Rewarding Learning

ADVANCED SUBSIDIARY (AS)
General Certificate of Education
2023

Further Mathematics *StemUp Model Solution*

Assessment Unit AS 2
assessing
Applied Mathematics

SFM21

[SFM21]
TUESDAY 23 MAY, AFTERNOON

TIME

1 hour 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided and on the Section D Supplementary Booklet (if applicable).
You must answer **all** questions from sections A and B **or** A and C **or** A and D **or** C and D.
You should spend equal time on each of the two sections.
Candidates taking Section D should use the insert provided and attach to your Answer Booklet using the treasury tag provided.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100.
The total mark for each section of this paper is 50.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
Answers should include diagrams where appropriate and marks may be awarded for them.
Take $g = 9.8\,\text{m}\,\text{s}^{-2}$, unless specified otherwise.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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SECTION A Mechanics 1

Answer all five questions in this section.

1 A particle P of mass 8 kg is projected with velocity u m s^{-1} up the line of greatest slope of a rough inclined plane, as shown in **Fig. 1** below.

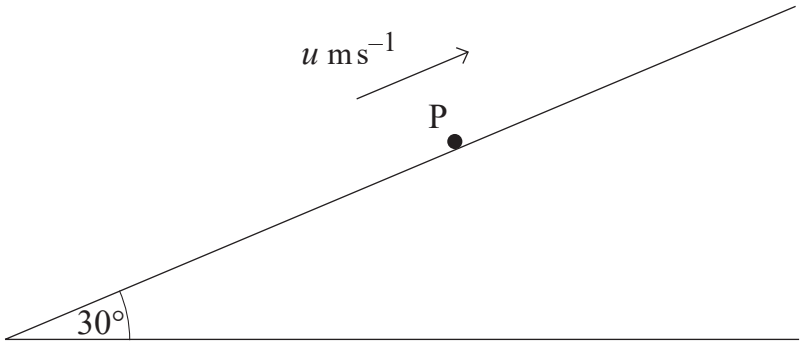


Fig. 1

The coefficient of friction between P and the surface of the plane is $\frac{\sqrt{3}}{6}$

The plane is inclined at 30° to the horizontal.

P moves a distance of 0.3 m up the plane before coming to rest.

Using work and energy considerations, find the value of u. [9]

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2 **(a)** A particle Q is moving along a horizontal line through O.

The distance from O to Q is x metres.

A variable force, H newtons, acts on Q where

$$H = 4x - \frac{56}{x^2}$$

Find the work done by H as the particle Q moves from $x = 4$ to $x = 7$ [4]

(b) A particle P of mass 1 kg moves under the action of two forces **F** and **G** newtons

$$\mathbf{F} = \begin{pmatrix} 5 \\ 2 \\ 8 \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} -1 + \lambda \\ 2 - 2\lambda \\ 3 + 3\lambda \end{pmatrix}$$

from a point A to a point B where

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \text{m} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 8 \\ 7 \\ 6 \end{pmatrix} \text{m}$$

(i) Find, in terms of λ, the work done by the resultant force as the particle moves from A to B. [4]

The speed of P changes from 11 m s^{-1} at A to 15 m s^{-1} at B.

(ii) Hence find the value of λ. [2]

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3 The vertical fountain in Granny's pond is 0.75 m tall as shown in **Fig. 2** below.

The fountain's pump lifts still water through a pipe to the top of the fountain where it is moving at $1.2\,m\,s^{-1}$

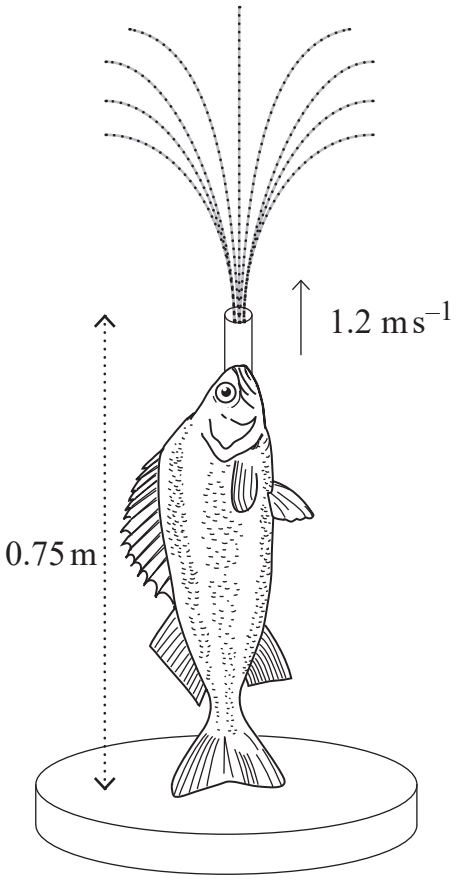


Fig. 2

The pipe has a circular cross-section with diameter of 0.02 m.

The density of water is $1000\,kg\,m^{-3}$

Find the effective power of the fountain's pump. [8]

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4 A particle P has mass m kg.

It is attached to two fixed points A and B in the same vertical line by means of two light inextensible strings, of equal length, each inclined at $\alpha°$ to the vertical.

P executes horizontal circles, centre C, of radius r m, with velocity v m s^{-1}, as shown in **Fig. 3** below.

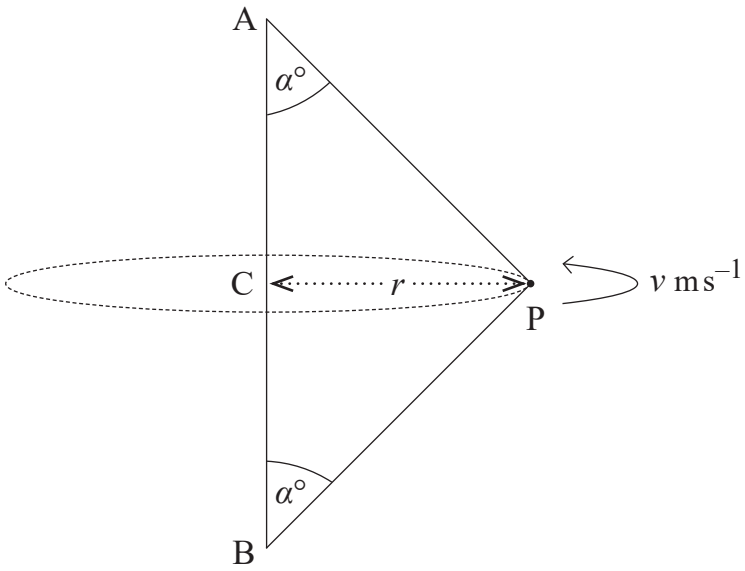


Fig. 3

Let the tension in AP be T_1 newtons and the tension in BP be T_2 newtons.

(i) Show that

$$T_1 = T_2 + \frac{mg}{\cos\alpha}$$ [2]

(ii) Show that

$$T_1 + T_2 = \frac{mv^2}{r\sin\alpha}$$ [3]

(iii) Using **(i)** and **(ii)**, deduce that

$$T_2 = \frac{m}{2}\left(\frac{v^2}{r\sin\alpha} - \frac{g}{\cos\alpha}\right)$$ [2]

(iv) Hence deduce that the minimum velocity with which P can move in such a circle is given by

$$v = \sqrt{rg\tan\alpha}$$ [2]

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5 A block of mass $5m$ kg rests on a rough, horizontal plane at point P.

The coefficient of friction between the block and the plane is $\mu = \frac{1}{9}$

The block is attached to a light elastic string with natural length 0.18 m and modulus of elasticity λ newtons.

The string passes over a smooth fixed pulley at point A, 0.02 m from P and is attached to a fixed point B, 0.18 m from P.

A and B lie on the same horizontal level as the block.

A smooth ring of mass m kg is threaded onto the string and is held at rest at the mid-point of AB as shown in **Fig. 4** below.

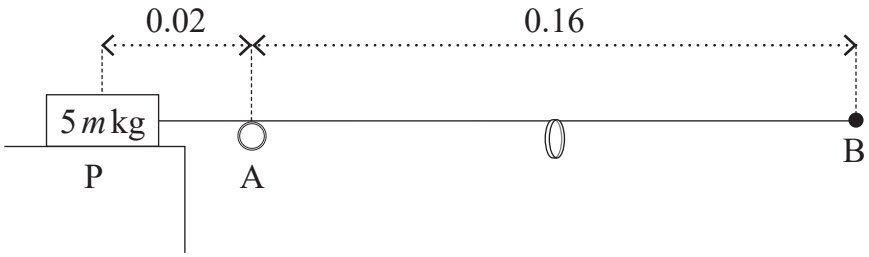


Fig. 4

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The ring is released from rest and falls 0.15 m vertically to the point C where it is moving at v m s^{-1}, as shown in **Fig. 5** below.

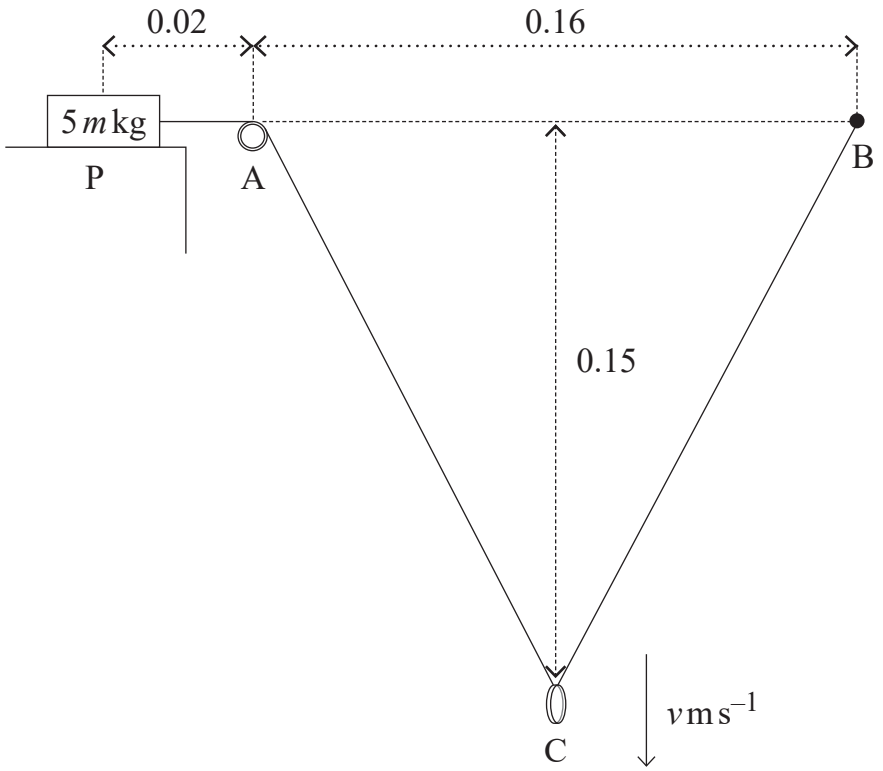


Fig. 5

The block is on the point of moving.

Model the block and the ring as particles.

(Note that the ring can slide freely along the string.)

(i) By considering the forces acting on the block at the instant when the ring is at C, show that $\lambda = \frac{5}{9} mg$ [6]

(ii) Using the Principle of Conservation of Mechanical Energy, show that $v = 1.4$ [8]

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SECTION B Mechanics 2

Answer all five questions in this section.

1 A rollercoaster designer is planning a vertical circular loop in a track as shown in **Fig. 1** below.

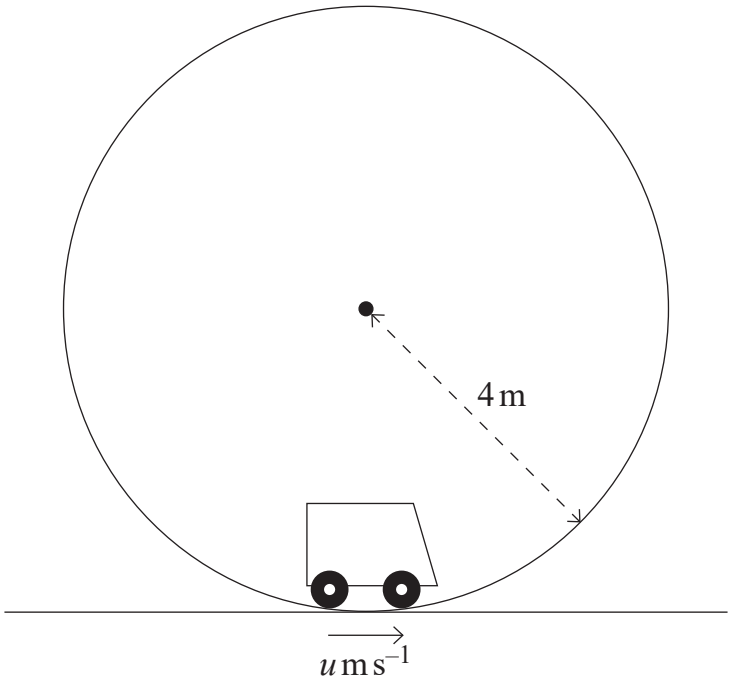


Fig. 1

The engineless car is moving with velocity u m s^{-1} at the lowest point in the loop.

A typical car including passengers has a combined mass of 1120 kg.

The car must travel along the track and around the inside of the loop without losing contact.

As a safety measure, the downward normal reaction exerted by the track on the car at the highest point must be greater than or equal to 2500 N.

The radius of the loop is 4 m.

Model the car and passengers as a particle and the track as smooth.

(i) Find the velocity of the car at the top of the loop when the normal reaction is 2500 N. [4]

(ii) Hence find the least possible value of u for the safety requirement to be satisfied. [4]

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2 A satellite of mass m kg at rest on the earth's surface has weight mg newtons.

The mass of the earth is M kg and its radius is R metres.

(i) Find an expression for the Universal Gravitational Constant G in terms of M, R and g. [2]

The satellite orbits the earth in a circular orbit around the equator travelling at 7200 m s^{-1}

(ii) If $R = 6.37 \times 10^6$, find the height of this orbit above the earth's surface. [4]

3 When liquid rises up a capillary tube, the height it reaches is given by the equation

$$h = \frac{2S}{\rho g a}$$

where S is the surface tension of the liquid,
ρ is the density of the liquid,
g is the acceleration due to gravity and
a is the radius of the capillary tube.

(i) Find the dimensions of surface tension, S, given that density has dimensions ML^{-3} [3]

Liquid resting on a horizontal surface forms bubbles of depth y, where y is given by the relationship

$$y = k\rho^{\alpha} g^{\beta} S^{\gamma}$$

where k is a dimensionless constant.

(ii) Find α, β and γ using the Method of Dimensions. [7]

Two liquids L_0 and L_1 form bubbles of depth y_0 and y_1 respectively.

The density of L_1 is 60% less than that of L_0

The surface tension of L_1 is 37.5% less than that of L_0

(iii) Find y_1 in terms of y_0 [3]

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4 An aeroplane flies a return trip from Belfast City Airport (BHD) to London Heathrow (LHR).

LHR is 513 km from BHD.

The flight from BHD to LHR happens in still air and takes one hour and five minutes.

(i) Find the speed of the aeroplane in $km\,h^{-1}$ in still air. [2]

LHR is on a bearing of 120° from BHD.

During the return journey from LHR to BHD, a wind blows from a bearing of 280° at 41 $km\,h^{-1}$ as shown in **Fig. 2** below.

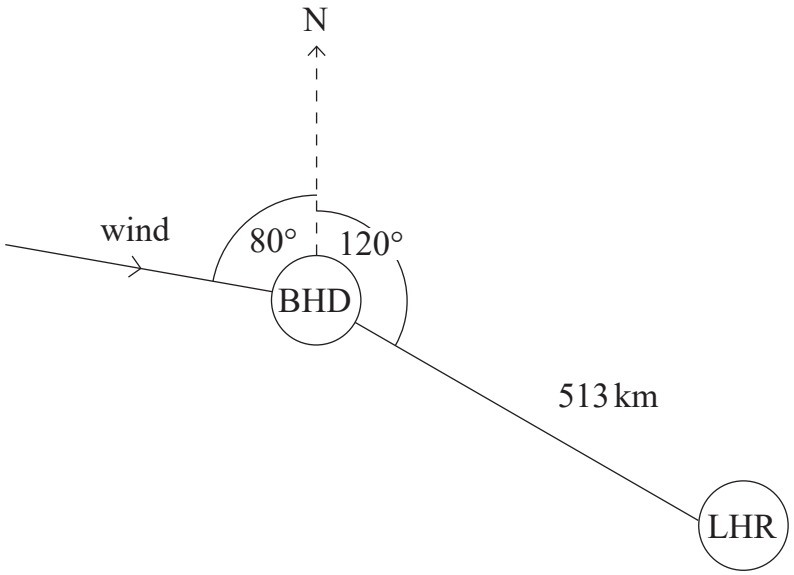


Fig. 2

The speed of the aeroplane in still air is constant for both parts of the journey.

(ii) Find the time taken for the return journey. [9]

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5 The ends of a light elastic string of natural length 0.6 m are attached to two fixed points P and Q which are 0.65 m apart on the same horizontal level.

A smooth ring of mass 2.67 kg is threaded on the string.

The length of the extended string is 0.89 m.

The ring is held in place at R by a horizontal force F newtons acting so that RP and RQ are at right angles, as shown in **Fig. 3** below, with RQ < PR.

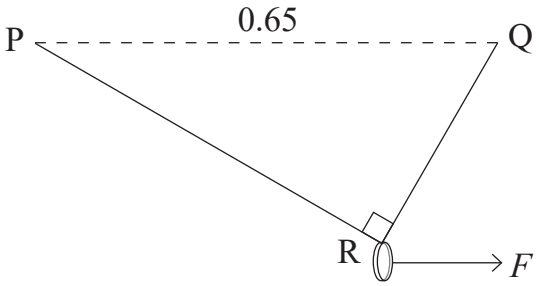


Fig. 3

(i) Using Pythagoras' theorem, show that RQ = 0.33 m. [4]

(ii) Show that the modulus of elasticity of the string is approximately 39.5 N. [6]

(iii) Calculate the value of the force F. [2]

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SECTION C Statistics

Answer all five questions in this section.

1 **(a)** The random variable Y is found directly from the random variable X using the formula

$$Y = aX + b$$

where a and b are positive constants.

X has expected value 15 and variance 6
Y has expected value 25 and variance 13.5

Find the values of a and b. [5]

(b) A researcher is investigating a possible relationship between IQ score and head circumference in adult humans.

She randomly selects a number of adults to participate in the study and asks them to complete an IQ test.

She records the IQ score and head circumference for each adult.

From the data, the researcher found the equation of the least squares regression line of IQ score on head circumference.

(i) Write down the response variable in this study. [1]

(ii) Comment on the reliability of using the researcher's regression line to estimate the head circumference of an adult with a given IQ score. [1]

The researcher calculated the product–moment correlation coefficient to be + 0.92

(iii) Comment on whether or not this provides sufficient evidence to show that head circumference affects IQ score. [2]

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2 A die is biased such that the probability of it landing on a two in a single throw is 0.2

Let the number of throws needed for the die to land on a two for the first time be modelled by the random variable X.

(i) Copy and complete **Table 1** below showing the probability distribution of X. [5]

Table 1

x	1	2	3	4 or more
$P(X = x)$				

(ii) Find the variance of X.

$\left[\text{If } X \sim \text{Geo}(p), \text{ then } \text{Var}(X) = \frac{1-p}{p^2}\right]$ [2]

3 Six racehorses are being entered into a charity race.

Each horse has a different one of the first six letters of the alphabet printed on its saddle.

At the start of the race they are randomly placed into a line of six traps.

(i) Find the number of possible arrangements. [1]

The race now begins with each horse having an equal chance of winning.

Find the probability that the horses labelled A, B, and C finish:

(ii) first, second and third respectively; [2]

(iii) in the top three places (in any order). [2]

After the race the six horses are allowed to cool down by walking around a circular paddock.

They enter the paddock in no particular order and are equally spaced around the closed circle.

(iv) Find the number of possible arrangements if there are no further restrictions. [2]

(v) Find the number of arrangements in which horses labelled A, B, and C are next to each other. [3]

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4 During the peak of a meteor shower the number of meteors detected per hour follows a Poisson distribution.

On average, during the peak of any Orionids meteor shower, 20 meteors are detected per hour.

During the peak of the next Orionids meteor shower find the probability that:

(i) exactly 15 meteors are detected in a one-hour period; [2]

(ii) more than 12 meteors are detected in a 30-minute period. [3]

The Lyrids meteor shower has a lower average peak rate of meteors.

The probability that more than ten meteors are detected in a one-hour period during the peak of the Lyrids meteor shower is 0.3 (to 1 dp).

(iii) Using the Poisson Cumulative Distribution Function table, find the average hourly rate of detected meteors for the peak of the Lyrids meteor shower.

Give your answer to the nearest whole number. [3]

5 The random variable X has the continuous probability density function f(x) defined by

$$\text{f}(x) = \begin{cases} ax^2 + bx + \frac{1}{6} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants.

(i) Given that $\text{E}(X) = \frac{13}{9}$ find the values of a and b. [10]

(ii) Find the modal value of X. [2]

(iii) Show that M, the median value of X, satisfies the equation

$$M^3 - M^2 + M - 3 = 0$$ [4]

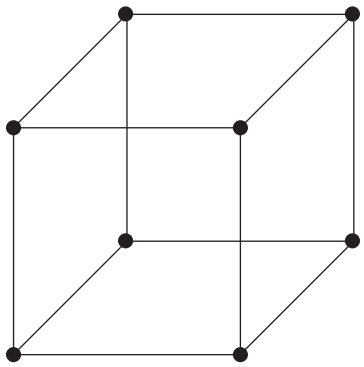
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SECTION D Discrete and Decision Mathematics

Answer all five questions in this section.

Note: A Supplementary Answer Booklet has been provided for Questions **1(b)** and **4(b)(i)**
Please attach this to your Answer Booklet using the treasury tag provided.

1 **(a)** The graph G_c of the cube is shown in **Fig. 1** below



G_c

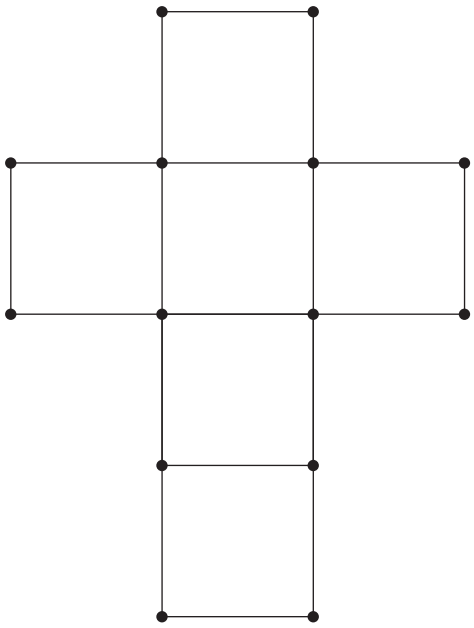
Fig. 1

(i) State whether each of the graphs G_c or $K_{3,3}$ is planar or not. [2]

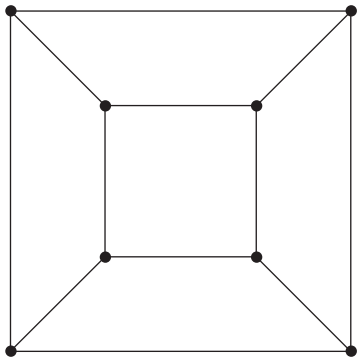
(ii) If either G_c or $K_{3,3}$ is planar then redraw that graph with no edge crossings. [2]

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(b) The graphs R and S are shown in **Fig. 2** below.



R



S

Fig. 2

Copy and complete in the **Supplementary Answer Booklet** the table shown in **Fig. 3** below by

- ticking in the 2nd column if the graph has an Eulerian trail – one that need not start and finish on the same vertex
- writing the minimum number of edges that need to be removed to leave a subgraph with an Eulerian circuit in the 3rd column.

Graph	Has an Eulerian trail	Number of edges that must be removed to leave an Eulerian circuit
R		
S		

[4]

Fig. 3

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2 An entomologist discovers a new variety of locust.

The number of locusts in any generation is found to be twice the number in the parent generation plus three times the number in the grandparent generation.

Let u_n be the number of locusts, in 1000s, in generation n.

(i) Show that u_n satisfies the recurrence relationship

$$u_{n+2} = 2u_{n+1} + 3u_n$$

where $n \geqslant 0$ is an integer. [1]

There are 1000 locusts in each of the two generations ($n = 0$, 1)

(ii) Show that

$$u_n = k(\alpha^n + \beta^n)$$

where α and β are integers to be determined. [6]

(iii) Calculate how many locusts are in generation 10 [2]

3 Let p, q and r be propositional statements.

Use truth tables to prove that

$$(p \text{ and } \sim q) \text{ or } \sim r$$

and

$$\sim[\ (\sim p \text{ or } q) \text{ and } r\ ]$$

are equivalent. [8]

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4 **(a)** The distances, in km, between 9 towns are given in the route map **Fig. 4** below.

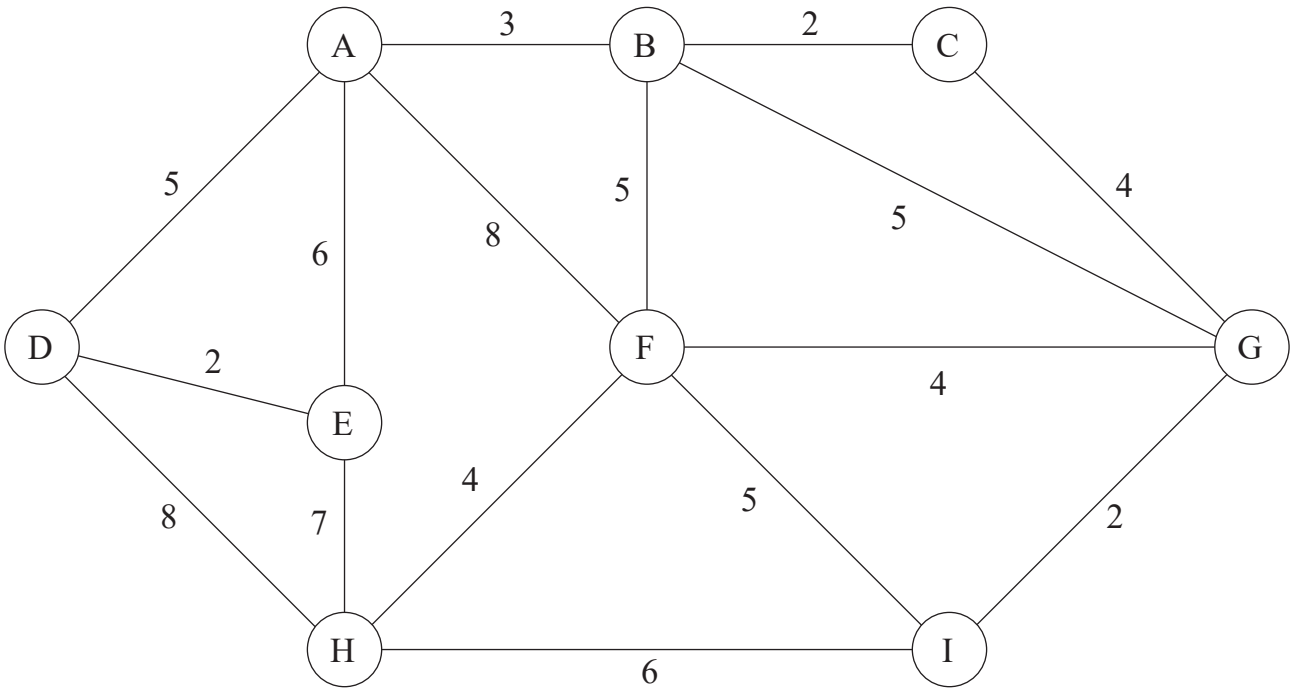


Fig. 4

A delivery company wants to find a minimum spanning tree for this route map.

(i) Using Prim's algorithm and starting at vertex A, find a minimum spanning tree for this route map.

List the arcs in the order in which they are selected. [4]

(ii) What is the distance of this spanning tree? [1]

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(b) **Fig. 5** below represents the activity network of a set-up for a conference.

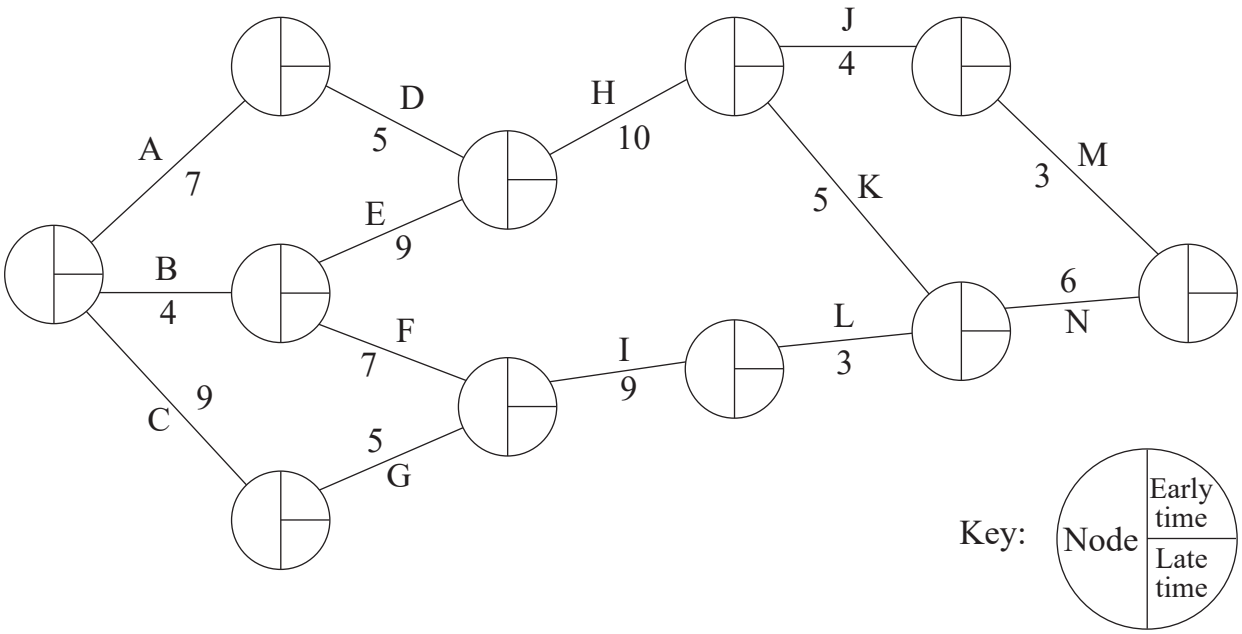


Fig. 5

Activities are represented by the arcs A, B, C, ..., M and N.

The adjacent numbers give the time in days for the associated activity.

(i) Calculate the early event time and the late event time for each event.

Record your answers in the **Supplementary Answer Booklet** provided. [4]

(ii) Write down the critical path. [1]

(iii) State the minimum number of days for the completion of the conference set-up. [1]

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5 The set G = {5, 15, 25, 35} forms a group under multiplication modulo 40, $\times_{40}$

(i) Copy and complete the group table of $(G, \times_{40})$ started in **Fig. 6** below.

$\times_{40}$	5	15	25	35
5	25	35	5	15
15	35			
25	5			
35	15			

Fig. 6 [3]

(ii) Write down the identity element of $(G, \times_{40})$. [1]

(iii) Copy and complete the table of elements and their periods of $(G, \times_{40})$ in **Fig. 7** below.

Element	5	15	25	35
Period				

Fig. 7 [2]

(iv) List all the subgroups of $(G, \times_{40})$. [3]

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The set H = $\{e, a, b, c\}$ forms a group under the operation $\circ$ defined by the Cayley table in **Fig. 8** below.

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Element	e	a	b	c
Period	1	4	2	4

Fig. 8

The set K = $\left\{ p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ r = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix},\ s = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \right\}$

forms a group under matrix multiplication, where $\mathrm{i} = \sqrt{-1}$

(v) Copy and complete the Cayley table of (K, $\times$) started in **Fig. 9** below.

$\times$	p	q	r	s
p	p	q	r	s
q	q			
r	r			
s	s			

Fig. 9 [3]

(vi) State with a reason which of the two groups $(G, \times_{40})$ or (H, $\circ$) is isomorphic to (K, $\times$). [2]

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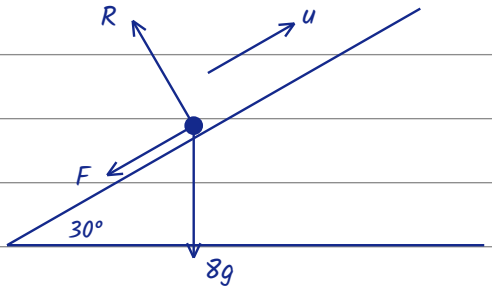
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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

Section A Mechanics 1

A1



Resolve perpendicular to the plane: $R = 8g \cos 30° = 4\sqrt{3}g$ N

Friction is limiting: $F = \mu R = \frac{\sqrt{3}}{6} \times 4\sqrt{3}g = 2g$ N

Work done against friction $= 2g \times 0.3 = 0.6g$ J

Gain in GPE $= 8g \times 0.3 \sin 30° = 1.2g$ J

The normal reaction does no work.

Initial KE $= \frac{1}{2} \times 8 \times u^2 = 4u^2$, final KE $= 0$

Work-energy: loss in KE = gain in GPE + work done against friction

$4u^2 = 1.2g + 0.6g = 1.8g$

$u^2 = 0.45g = 4.41$

$u = 2.1$

A2 (a)

Work done $= \int_4^7 \left(4x - \frac{56}{x^2}\right) dx$

$= \left[2x^2 + \frac{56}{x}\right]_4^7$

Question number

$= (98 + 8) - (32 + 14)$

$= 60$ J

(b)(i) Resultant force $F + G = \begin{pmatrix} 5 + (-1 + \lambda) \\ 2 + (2 - 2\lambda) \\ 8 + (3 + 3\lambda) \end{pmatrix} = \begin{pmatrix} 4 + \lambda \\ 4 - 2\lambda \\ 11 + 3\lambda \end{pmatrix}$

Displacement $AB = OB - OA = \begin{pmatrix} 8 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$

Work done $= (F + G) \cdot AB = 6(4 + \lambda) + 4(4 - 2\lambda) + 1(11 + 3\lambda)$

$= 24 + 6\lambda + 16 - 8\lambda + 11 + 3\lambda$

$= (51 + \lambda)$ J

(ii) Work done = change in kinetic energy:

$51 + \lambda = \frac{1}{2} \times 1 \times 15^2 - \frac{1}{2} \times 1 \times 11^2$

$51 + \lambda = \frac{1}{2}(225 - 121) = 52$

$\lambda = 1$

A3 Radius of pipe = 0.01 m

Cross-sectional area $= \pi \times 0.01^2 = 0.0001\pi$ m^2

Volume of water per second $= 0.0001\pi \times 1.2 = 0.00012\pi$ m^3 s^{-1}

Mass of water per second $= 1000 \times 0.00012\pi = 0.12\pi$ kg s^{-1}

Water starts at rest and is lifted 0.75 m to leave at 1.2 m s^{-1}.

GPE gained per second $= 0.12\pi \times 9.8 \times 0.75 = 2.7709$ J s^{-1}

KE gained per second $= \frac{1}{2} \times 0.12\pi \times 1.2^2 = 0.27143$ J s^{-1}

Question number

Power = work done per second = 2.7709 + 0.27143

= 3.0423 W

Effective power = 3.04 W (3 s.f.)

A4 (i) No vertical acceleration, so resolve vertically at P:

$T_1 \cos\alpha = T_2 \cos\alpha + mg$

$T_1 = T_2 + \frac{mg}{\cos\alpha}$ as required

(ii) Horizontally, Newton's second law towards C:

$T_1 \sin\alpha + T_2 \sin\alpha = \frac{mv^2}{r}$

$T_1 + T_2 = \frac{mv^2}{r \sin\alpha}$ as required

(iii) Substitute (i) into (ii):

$T_2 + \frac{mg}{\cos\alpha} + T_2 = \frac{mv^2}{r \sin\alpha}$

$2T_2 = \frac{mv^2}{r \sin\alpha} - \frac{mg}{\cos\alpha}$

$T_2 = \frac{m}{2}\left(\frac{v^2}{r \sin\alpha} - \frac{g}{\cos\alpha}\right)$ as required

(iv) String BP must stay taut, so $T_2 \geq 0$:

$\frac{m}{2}\left(\frac{v^2}{r \sin\alpha} - \frac{g}{\cos\alpha}\right) \geq 0$

$\frac{v^2}{r \sin\alpha} \geq \frac{g}{\cos\alpha}$

Question number

$v^2 \geqslant \dfrac{rg \sin\alpha}{\cos\alpha} = rg \tan\alpha$

Minimum velocity $v = \sqrt{rg \tan\alpha}$ as required

AS (i) Block: resolve vertically: $R = 5mg$

On the point of moving, friction is limiting: $F = \mu R = \frac{1}{9} \times 5mg = \frac{5}{9}mg$

Resolve horizontally for the block: $T = F$

$T = \frac{5}{9}mg$

M is the midpoint of AB: $AM = MB = 0.08$ m, $MC = 0.15$ m

$AC = BC = \sqrt{(0.08^2 + 0.15^2)} = \sqrt{0.0289} = 0.17$ m

Stretched length of string $= PA + AC + CB = 0.02 + 0.17 + 0.17 = 0.36$ m

Extension $x = 0.36 - 0.18 = 0.18$ m

Hooke's law: $T = \dfrac{\lambda x}{l} = \dfrac{\lambda \times 0.18}{0.18} = \lambda$

So $\lambda = \frac{5}{9}mg$ as required

(ii) Take the level of AB as zero GPE.

Initially: ring at rest, string length $0.02 + 0.16 = 0.18$ m = natural length,

so KE = 0, GPE = 0, EPE = 0: initial energy = 0

At C (block still at rest):

$KE = \frac{1}{2}mv^2$, $GPE = -0.15mg$

$EPE = \dfrac{\lambda x^2}{2l} = \dfrac{\lambda \times 0.18^2}{2 \times 0.18} = 0.09\lambda = 0.09 \times \frac{5}{9}mg = 0.05mg$

Conservation of mechanical energy:

Question number

$0 = \frac{1}{2}mv^2 - 0.15mg + 0.05mg$

$\frac{1}{2}v^2 = 0.1g$

$v^2 = 0.2g = 1.96$

$v = 1.4$ as required

Section B Mechanics 2

B1 (i) At the top, the reaction (2500 N) and the weight (1120g N) both act downwards, towards the centre.

Newton's second law: $1120g + 2500 = \frac{1120v^2}{4}$

$10\,976 + 2500 = 280v^2$

$v^2 = 48.129$

$v = 6.94 \text{ m s}^{-1}$ (3 s.f.)

(ii) The track is smooth, so mechanical energy is conserved.

The top is $2 \times 4 = 8$ m above the lowest point.

$\frac{1}{2} \times 1120u^2 = \frac{1}{2} \times 1120v^2 + 1120g \times 8$

$u^2 = v^2 + 16g = 48.129 + 156.8 = 204.93$

$u = 14.3 \text{ m s}^{-1}$ (3 s.f.)

(reaction $\geq$ 2500 N at the top requires $u \geq 14.3 \text{ m s}^{-1}$)

B2 (i) On the surface, weight = gravitational attraction:

Question number

$\frac{GMm}{R^2} = mg$

$G = \frac{gR^2}{M}$

(ii) Let h be the height of the orbit, radius $R + h$.

Newton's second law: $\frac{GMm}{(R+h)^2} = \frac{m \times 7200^2}{R+h}$

$R + h = \frac{GM}{7200^2} = \frac{gR^2}{7200^2}$

$R + h = \frac{9.8 \times (6.37 \times 10^6)^2}{7200^2} = 7.6708 \times 10^6$

$h = 7.6708 \times 10^6 - 6.37 \times 10^6 = 1.3008 \times 10^6$ m

Height $= 1.30 \times 10^6$ m $= 1300$ km (3 s.f.)

B3 (i) $S = \frac{\rho g a h}{2}$

$[h] = L$, $[\rho] = ML^{-3}$, $[g] = LT^{-2}$, $[a] = L$

$[S] = ML^{-3} \times LT^{-2} \times L \times L$

$[S] = MT^{-2}$

(ii) $[y] = [\rho]^\alpha [g]^\beta [S]^\gamma$ (k is dimensionless)

$L = (ML^{-3})^\alpha (LT^{-2})^\beta (MT^{-2})^\gamma$

Compare indices:

M: $0 = \alpha + \gamma$

L: $1 = -3\alpha + \beta$

T: $0 = -2\beta - 2\gamma \Rightarrow \beta = -\gamma$

From M: $\alpha = -\gamma$, so L: $1 = 3\gamma - \gamma = 2\gamma$

Question number

$\gamma = \frac{1}{2}, \quad \alpha = -\frac{1}{2}, \quad \beta = -\frac{1}{2}$

So $y = k\sqrt{\frac{S}{\rho g}}$

(iii) $\rho_1 = 0.4\rho_0$ and $S_1 = 0.625S_0$

$y_1 = k\sqrt{\frac{0.625S_0}{0.4\rho_0 g}} = \sqrt{\frac{0.625}{0.4}} \times k\sqrt{\frac{S_0}{\rho_0 g}}$

$y_1 = \sqrt{1.5625}\ y_0$

$y_1 = 1.25y_0 = \frac{5}{4}y_0$

B4 (i) Time = 1 h 5 min = $\frac{13}{12}$ h

Speed = $513 \div \frac{13}{12} = 473.54$ km h^{-1}

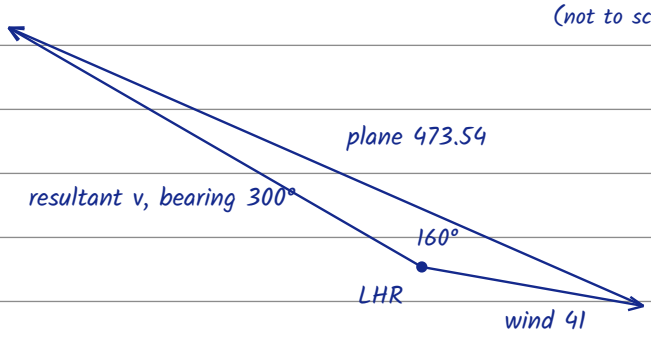
= 474 km h^{-1} (3 s.f.)

(ii) Return track from LHR to BHD is on a bearing of 300°.

The wind blows from 280°, i.e. towards a bearing of 100°.

Angle between the track (300°) and the wind velocity (100°) = 160°

(not to scale)



Triangle of velocities, sine rule (α opposite the wind side):

Question number

$\frac{\sin \alpha}{41} = \frac{\sin 160°}{473.54}$

$\sin \alpha = 0.029613, \quad \alpha = 1.6969°$

Third angle $\beta = 180° - 160° - 1.6969° = 18.3031°$

$\frac{v}{\sin \beta} = \frac{473.54}{\sin 160°}$

$v = \frac{473.54 \times \sin 18.3031°}{\sin 160°} = 434.80 \text{ km h}^{-1}$

Time $= \frac{513}{434.80} = 1.18$ h (3 s.f.)

(about 1 hour 11 minutes)

B5 (i) Let RQ = x, then PR = 0.89 − x (total string length 0.89 m).

Angle PRQ = 90°, so by Pythagoras:

$(0.89 - x)^2 + x^2 = 0.65^2$

$0.7921 - 1.78x + 2x^2 = 0.4225$

$2x^2 - 1.78x + 0.3696 = 0$

$x^2 - 0.89x + 0.1848 = 0$

$(x - 0.33)(x - 0.56) = 0$

$x = 0.33$ or $x = 0.56$

RQ < PR, so RQ = 0.33 m as required (and PR = 0.56 m)

(ii) The ring is smooth, so the tension T is the same in both parts.

Angle RPQ = θ: $\sin \theta = \frac{0.33}{0.65}, \quad \cos \theta = \frac{0.56}{0.65}$

Angle RQP = φ = 90° − θ: $\sin \varphi = \frac{0.56}{0.65}, \quad \cos \varphi = \frac{0.33}{0.65}$

Resolve vertically (F is horizontal):

Question number

$T \sin\theta + T \sin\varphi = 2.67g$

$T\left(\frac{0.33}{0.65} + \frac{0.56}{0.65}\right) = 2.67 \times 9.8$

$T = \frac{2.67 \times 9.8 \times 0.65}{0.89} = 19.110 \text{ N}$

Extension $= 0.89 - 0.6 = 0.29$ m

Hooke's law: $T = \frac{\lambda \times 0.29}{0.6}$

$\lambda = \frac{19.110 \times 0.6}{0.29} = 39.54$

$\lambda \approx 39.5$ N as required

(iii) Resolve horizontally:

$F + T\cos\varphi = T\cos\theta$

$F = 19.110 \times \left(\frac{0.56}{0.65} - \frac{0.33}{0.65}\right)$

$F = 19.110 \times \frac{0.23}{0.65} = 6.76$ N (3 s.f.)

Section C Statistics

C1 (a) $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X)$

$13.5 = 6a^2$

$a^2 = 2.25$, $a = 1.5$ $(a > 0)$

$E(Y) = aE(X) + b$

$25 = 1.5 \times 15 + b$

$b = 25 - 22.5 = 2.5$

Question number	
(b)(i)	IQ score
(ii)	Not reliable: this is the regression line of IQ score on head circumference, so it should only be used to estimate IQ score. To estimate head circumference from IQ score the regression line of head circumference on IQ score would be needed.
(iii)	It does not provide sufficient evidence. $r = 0.92$ shows strong positive correlation, but correlation does not imply causation (both could be linked to another factor).
C2 (i)	$X \sim Geo(0.2)$, $p = 0.2$, $q = 0.8$
	$P(X = 1) = 0.2$
	$P(X = 2) = 0.8 \times 0.2 = 0.16$
	$P(X = 3) = 0.8^2 \times 0.2 = 0.128$
	$P(X \geq 4) = 1 - (0.2 + 0.16 + 0.128) = 1 - 0.488 = 0.512$

x	1	2	3	4 or more
$P(X = x)$	0.2	0.16	0.128	0.512

(ii)	$\text{Var}(X) = \frac{1-p}{p^2} = \frac{0.8}{0.2^2} = \frac{0.8}{0.04} = 20$
C3 (i)	$6! = 720$
(ii)	$P = \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{120}$

Question number

(iii) $P = \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} = \frac{6}{120} = \frac{1}{20}$

(iv) Arrangements in a circle = $(6 - 1)! = 5! = 120$

(v) Treat A, B, C as one block: 4 units around the circle,

$(4 - 1)! = 3! = 6$ ways.

A, B, C can be arranged within the block in $3! = 6$ ways.

Number of arrangements = $6 \times 6 = 36$

C4 (i) X = number of meteors detected in one hour, $X \sim Po(20)$

$P(X = 15) = \frac{e^{-20} \times 20^{15}}{15!}$

$= 0.0516$ (3 s.f.)

(ii) Y = number detected in 30 minutes, $Y \sim Po(10)$

$P(Y > 12) = 1 - P(Y \leq 12)$

$= 1 - 0.7916$

$= 0.208$ (3 s.f.)

(iii) $W \sim Po(\lambda)$, $P(W > 10) = 0.3$, so $P(W \leq 10) = 0.7$

From tables: $\lambda = 8$: $P(W \leq 10) = 0.8159$

$\lambda = 9$: $P(W \leq 10) = 0.7060$ $(P(W > 10) = 0.294 \approx 0.3)$

$\lambda = 10$: $P(W \leq 10) = 0.5830$

Average hourly rate = 9 meteors per hour

C5 (i) $\int_0^2 (ax^2 + bx + \frac{1}{6})\, dx = 1$

Question number

$\left[\frac{ax^3}{3} + \frac{bx^2}{2} + \frac{x}{6}\right]_0^2 = 1$

$\frac{8a}{3} + 2b + \frac{1}{3} = 1$

$\frac{8a}{3} + 2b = \frac{2}{3} \quad \Rightarrow \quad 4a + 3b = 1 \quad ①$

$E(X) = \int_0^2 x(ax^2 + bx + \frac{1}{6})\,dx = \frac{13}{9}$

$\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{x^2}{12}\right]_0^2 = \frac{13}{9}$

$4a + \frac{8b}{3} + \frac{1}{3} = \frac{13}{9}$

$4a + \frac{8b}{3} = \frac{10}{9} \quad \Rightarrow \quad 18a + 12b = 5 \quad ②$

① × 4: $16a + 12b = 4$

② − this: $2a = 1$

$a = \frac{1}{2}, \quad b = \frac{1 - 4a}{3} = -\frac{1}{3}$

(ii) $f(x) = \frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}$ is a quadratic with a minimum point

(at $x = \frac{1}{3}$), so the maximum on $0 \leq x \leq 2$ is at an end point.

$f(0) = \frac{1}{6}, \quad f(2) = 2 - \frac{2}{3} + \frac{1}{6} = \frac{3}{2}$

Mode = 2

Question number

(iii) $\int_0^M \left(\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}\right) dx = 0.5$

$\left[\frac{x^3}{6} - \frac{x^2}{6} + \frac{x}{6}\right]_0^M = \frac{1}{2}$

$\frac{M^3}{6} - \frac{M^2}{6} + \frac{M}{6} = \frac{1}{2}$

Multiply by 6: $M^3 - M^2 + M = 3$

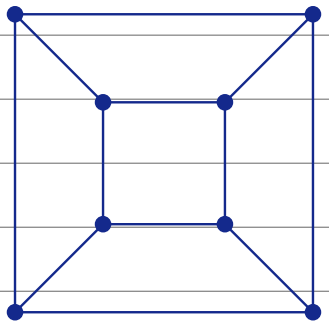
$M^3 - M^2 + M - 3 = 0$ as required

Section D Discrete and Decision Mathematics

D1 (a)(i) G_c is planar.

$K_{3,3}$ is not planar.

(ii)



G_c redrawn with no edge crossings (8 vertices, 12 edges).

(b) See the table in the Supplementary Answer Booklet:

R: every vertex has even degree except two, so R has an Eulerian trail (yes); removing 1 edge (joining the two odd vertices) leaves an Eulerian circuit.

S: all 8 vertices have degree 3 (odd), so no Eulerian trail;

Question number

removing 4 edges (a matching whose removal leaves a single 8-cycle) leaves an Eulerian circuit.

D2 (i) Generation $n + 2$ has parent generation $n + 1$ and grandparent generation n, so

$u_{n+2} = 2u_{n+1} + 3u_n$

(ii) Try $u_n = kx^n$:

$kx^{n+2} = 2kx^{n+1} + 3kx^n$

$kx^n(x^2 - 2x - 3) = 0$

$x \neq 0$, so $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

General solution: $u_n = A \times 3^n + B(-1)^n$

$u_0 = 1$: $1 = A + B$

$u_1 = 1$: $1 = 3A - B$

Add: $4A = 2$, $A = \frac{1}{2}$, $B = \frac{1}{2}$

$u_n = \frac{1}{2}(3^n + (-1)^n)$

So $k = \frac{1}{2}$, $\alpha = 3$, $\beta = -1$

(iii) $u_{10} = \frac{1}{2}(3^{10} + (-1)^{10}) = \frac{1}{2}(59\,049 + 1) = 29\,525$

Number of locusts in generation 10 = 29 525 000

Question number

D3 Let A = p and ~q, B = ~p or q, C = B and r

p	q	r	~p	~q	~r	A	A or ~r	B	C	~C
T	T	T	F	F	F	F	F	T	T	F
T	T	F	F	F	T	F	T	T	F	T
T	F	T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	T	T	F	F	T
F	T	T	T	F	F	F	F	T	T	F
F	T	F	T	F	T	F	T	T	F	T
F	F	T	T	T	F	F	F	T	T	F
F	F	F	T	T	T	F	T	T	F	T

The columns for (p and ~q) or ~r and ~[(~p or q) and r] are identical (F T T T F T F T), so the two statements are equivalent.

D4 (a)(i) Prim's algorithm from A, each time choosing the shortest arc joining a vertex in the tree to a vertex not yet in the tree:

AB (3), BC (2), CG (4), GI (2), GF (4), FH (4), AD (5), DE (2)

(ii) Distance = 3 + 2 + 4 + 2 + 4 + 4 + 5 + 2 = 26 km

(b)(i) Early and late times are recorded on Fig. 5 in the Supplementary Answer Booklet (nodes numbered 1 to 11):

Node	1	2	3	4	5	6	7	8	9	10	11
Early	0	7	4	9	13	14	23	23	27	28	34
Late	0	8	4	11	13	16	23	25	31	28	34

(ii) Critical path: B, E, H, K, N

Question number

(iii) Minimum time = 4 + 9 + 10 + 5 + 6 = 34 days

DS (i)

$\times_{40}$	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

(ii) Identity element = 25

(iii)

Element	5	15	25	35
Period	2	2	1	2

(iv) Subgroups: {25}, {25, 5}, {25, 15}, {25, 35},
and G = {5, 15, 25, 35} itself

(v)

x	p	q	r	s
p	p	q	r	s
q	q	p	s	r
r	r	s	q	p
s	s	r	p	q

(vi)

Element	p	q	r	s
Period	1	2	4	4

(K, x) has elements of period 4, as does (H, °) (periods 1, 4, 2, 4),
but every element of $(G, \times_{40})$ has period 1 or 2.
So (H, °) is isomorphic to (K, x), e.g. e ↔ p, b ↔ q, a ↔ r, c ↔ s

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13350/4

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2023

Further Mathematics *StemUp Model Solution*

Assessment Unit AS 2
assessing
Applied Mathematics

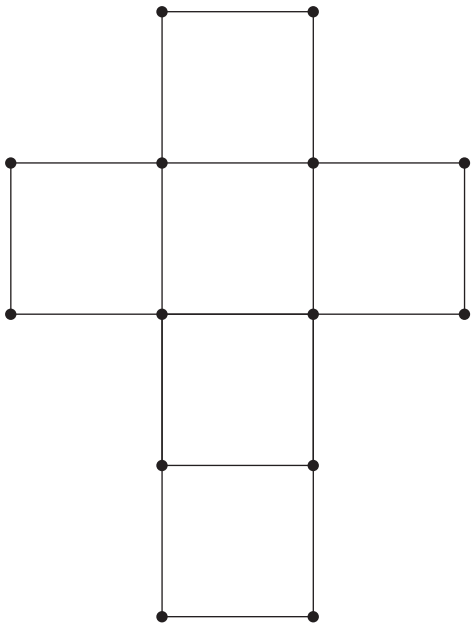
[SFM21]
TUESDAY 23 MAY, AFTERNOON

SECTION D

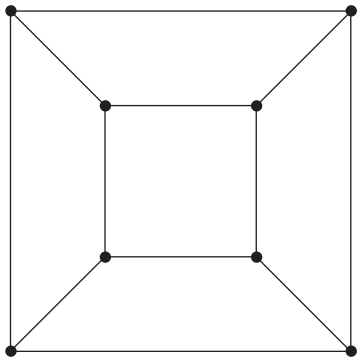
SUPPLEMENTARY ANSWER BOOKLET

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1 (b) The graphs R and S are shown in **Fig. 2** below.



R



S

Fig. 2

Copy and complete the table shown in **Fig. 3** below by

- ticking in the 2nd column if the graph has an Eulerian trail – one that need not start and finish on the same vertex
- writing the minimum number of edges that need to be removed to leave a subgraph with an Eulerian circuit in the 3rd column.

Graph	Has an Eulerian trail	Number of edges that must be removed to leave an Eulerian circuit
R	✓ (yes)	1
S	✗ (no)	4

[4]

Fig. 3

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4 (b) (i) **Fig. 5** below represents the activity network of a set-up for a conference.

Activities are represented by the arcs A, B, C, ..., M.

The adjacent numbers give the time in days for the associated activity.

Record your answers in the figure below, calculating the early event time and the late event time for each event. [4]

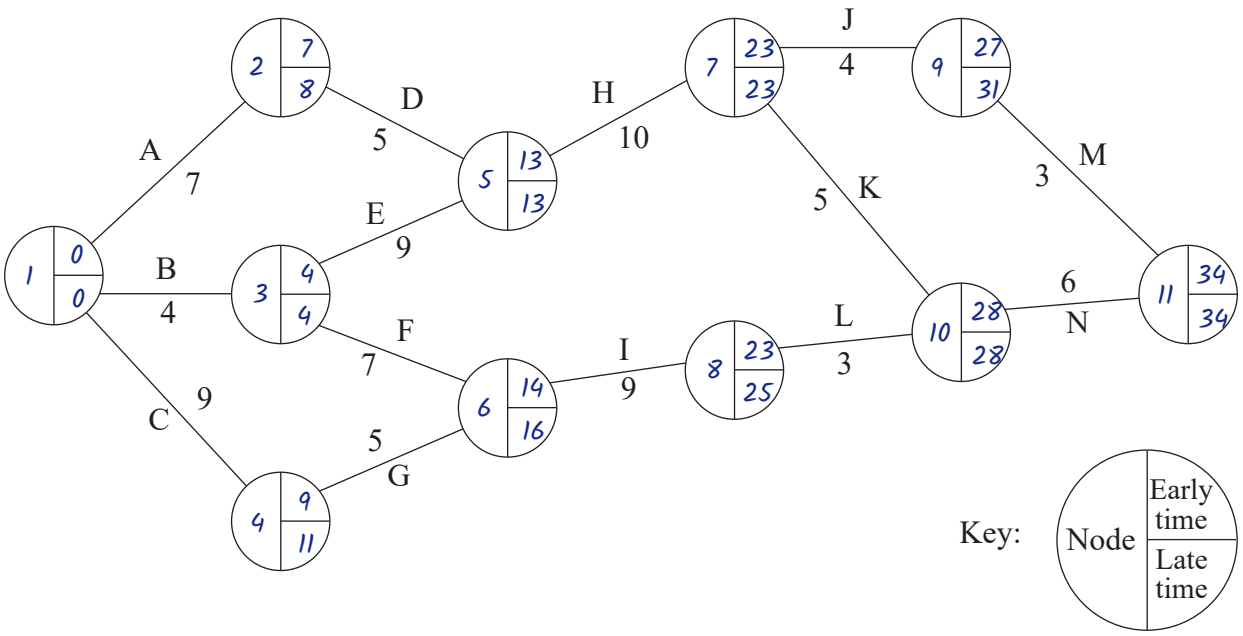


Fig. 5

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13350.02/1

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Mathematical Formulae and Tables

For use by candidates taking the Advanced Subsidiary and Advanced GCE examinations in Mathematics and Further Mathematics

For use from 2019

Appendix 1

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PURE MATHEMATICS

Mensuration

Surface area of sphere = $4\pi r^2$
Area of curved surface of cone = $\pi r \times \text{slant height}$

Summations

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Arithmetic Series

$u_n = a + (n-1)d$

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\left[2a+(n-1)d\right]$$

Geometric Series

$u_n = ar^{n-1}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial Series

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \quad (|x|<1, n \in \mathbb{R})$$

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Logarithms and exponentials

$$e^{x \ln a} = a^x$$

Complex Numbers

$$\{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The roots of $z^n = 1$ are given by $z = e^{\frac{2\pi k i}{n}}$, for $k = 0, 1, 2, \ldots, n-1$

Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \text{ for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \ldots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots \text{ for all } x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots (-1)^r\frac{x^{2r+1}}{2r+1} + \ldots \quad (-1 < x < 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \text{ for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \text{ for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \quad (-1 < x < 1)$$

Hyperbolic Functions

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2\sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad (x \geq 1)$

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$

Trigonometry

In the triangle ABC: $a^2 = b^2 + c^2 - 2bc \cos A$

Trigonometric Identities

$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \frac{1}{2})\,\pi)$

Small angle approximations

$\sin \theta \approx \theta$

$\cos \theta \approx 1 - \dfrac{\theta^2}{2}$

$\tan \theta \approx \theta$

where θ is measured in radians.

Vectors

Vector product: $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}} = \begin{vmatrix}\mathbf{i} & a_1 & b_1\\ \mathbf{j} & a_2 & b_2\\ \mathbf{k} & a_3 & b_3\end{vmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{pmatrix}$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = \mathbf{b}.(\mathbf{c}\times\mathbf{a}) = \mathbf{c}.(\mathbf{a}\times\mathbf{b})$$

If A is the point with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and the direction vector $\mathbf{b}$ is given by $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then the straight line through A with direction vector $\mathbf{b}$ has cartesian equation

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} \quad (= \lambda)$$

The plane through A with normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ has cartesian equation $n_1x + n_2y + n_3z + d = 0$ where $d = -\mathbf{a}.\mathbf{n}$

The plane through non-collinear points *A*, *B* and *C* has vector equation
$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

The plane through the point with position vector $\mathbf{a}$ and parallel to vectors $\mathbf{b}$ and $\mathbf{c}$ has equation
$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$

The perpendicular distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$

Matrix transformations

Anticlockwise rotation through θ about the origin: $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

Reflection in the line $y = (\tan\theta)\,x$: $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k \sec^2 kx$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Integration

(+ constant; a > 0 where relevant)

$f(x)$	$\int f(x)\,dx$
$\tan x$	$\ln \lvert\sec x\rvert$
$\cot x$	$\ln \lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln \lvert\operatorname{cosec} x + \cot x\rvert = \ln\left\lvert\tan\left(\frac{x}{2}\right)\right\rvert$
$\sec x$	$\ln \lvert\sec x + \tan x\rvert = \ln\left\lvert\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\rvert$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \lvert\cosh x\rvert$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right), (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$ or $\ln\left(x+\sqrt{x^2-a^2}\right), (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$ or $\ln\left(x+\sqrt{x^2+a^2}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right), (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert, (\lvert x\rvert > a)$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Area of a Sector

$A = \frac{1}{2}\int r^2 d\theta$ (polar coordinates)

NUMERICAL MATHEMATICS

Numerical integration

The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n\text{-}1})\}$, where $h = \frac{b-a}{n}$

Numerical Solution of Equations

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

MECHANICS

Motion in a circle

Transverse velocity: $v = r\dot{\theta}$

Transverse acceleration: $\dot{v} = r\ddot{\theta}$

Radial acceleration: $-r\dot{\theta}^2 = -\frac{v^2}{r}$

Centres of Mass

For uniform bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere, radius *r* : $\frac{3}{8}r$ from centre

Hemispherical shell, radius *r* : $\frac{1}{2}r$ from centre

Circular arc, radius *r*, angle at centre 2α: $\frac{r\sin\alpha}{\alpha}$ from centre

Sector of circle, radius *r*, angle at centre 2α: $\frac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height *h*: $\frac{1}{4}h$ above the base on the line from centre of base to vertex

Conical shell of height *h*: $\frac{1}{3}h$ above the base on the line from centre of base to vertex

Universal law of gravitation

Force = $\frac{Gm_1m_2}{d^2}$

PROBABILITY AND STATISTICS

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B|A)$

$$P(A|B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}$$

Discrete distributions

For a discrete random variable X taking values x_i with probabilities p_i

Expectation (mean): $E(X) = \mu = \sum_i x_i p_i$

Variance: $\mathrm{Var}(X) = \sigma^2 = \Sigma_i (x_i - \mu)^2 p_i = \sum_i x_i^2 p_i - \mu^2$

For a function $g(X)$: $E(g(X)) = \sum_i g(x_i) p_i$

Standard discrete distributions:

Distribution of X	$P(X = x)$	Mean	Variance
Binomial $B(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson $Po(\lambda)$	$e^{-\lambda} \frac{\lambda^x}{x!}$	λ	λ

Continuous distributions

For a continuous random variable X having probability density function $f(x)$:

Expectation (mean): $E(X) = \mu = \int x f(x) dx$

Variance: $\mathrm{Var}(X) = \sigma^2 = \int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$

For a function $g(X)$: $E(g(X)) = \int g(x) f(x) dx$

Standard continuous distributions

Distribution of X	P.D.F.	Mean	Variance
Uniform (Rectangular) on $[a, b]$	$\frac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Normal N(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

Expectation algebra

For independent random variables X and Y

$\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$; $\mathrm{Var}(aX \pm bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$

Sampling distributions

For a random sample $x_1, x_2, \ldots, x_n$ of n independent observations from a distribution having mean μ and variance σ^2

$\bar{x}$ is an unbiased estimator of μ, with $\mathrm{Var}(\bar{x}) = \frac{\sigma^2}{n}$

S^2 is an unbiased estimator of σ^2, where $S^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$

If X is the observed number of successes in n independent Bernoulli trials in each of which the probability of success is p, and $Y = \frac{x}{n}$, then

$\mathrm{E}(Y) = p$ and $\mathrm{Var}(Y) = \frac{p(1-p)}{n}$

For a random sample of n_X observations from N(μ_x, σ_x^2) and, independently, a random sample of n_y observations from N(μ_y, σ_y^2)

$$\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \sim \mathrm{N}(0,1)$$

If $\sigma_x^2 = \sigma_y^2 = \sigma^2$ (unknown) then

$$\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{S_\mathrm{p}^2\left(\frac{1}{n_x} + \frac{1}{n_y}\right)}} \sim t_{n_x + n_y - 2} \text{ where } S_\mathrm{p}^2 = \frac{(n_x - 1)S_x^2 + (n_y - 1)S_y^2}{n_x + n_y - 2}$$

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Correlation and regression

For a set of n pairs of values (x_i, y_i)

$$S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

$$S_{yy} = \Sigma(y_i - \bar{y})^2 = \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}$$

$$S_{xy} = \Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}$$

The product moment correlation coefficient is

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\{\Sigma(x_i - \bar{x})^2\}\{\Sigma(y_i - \bar{y})^2\}}} = \frac{\Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}}{\sqrt{\left(\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right)\left(\Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}\right)}}$$

The regression coefficient of y on x is $b = \frac{S_{xy}}{S_{xx}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$

Least squares regression line of y on x is $y = a + bx$ where $a = \bar{y} - b\bar{x}$

Non-parametric tests

Goodness-of-fit test and contingency tables:

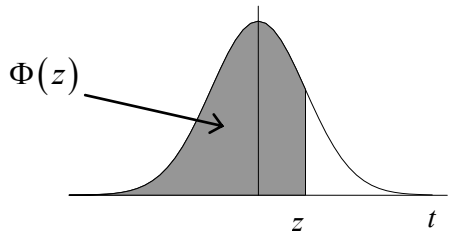
$$\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi_v^2$$

NORMAL PROBABILITY TABLE

Table of $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	(ADD) 1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	8	11	15	19	23	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	21	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	6	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	6	8	11	14	17	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	18	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	19
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	16
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	3	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990	0	0	0	0	0	0	0	0	0

The function tabulated is $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, \mathrm{d}t$. $\Phi(z)$ is the probability that a random variable having a Normal frequency density, with mean zero and variance unity, will be less than z.



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BINOMIAL CUMULATIVE DISTRIBUTION FUNCTION

The tabulated value is $P(X \leq x)$, where X has a binomial distribution with index n and parameter p.

p =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$n = 5, x = 0$	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0313
1	0.9774	0.9185	0.8352	0.7373	0.6328	0.5282	0.4284	0.3370	0.2562	0.1875
2	0.9988	0.9914	0.9734	0.9421	0.8965	0.8369	0.7648	0.6826	0.5931	0.5000
3	1.0000	0.9995	0.9978	0.9933	0.9844	0.9692	0.9460	0.9130	0.8688	0.8125
4	1.0000	1.0000	0.9999	0.9997	0.9990	0.9976	0.9947	0.9898	0.9815	0.9688
$n = 6, x = 0$	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156
1	0.9672	0.8857	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094
2	0.9978	0.9842	0.9527	0.9011	0.8306	0.7443	0.6471	0.5443	0.4415	0.3438
3	0.9999	0.9987	0.9941	0.9830	0.9624	0.9295	0.8826	0.8208	0.7447	0.6563
4	1.0000	0.9999	0.9996	0.9984	0.9954	0.9891	0.9777	0.9590	0.9308	0.8906
5	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9982	0.9959	0.9917	0.9844
$n = 7, x = 0$	0.6983	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078
1	0.9556	0.8503	0.7166	0.5767	0.4449	0.3294	0.2338	0.1586	0.1024	0.0625
2	0.9962	0.9743	0.9262	0.8520	0.7564	0.6471	0.5323	0.4199	0.3164	0.2266
3	0.9998	0.9973	0.9879	0.9667	0.9294	0.8740	0.8002	0.7102	0.6083	0.5000
4	1.0000	0.9998	0.9988	0.9953	0.9871	0.9712	0.9444	0.9037	0.8471	0.7734
5	1.0000	1.0000	0.9999	0.9996	0.9987	0.9962	0.9910	0.9812	0.9643	0.9375
6	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9984	0.9963	0.9922
$n = 8, x = 0$	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039
1	0.9428	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352
2	0.9942	0.9619	0.8948	0.7969	0.6785	0.5518	0.4278	0.3154	0.2201	0.1445
3	0.9996	0.9950	0.9786	0.9437	0.8862	0.8059	0.7064	0.5941	0.4770	0.3633
4	1.0000	0.9996	0.9971	0.9896	0.9727	0.9420	0.8939	0.8263	0.7396	0.6367
5	1.0000	1.0000	0.9998	0.9988	0.9958	0.9887	0.9747	0.9502	0.9115	0.8555
6	1.0000	1.0000	1.0000	0.9999	0.9996	0.9987	0.9964	0.9915	0.9819	0.9648
7	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9983	0.9961
$n = 9, x = 0$	0.6302	0.3874	0.2316	0.1342	0.0751	0.0404	0.0207	0.0101	0.0046	0.0020
1	0.9288	0.7748	0.5995	0.4362	0.3003	0.1960	0.1211	0.0705	0.0385	0.0195
2	0.9916	0.9470	0.8591	0.7382	0.6007	0.4628	0.3373	0.2318	0.1495	0.0898
3	0.9994	0.9917	0.9661	0.9144	0.8343	0.7297	0.6089	0.4826	0.3614	0.2539
4	1.0000	0.9991	0.9944	0.9804	0.9511	0.9012	0.8283	0.7334	0.6214	0.5000
5	1.0000	0.9999	0.9994	0.9969	0.9900	0.9747	0.9464	0.9006	0.8342	0.7461
6	1.0000	1.0000	1.0000	0.9997	0.9987	0.9957	0.9888	0.9750	0.9502	0.9102
7	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9986	0.9962	0.9909	0.9805
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9980
$n = 10, x = 0$	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
1	0.9139	0.7361	0.5443	0.3758	0.2440	0.1493	0.0860	0.0464	0.0233	0.0107
2	0.9885	0.9298	0.8202	0.6778	0.5256	0.3828	0.2616	0.1673	0.0996	0.0547
3	0.9990	0.9872	0.9500	0.8791	0.7759	0.6496	0.5138	0.3823	0.2660	0.1719
4	0.9999	0.9984	0.9901	0.9672	0.9219	0.8497	0.7515	0.6331	0.5044	0.3770
5	1.0000	0.9999	0.9986	0.9936	0.9803	0.9527	0.9051	0.8338	0.7384	0.6230
6	1.0000	1.0000	0.9999	0.9991	0.9965	0.9894	0.9740	0.9452	0.8980	0.8281
7	1.0000	1.0000	1.0000	0.9999	0.9996	0.9984	0.9952	0.9877	0.9726	0.9453
8	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9983	0.9955	0.9893
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990

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p =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$n = 12, x = 0$	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002
1	0.8816	0.6590	0.4435	0.2749	0.1584	0.0850	0.0424	0.0196	0.0083	0.0032
2	0.9804	0.8891	0.7358	0.5583	0.3907	0.2528	0.1513	0.0834	0.0421	0.0193
3	0.9978	0.9744	0.9078	0.7946	0.6488	0.4925	0.3467	0.2253	0.1345	0.0730
4	0.9998	0.9957	0.9761	0.9274	0.8424	0.7237	0.5833	0.4382	0.3044	0.1938
5	1.0000	0.9995	0.9954	0.9806	0.9456	0.8822	0.7873	0.6652	0.5269	0.3872
6	1.0000	0.9999	0.9993	0.9961	0.9857	0.9614	0.9154	0.8418	0.7393	0.6128
7	1.0000	1.0000	0.9999	0.9994	0.9972	0.9905	0.9745	0.9427	0.8883	0.8062
8	1.0000	1.0000	1.0000	0.9999	0.9996	0.9983	0.9944	0.9847	0.9644	0.9270
9	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9992	0.9972	0.9921	0.9807
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9968
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
$n = 15, x = 0$	0.4633	0.2059	0.0874	0.0352	0.0134	0.0047	0.0016	0.0005	0.0001	0.0000
1	0.8290	0.5490	0.3186	0.1671	0.0802	0.0353	0.0142	0.0052	0.0017	0.0005
2	0.9638	0.8159	0.6042	0.3980	0.2361	0.1268	0.0617	0.0271	0.0107	0.0037
3	0.9945	0.9444	0.8227	0.6482	0.4613	0.2969	0.1727	0.0905	0.0424	0.0176
4	0.9994	0.9873	0.9383	0.8358	0.6865	0.5155	0.3519	0.2173	0.1204	0.0592
5	0.9999	0.9978	0.9832	0.9389	0.8516	0.7216	0.5643	0.4032	0.2608	0.1509
6	1.0000	0.9997	0.9964	0.9819	0.9434	0.8689	0.7548	0.6098	0.4522	0.3036
7	1.0000	1.0000	0.9994	0.9958	0.9827	0.9500	0.8868	0.7869	0.6535	0.5000
8	1.0000	1.0000	0.9999	0.9992	0.9958	0.9848	0.9578	0.9050	0.8182	0.6964
9	1.0000	1.0000	1.0000	0.9999	0.9992	0.9963	0.9876	0.9662	0.9231	0.8491
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9972	0.9907	0.9745	0.9408
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9981	0.9937	0.9824
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9963
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$n = 20, x = 0$	0.3585	0.1216	0.0388	0.0115	0.0032	0.0008	0.0002	0.0000	0.0000	0.0000
1	0.7358	0.3917	0.1756	0.0692	0.0243	0.0076	0.0021	0.0005	0.0001	0.0000
2	0.9245	0.6769	0.4049	0.2061	0.0913	0.0355	0.0121	0.0036	0.0009	0.0002
3	0.9841	0.8670	0.6477	0.4114	0.2252	0.1071	0.0444	0.0160	0.0049	0.0013
4	0.9974	0.9568	0.8298	0.6296	0.4148	0.2375	0.1182	0.0510	0.0189	0.0059
5	0.9997	0.9887	0.9327	0.8042	0.6172	0.4164	0.2454	0.1256	0.0553	0.0207
6	1.0000	0.9976	0.9781	0.9133	0.7858	0.6080	0.4166	0.2500	0.1299	0.0577
7	1.0000	0.9996	0.9941	0.9679	0.8982	0.7723	0.6010	0.4159	0.2520	0.1316
8	1.0000	0.9999	0.9987	0.9900	0.9591	0.8867	0.7624	0.5956	0.4143	0.2517
9	1.0000	1.0000	0.9998	0.9974	0.9861	0.9520	0.8782	0.7553	0.5914	0.4119
10	1.0000	1.0000	1.0000	0.9994	0.9961	0.9829	0.9468	0.8725	0.7507	0.5881
11	1.0000	1.0000	1.0000	0.9999	0.9991	0.9949	0.9804	0.9435	0.8692	0.7483
12	1.0000	1.0000	1.0000	1.0000	0.9998	0.9987	0.9940	0.9790	0.9420	0.8684
13	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9935	0.9786	0.9423
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9936	0.9793
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9941
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9987
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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p =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$n = 25, x = 0$	0.2774	0.0718	0.0172	0.0038	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
1	0.6424	0.2712	0.0931	0.0274	0.0070	0.0016	0.0003	0.0001	0.0000	0.0000
2	0.8729	0.5371	0.2537	0.0982	0.0321	0.0090	0.0021	0.0004	0.0001	0.0000
3	0.9659	0.7636	0.4711	0.2340	0.0962	0.0332	0.0097	0.0024	0.0005	0.0001
4	0.9928	0.9020	0.6821	0.4207	0.2137	0.0905	0.0320	0.0095	0.0023	0.0005
5	0.9988	0.9666	0.8385	0.6167	0.3783	0.1935	0.0826	0.0294	0.0086	0.0020
6	0.9998	0.9905	0.9305	0.7800	0.5611	0.3407	0.1734	0.0736	0.0258	0.0073
7	1.0000	0.9977	0.9745	0.8909	0.7265	0.5118	0.3061	0.1536	0.0639	0.0216
8	1.0000	0.9995	0.9920	0.9532	0.8506	0.6769	0.4668	0.2735	0.1340	0.0539
9	1.0000	0.9999	0.9979	0.9827	0.9287	0.8106	0.6303	0.4246	0.2424	0.1148
10	1.0000	1.0000	0.9995	0.9944	0.9703	0.9022	0.7712	0.5858	0.3843	0.2122
11	1.0000	1.0000	0.9999	0.9985	0.9893	0.9558	0.8746	0.7323	0.5426	0.3450
12	1.0000	1.0000	1.0000	0.9996	0.9966	0.9825	0.9396	0.8462	0.6937	0.5000
13	1.0000	1.0000	1.0000	0.9999	0.9991	0.9940	0.9745	0.9222	0.8173	0.6550
14	1.0000	1.0000	1.0000	1.0000	0.9998	0.9982	0.9907	0.9656	0.9040	0.7878
15	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9971	0.9868	0.9560	0.8852
16	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9957	0.9826	0.9461
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9988	0.9942	0.9784
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9927
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9980
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$n = 30, x = 0$	0.2146	0.0424	0.0076	0.0012	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.5535	0.1837	0.0480	0.0105	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000
2	0.8122	0.4114	0.1514	0.0442	0.0106	0.0021	0.0003	0.0000	0.0000	0.0000
3	0.9392	0.6474	0.3217	0.1227	0.0374	0.0093	0.0019	0.0003	0.0000	0.0000
4	0.9844	0.8245	0.5245	0.2552	0.0979	0.0302	0.0075	0.0015	0.0002	0.0000
5	0.9967	0.9268	0.7106	0.4275	0.2026	0.0766	0.0233	0.0057	0.0011	0.0002
6	0.9994	0.9742	0.8474	0.6070	0.3481	0.1595	0.0586	0.0172	0.0040	0.0007
7	0.9999	0.9922	0.9302	0.7608	0.5143	0.2814	0.1238	0.0435	0.0121	0.0026
8	1.0000	0.9980	0.9722	0.8713	0.6736	0.4315	0.2247	0.0940	0.0312	0.0081
9	1.0000	0.9995	0.9903	0.9389	0.8034	0.5888	0.3575	0.1763	0.0694	0.0214
10	1.0000	0.9999	0.9971	0.9744	0.8943	0.7304	0.5078	0.2915	0.1350	0.0494
11	1.0000	1.0000	0.9992	0.9905	0.9493	0.8407	0.6548	0.4311	0.2327	0.1002
12	1.0000	1.0000	0.9998	0.9969	0.9784	0.9155	0.7802	0.5785	0.3592	0.1808
13	1.0000	1.0000	1.0000	0.9991	0.9918	0.9599	0.8737	0.7145	0.5025	0.2923
14	1.0000	1.0000	1.0000	0.9998	0.9973	0.9831	0.9348	0.8246	0.6448	0.4278
15	1.0000	1.0000	1.0000	0.9999	0.9992	0.9936	0.9699	0.9029	0.7691	0.5722
16	1.0000	1.0000	1.0000	1.0000	0.9998	0.9979	0.9876	0.9519	0.8644	0.7077
17	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9955	0.9788	0.9286	0.8192
18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9917	0.9666	0.8998
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9971	0.9862	0.9506
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9991	0.9950	0.9786
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9984	0.9919
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9974
23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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p =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$n = 40, x = 0$	0.1285	0.0148	0.0015	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.3991	0.0805	0.0121	0.0015	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.6767	0.2228	0.0486	0.0079	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000
3	0.8619	0.4231	0.1302	0.0285	0.0047	0.0006	0.0001	0.0000	0.0000	0.0000
4	0.9520	0.6290	0.2633	0.0759	0.0160	0.0026	0.0003	0.0000	0.0000	0.0000
5	0.9861	0.7937	0.4325	0.1613	0.0433	0.0086	0.0013	0.0001	0.0000	0.0000
6	0.9966	0.9005	0.6067	0.2859	0.0962	0.0238	0.0044	0.0006	0.0001	0.0000
7	0.9993	0.9581	0.7559	0.4371	0.1820	0.0553	0.0124	0.0021	0.0002	0.0000
8	0.9999	0.9845	0.8646	0.5931	0.2998	0.1110	0.0303	0.0061	0.0009	0.0001
9	1.0000	0.9949	0.9328	0.7318	0.4395	0.1959	0.0644	0.0156	0.0027	0.0003
10	1.0000	0.9985	0.9701	0.8392	0.5839	0.3087	0.1215	0.0352	0.0074	0.0011
11	1.0000	0.9996	0.9880	0.9125	0.7151	0.4406	0.2053	0.0709	0.0179	0.0032
12	1.0000	0.9999	0.9957	0.9568	0.8209	0.5772	0.3143	0.1285	0.0386	0.0083
13	1.0000	1.0000	0.9986	0.9806	0.8968	0.7032	0.4408	0.2112	0.0751	0.0192
14	1.0000	1.0000	0.9996	0.9921	0.9456	0.8074	0.5721	0.3174	0.1326	0.0403
15	1.0000	1.0000	0.9999	0.9971	0.9738	0.8849	0.6946	0.4402	0.2142	0.0769
16	1.0000	1.0000	1.0000	0.9990	0.9884	0.9367	0.7978	0.5681	0.3185	0.1341
17	1.0000	1.0000	1.0000	0.9997	0.9953	0.9680	0.8761	0.6885	0.4391	0.2148
18	1.0000	1.0000	1.0000	0.9999	0.9983	0.9852	0.9301	0.7911	0.5651	0.3179
19	1.0000	1.0000	1.0000	1.0000	0.9994	0.9937	0.9637	0.8702	0.6844	0.4373
20	1.0000	1.0000	1.0000	1.0000	0.9998	0.9976	0.9827	0.9256	0.7870	0.5627
21	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9925	0.9608	0.8669	0.6821
22	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9970	0.9811	0.9233	0.7852
23	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9989	0.9917	0.9595	0.8659
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9966	0.9804	0.9231
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9988	0.9914	0.9597
26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9966	0.9808
27	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9988	0.9917
28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9968
29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9989
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997
31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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p =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$n = 50$, $x = 0$	0.0769	0.0052	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.2794	0.0338	0.0029	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.5405	0.1117	0.0142	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.7604	0.2503	0.0460	0.0057	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.8964	0.4312	0.1121	0.0185	0.0021	0.0002	0.0000	0.0000	0.0000	0.0000
5	0.9622	0.6161	0.2194	0.0480	0.0070	0.0007	0.0001	0.0000	0.0000	0.0000
6	0.9882	0.7702	0.3613	0.1034	0.0194	0.0025	0.0002	0.0000	0.0000	0.0000
7	0.9968	0.8779	0.5188	0.1904	0.0453	0.0073	0.0008	0.0001	0.0000	0.0000
8	0.9992	0.9421	0.6681	0.3073	0.0916	0.0183	0.0025	0.0002	0.0000	0.0000
9	0.9998	0.9755	0.7911	0.4437	0.1637	0.0402	0.0067	0.0008	0.0001	0.0000
10	1.0000	0.9906	0.8801	0.5836	0.2622	0.0789	0.0160	0.0022	0.0002	0.0000
11	1.0000	0.9968	0.9372	0.7107	0.3816	0.1390	0.0342	0.0057	0.0006	0.0000
12	1.0000	0.9990	0.9699	0.8139	0.5110	0.2229	0.0661	0.0133	0.0018	0.0002
13	1.0000	0.9997	0.9868	0.8894	0.6370	0.3279	0.1163	0.0280	0.0045	0.0005
14	1.0000	0.9999	0.9947	0.9393	0.7481	0.4468	0.1878	0.0540	0.0104	0.0013
15	1.0000	1.0000	0.9981	0.9692	0.8369	0.5692	0.2801	0.0955	0.0220	0.0033
16	1.0000	1.0000	0.9993	0.9856	0.9017	0.6839	0.3889	0.1561	0.0427	0.0077
17	1.0000	1.0000	0.9998	0.9937	0.9449	0.7822	0.5060	0.2369	0.0765	0.0164
18	1.0000	1.0000	0.9999	0.9975	0.9713	0.8594	0.6216	0.3356	0.1273	0.0325
19	1.0000	1.0000	1.0000	0.9991	0.9861	0.9152	0.7264	0.4465	0.1974	0.0595
20	1.0000	1.0000	1.0000	0.9997	0.9937	0.9522	0.8139	0.5610	0.2862	0.1013
21	1.0000	1.0000	1.0000	0.9999	0.9974	0.9749	0.8813	0.6701	0.3900	0.1611
22	1.0000	1.0000	1.0000	1.0000	0.9990	0.9877	0.9290	0.7660	0.5019	0.2399
23	1.0000	1.0000	1.0000	1.0000	0.9996	0.9944	0.9604	0.8438	0.6134	0.3359
24	1.0000	1.0000	1.0000	1.0000	0.9999	0.9976	0.9793	0.9022	0.7160	0.4439
25	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9900	0.9427	0.8034	0.5561
26	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9955	0.9686	0.8721	0.6641
27	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9981	0.9840	0.9220	0.7601
28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9924	0.9556	0.8389
29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9966	0.9765	0.8987
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9986	0.9884	0.9405
31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9947	0.9675
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9978	0.9836
33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9991	0.9923
34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9967
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987
36	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995
37	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
38	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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POISSON CUMULATIVE DISTRIBUTION FUNCTION

The tabulated value is $P(X \leq x)$, where X has a Poisson distribution with parameter λ.

$\lambda =$	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$x = 0$	0.6065	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067
1	0.9098	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404
2	0.9856	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247
3	0.9982	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650
4	0.9998	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405
5	1.0000	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	1.0000	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622
7	1.0000	1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666
8	1.0000	1.0000	1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319
9	1.0000	1.0000	1.0000	1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990	0.9972	0.9933	0.9863
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9976	0.9945
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9980
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$\lambda =$	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$x = 0$	0.0041	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
1	0.0266	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005
2	0.0884	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028
3	0.2017	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103
4	0.3575	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293
5	0.5289	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671
6	0.6860	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301
7	0.8095	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202
8	0.8944	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328
9	0.9462	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579
10	0.9747	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830
11	0.9890	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968
12	0.9955	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916
13	0.9983	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645
14	0.9994	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165
15	0.9998	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513
16	0.9999	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730
17	1.0000	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857
18	1.0000	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928
19	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965
20	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9991	0.9984
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9993
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997

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PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

The values in the table are those which a random variable with the χ^2 distribution on ν degrees of freedom exceeds with the probability shown.

ν	0.995	0.990	0.975	0.950	0.900	0.100	0.050	0.025	0.010	0.005
1	0.000	0.000	0.001	0.004	0.016	2.705	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.580	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.042	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.088	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672

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PERCENTAGE POINTS OF STUDENT'S *t* DISTRIBUTION

The values in the table are those which a random variable with student's *t* distribution on *v* degrees of freedom exceeds with the probability shown.

v	0.10	0.05	0.025	0.01	0.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
32	1.309	1.694	2.037	2.449	2.738
34	1.307	1.691	2.032	2.441	2.728
36	1.306	1.688	2.028	2.435	2.719
38	1.304	1.686	2.024	2.429	2.712
40	1.303	1.684	2.021	2.423	2.704
45	1.301	1.679	2.014	2.412	2.690
50	1.299	1.676	2.009	2.403	2.678
55	1.297	1.673	2.004	2.396	2.668
60	1.296	1.671	2.000	2.390	2.660
70	1.294	1.667	1.994	2.381	2.648
80	1.292	1.664	1.990	2.374	2.639
90	1.291	1.662	1.987	2.369	2.632
100	1.290	1.660	1.984	2.364	2.626
110	1.289	1.659	1.982	2.361	2.621
120	1.289	1.658	1.980	2.358	2.617

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CRITICAL VALUES FOR CORRELATION COEFFICIENTS

These tables concern tests of the hypothesis that a population correlation coefficient ρ is 0. The values in the tables are the minimum values which need to be reached by a sample correlation coefficient in order to be significant at the level shown, on a one-tailed test.

Sample	Product Moment Coefficient				
	Level				
Level	0.10	0.05	0.025	0.01	0.005
4	0.8000	0.9000	0.9500	0.9800	0.9900
5	0.6870	0.8054	0.8783	0.9343	0.9587
6	0.6084	0.7293	0.8114	0.8822	0.9172
7	0.5509	0.6694	0.7545	0.8329	0.8745
8	0.5067	0.6215	0.7067	0.7887	0.8343
9	0.4716	0.5822	0.6664	0.7498	0.7977
10	0.4428	0.5494	0.6319	0.7155	0.7646
11	0.4187	0.5214	0.6021	0.6851	0.7348
12	0.3981	0.4973	0.5760	0.6581	0.7079
13	0.3802	0.4762	0.5529	0.6339	0.6835
14	0.3646	0.4575	0.5324	0.6120	0.6614
15	0.3507	0.4409	0.5140	0.5923	0.6411
16	0.3383	0.4259	0.4973	0.5742	0.6226
17	0.3271	0.4124	0.4821	0.5577	0.6055
18	0.3170	0.4000	0.4683	0.5425	0.5897
19	0.3077	0.3887	0.4555	0.5285	0.5751
20	0.2992	0.3783	0.4438	0.5155	0.5614
21	0.2914	0.3687	0.4329	0.5034	0.5487
22	0.2841	0.3598	0.4227	0.4921	0.5368
23	0.2774	0.3515	0.4133	0.4815	0.5256
24	0.2711	0.3438	0.4044	0.4716	0.5151
25	0.2653	0.3365	0.3961	0.4622	0.5052
26	0.2598	0.3297	0.3882	0.4534	0.4958
27	0.2546	0.3233	0.3809	0.4451	0.4869
28	0.2497	0.3172	0.3739	0.4372	0.4785
29	0.2451	0.3115	0.3673	0.4297	0.4705
30	0.2407	0.3061	0.3610	0.4226	0.4629
40	0.2070	0.2638	0.3120	0.3665	0.4026
50	0.1843	0.2353	0.2787	0.3281	0.3610
60	0.1678	0.2144	0.2542	0.2997	0.3301
70	0.1550	0.1982	0.2352	0.2776	0.3060
80	0.1448	0.1852	0.2199	0.2597	0.2864
90	0.1364	0.1745	0.2072	0.2449	0.2702
100	0.1292	0.1654	0.1966	0.2324	0.2565

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Cycle indices for 3D rotational symmetry groups acting on:

Vertices of a Tetrahedron	$\frac{1}{12}(x_1^4 + 8x_1^1x_3^1 + 3x_2^2)$
Faces of a Tetrahedron	$\frac{1}{12}(x_1^4 + 8x_1^1x_3^1 + 3x_2^2)$
Edges of a Tetrahedron	$\frac{1}{12}(x_1^6 + 8x_3^2 + 3x_1^2x_2^2)$
Vertices of a Cube	$\frac{1}{24}(x_1^8 + 8x_1^2x_3^2 + 9x_2^4 + 6x_4^2)$
Faces of a Cube	$\frac{1}{24}(x_1^6 + 6x_1^2x_4^1 + 3x_1^2x_2^2 + 6x_2^3 + 8x_3^2)$
Edges of a Cube	$\frac{1}{24}(x_1^{12} + 3x_2^6 + 6x_1^2x_2^5 + 6x_4^3 + 8x_3^4)$
Vertices of an Octahedron	$\frac{1}{24}(x_1^6 + 6x_1^2x_4^1 + 3x_1^2x_2^2 + 6x_2^3 + 8x_3^2)$
Faces of an Octahedron	$\frac{1}{24}(x_1^8 + 8x_1^2x_3^2 + 9x_2^4 + 6x_4^2)$
Edges of an Octahedron	$\frac{1}{24}(x_1^{12} + 3x_2^6 + 6x_1^2x_2^5 + 6x_4^3 + 8x_3^4)$

Cycle Indices for 3D rotational symmetry groups (rotation plus flip) acting on polygons.

Polygon with p (prime) vertices	$\frac{1}{2p}\left(x_1^p + (p-1)x_p^1 + px_1^1x_2^{\frac{p-1}{2}}\right)$
Square	$\frac{1}{8}(x_1^4 + 3x_2^2 + 2x_4^1 + 2x_1^2x_2^1)$
Hexagon	$\frac{1}{12}(x_1^6 + 4x_2^3 + 2x_3^2 + 2x_6^1 + 3x_1^2x_2^2)$
Octagon	$\frac{1}{16}(x_1^8 + 4x_1^2x_2^3 + 5x_2^4 + 2x_4^2 + 4x_8^1)$
Nonagon	$\frac{1}{18}(x_1^9 + 9x_1^1x_2^4 + 6x_9^1 + 2x_3^3)$
Decagon	$\frac{1}{20}(x_1^{10} + 6x_2^5 + 4x_5^2 + 4x_{10}^1 + 5x_1^2x_2^4)$
Dodecagon	$\frac{1}{24}(x_1^{12} + 7x_2^6 + 2x_3^4 + 2x_4^3 + 2x_6^2 + 4x_{12}^1 + 6x_1^2x_2^5)$

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