3 The matrix $\mathbf{P} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

(i) Describe fully the single transformation represented by **P** [3]

$\cos 45° = \sin 45° = \frac{\sqrt{2}}{2}$

so $P = \begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix}$

P is a rotation of 45° anticlockwise about the origin

The matrix $\mathbf{Q} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$

The matrix **S** represents the combined effect of the transformation represented by **P** followed by the transformation represented by **Q**

(ii) Find the matrix **S** [2]

S = QP (P first, then Q)

$P = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$S = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

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$= \frac{\sqrt{2}}{2}\begin{pmatrix} 1-1 & -1-1 \\ 0+3 & 0+3 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & -2 \\ 3 & 3 \end{pmatrix}$

$S = \begin{pmatrix} 0 & -\sqrt{2} \\ 3\sqrt{2}/2 & 3\sqrt{2}/2 \end{pmatrix}$

A region R is mapped to a new region T under the transformation represented by **S**

(iii) If the area of T is 60 cm^2, find the area of R. [3]

$\det S = (0)\left(\frac{3\sqrt{2}}{2}\right) - (-\sqrt{2})\left(\frac{3\sqrt{2}}{2}\right)$

$= 0 + \frac{3 \times 2}{2} = 3$

Area of T = 3 × Area of R

Area of R = 60 ÷ 3

= 20 cm^2

[Turn over

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4 **(a)** The complex numbers z_1 and z_2 are given by

$$z_1 = 2 + 2i \quad \text{and} \quad z_2 = \sqrt{3} - i$$

(i) Find the modulus and argument of each of the complex numbers z_1 and z_2 [5]

$|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\arg z_1 = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$ (first quadrant)

$|z_2| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

z_2 is in the fourth quadrant:

$\arg z_2 = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

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(ii) Verify that $|z_1 z_2| = |z_1||z_2|$ [5]

$z_1z_2 = (2 + 2i)(\sqrt{3} - i)$

$= 2\sqrt{3} - 2i + 2\sqrt{3}\,i - 2i^2$

$= (2\sqrt{3} + 2) + (2\sqrt{3} - 2)i$

$|z_1z_2| = \sqrt{(2\sqrt{3} + 2)^2 + (2\sqrt{3} - 2)^2}$

$= \sqrt{(12 + 8\sqrt{3} + 4) + (12 - 8\sqrt{3} + 4)}$

$= \sqrt{32} = 4\sqrt{2}$

$|z_1||z_2| = 2\sqrt{2} \times 2 = 4\sqrt{2}$

$\therefore |z_1z_2| = |z_1||z_2|$ as required

[Turn over

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(b) The equation

$$z^3 + z^2 + 11z + 51 = 0$$

has a root (1 – 4i).

Find the other two roots of the equation. [6]

Coefficients are real, so the conjugate 1 + 4i is also a root.

$(z - (1 - 4i))(z - (1 + 4i)) = ((z - 1) + 4i)((z - 1) - 4i)$

$= (z - 1)^2 - (4i)^2$

$= z^2 - 2z + 1 + 16 = z^2 - 2z + 17$

$z^3 + z^2 + 11z + 51 = (z^2 - 2z + 17)(z + c)$

Constant terms: $17c = 51 \Rightarrow c = 3$

Check z^2 terms: $c - 2 = 1$ ✓ z terms: $17 - 2c = 11$ ✓

$\therefore z^3 + z^2 + 11z + 51 = (z^2 - 2z + 17)(z + 3)$

$z + 3 = 0 \Rightarrow z = -3$

Other two roots: $z = 1 + 4i$ and $z = -3$

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5 Three lines are defined by

L_1 $\mathbf{r} = (-\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

L_2 $\mathbf{r} = (-2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + \lambda(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

L_3 $\dfrac{x-3}{2} = \dfrac{y-1}{-1} = \dfrac{z+2}{3}$

(i) Show that the lines L_1 and L_2 are skew. [6]

Directions $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ are not multiples of each other,

so L_1 and L_2 are not parallel.

If they meet: $\begin{pmatrix} -1+\mu \\ 3-\mu \\ 5+2\mu \end{pmatrix} = \begin{pmatrix} -2+3\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix}$

i: $-1 + \mu = -2 + 3\lambda$ ①

j: $3 - \mu = 2 - \lambda$ ②

k: $5 + 2\mu = -2 + 2\lambda$ ③

① + ②: $2 = 2\lambda \Rightarrow \lambda = 1$

②: $3 - \mu = 1 \Rightarrow \mu = 2$

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Check in ③: LHS = 5 + 2(2) = 9

RHS = −2 + 2(1) = 0

9 ≠ 0, so the equations are inconsistent and the lines do not meet.

The lines are not parallel and do not intersect,

so L_1 and L_2 are skew as required

[Turn over

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(ii) Find the coordinates of the point at which L_2 and L_3 intersect. [5]

L_3: $\frac{x-3}{2} = \frac{y-1}{-1} = \frac{z+2}{3} = \theta$

$x = 3 + 2\theta, \quad y = 1 - \theta, \quad z = -2 + 3\theta$

$$\begin{pmatrix} -2+3\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 3+2\theta \\ 1-\theta \\ -2+3\theta \end{pmatrix}$$

i: $-2 + 3\lambda = 3 + 2\theta$ ①

j: $2 - \lambda = 1 - \theta \Rightarrow \lambda = 1 + \theta$ ②

Sub ② into ①: $-2 + 3 + 3\theta = 3 + 2\theta \Rightarrow \theta = 2$

$\lambda = 3$

k: L_2: $-2 + 2(3) = 4$ L_3: $-2 + 3(2) = 4$ ✓ consistent

Point: $x = -2 + 3(3) = 7, \quad y = 2 - 3 = -1, \quad z = 4$

Intersect at $(7, -1, 4)$

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6 **(a)** The complex number z is such that

$$|z| = 6 \qquad \text{and} \qquad \arg z = -\frac{\pi}{3}$$

Express z in the form $a + b\mathrm{i}$ where a and b are real numbers. [4]

$a = |z| \cos(\arg z) = 6 \cos(-\frac{\pi}{3}) = 6 \times \frac{1}{2} = 3$

$b = |z| \sin(\arg z) = 6 \sin(-\frac{\pi}{3}) = 6 \times (-\frac{\sqrt{3}}{2}) = -3\sqrt{3}$

$z = 3 - 3\sqrt{3}\,i$

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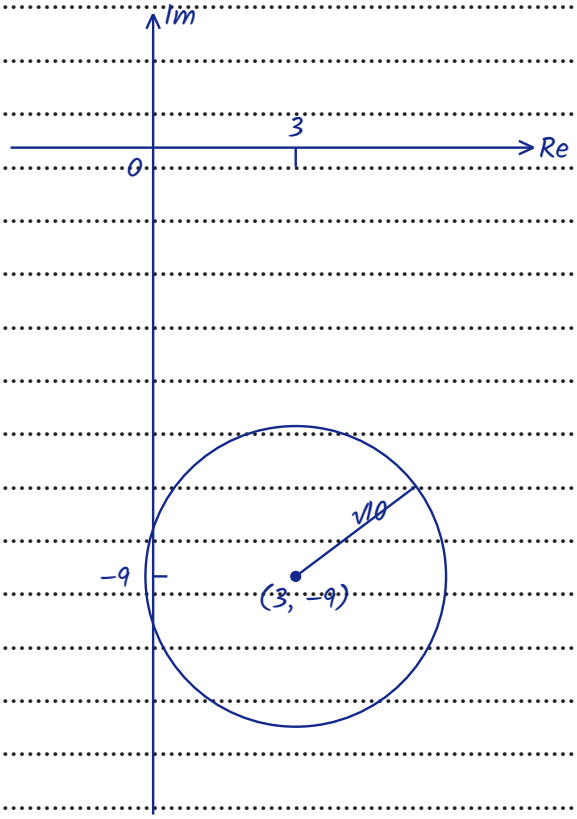
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(b) (i) Sketch on an Argand diagram the locus of those points representing complex numbers z which satisfy

$$|z - (3 - 9i)| = \sqrt{10} \qquad [3]$$

Circle, centre 3 – 9i, i.e. (3, –9), radius √10



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(ii) Hence, or otherwise, find the maximum value of $|z|$ for the complex numbers z which satisfy

$$|z - (3 - 9i)| = \sqrt{10} \qquad [4]$$

$|z|$ is the distance from O to the point z on the circle.
Maximum is at the point P where the line from O through the centre C meets the circle on the far side, so OP = OC + radius.
$OC = \sqrt{3^2 + (-9)^2} = \sqrt{90} = 3\sqrt{10}$
max $|z| = 3\sqrt{10} + \sqrt{10} = 4\sqrt{10}$
$= 12.6$ (3 s.f.)

[Turn over



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7 A matrix **N** is given by

$$\mathbf{N} = \begin{pmatrix} p & 1 & p+2 \\ 3 & -1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

(i) Find the value of p for which this matrix is singular. [5]

Singular $\Leftrightarrow$ det N = 0

det N = p((−1)(4) − (0)(1)) − 1((3)(4) − (0)(2))
+ (p + 2)((3)(1) − (−1)(2))

= p(−4) − 1(12) + (p + 2)(5)

= −4p − 12 + 5p + 10

= p − 2

p − 2 = 0 ⇒ p = 2

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(ii) If $p = -1$ find, without the use of a calculator, the inverse of **N** [5]

$p = -1$: $N = \begin{pmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$

$\det N = p - 2 = -1 - 2 = -3$

Matrix of minors $= \begin{pmatrix} -4 & 12 & 5 \\ 3 & -6 & -3 \\ 1 & -3 & -2 \end{pmatrix}$

Matrix of cofactors $= \begin{pmatrix} -4 & -12 & 5 \\ -3 & -6 & 3 \\ 1 & 3 & -2 \end{pmatrix}$

Adjoint (transpose) $= \begin{pmatrix} -4 & -3 & 1 \\ -12 & -6 & 3 \\ 5 & 3 & -2 \end{pmatrix}$

$N^{-1} = \frac{1}{\det N} \times \text{adj } N = -\frac{1}{3}\begin{pmatrix} -4 & -3 & 1 \\ -12 & -6 & 3 \\ 5 & 3 & -2 \end{pmatrix}$

$N^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 3 & -1 \\ 12 & 6 & -3 \\ -5 & -3 & 2 \end{pmatrix}$

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(iii) Hence solve the following system of linear equations.

$$\begin{aligned} -x + y + z &= -3 \\ 3x - y &= 14 \\ 2x + y + 4z &= 18 \end{aligned}$$

[4]

$N\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \\ 18 \end{pmatrix}$ with $p = -1$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = N^{-1}\begin{pmatrix} -3 \\ 14 \\ 18 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & 3 & -1 \\ 12 & 6 & -3 \\ -5 & -3 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 14 \\ 18 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} -12 + 42 - 18 \\ -36 + 84 - 54 \\ 15 - 42 + 36 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 12 \\ -6 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$

$x = 4, \quad y = -2, \quad z = 3$

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(iv) If $p = 2$, give a geometrical interpretation for the system of equations

$$\mathbf{N}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \\ 18 \end{pmatrix}$$

[3]

$p = 2$: $\begin{pmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \\ 18 \end{pmatrix}$

Planes: $2x + y + 4z = -3$, $3x - y = 14$, $2x + y + 4z = 18$

The first and third planes have the same normal but different constants, so they are two separate parallel planes.

The second plane is not parallel to them, so it cuts both of them (in two parallel lines).

The system is inconsistent: there is no solution.

[Turn over

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8 The tetrahedron WXYZ has vertices (1, 2, 3), (–1, 0, 4), (2, 2, 2) and (3, –2, 5) respectively.

(i) Find a Cartesian equation of the plane WXY. [9]

$W(1, 2, 3)$, $X(-1, 0, 4)$, $Y(2, 2, 2)$

$\overrightarrow{WX} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

$\overrightarrow{WY} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Normal $n = \overrightarrow{WX} \times \overrightarrow{WY}$

$= i((-2)(-1) - (1)(0)) - j((-2)(-1) - (1)(1)) + k((-2)(0) - (-2)(1))$

$= 2i - j + 2k$

Plane: $r \cdot n = a \cdot n$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 2 - 2 + 6 = 6$

$2x - y + 2z = 6$

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(ii) Find the volume of the tetrahedron. [6]

$Z(3, -2, 5)$: $\overrightarrow{WZ} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$

Volume $= \frac{1}{6} \left| \overrightarrow{WZ} \cdot (\overrightarrow{WX} \times \overrightarrow{WY}) \right|$

$= \frac{1}{6} \left| \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right|$

$= \frac{1}{6} | 4 + 4 + 4 |$

$= \frac{1}{6} \times 12$

$= 2 \text{ units}^3$

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(iii) Calculate the acute angle between the line WZ and the plane WXY. [5]

Let θ be the angle between WZ and the normal n:

$$\begin{pmatrix}2\\-4\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\2\end{pmatrix} = \left|\begin{pmatrix}2\\-4\\2\end{pmatrix}\right| \left|\begin{pmatrix}2\\-1\\2\end{pmatrix}\right| \cos\theta$$

$$12 = \sqrt{24} \times \sqrt{9} \times \cos\theta$$

$$\cos\theta = \frac{12}{3\sqrt{24}} = 0.8165\ldots$$

$$\theta = 35.26°$$

Angle between line and plane $= 90° - 35.26°$

$= 54.7°$ (3 s.f.)

THIS IS THE END OF THE QUESTION PAPER

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Question Number	**Marks**
1	
2	
3	
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7	
8	
Total Marks	

Examiner Number

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