Rewarding Learning

ADVANCED SUBSIDIARY (AS)
General Certificate of Education
2023

Further Mathematics

Assessment Unit AS 2
assessing
Applied Mathematics

[SFM21]

TUESDAY 23 MAY, AFTERNOON

MARK SCHEME

GCE ADVANCED/ADVANCED SUBSIDIARY (AS) FURTHER MATHEMATICS

Introduction

The mark scheme normally provides the most popular solution to each question. Other solutions given by candidates are evaluated and credit given as appropriate; these alternative methods are not usually illustrated in the published mark scheme.

The marks awarded for each question are shown in the right-hand column and they are prefixed by the letters **M**, **W** and **MW** as appropriate. The key to the mark scheme is given below:

M indicates marks for correct method.

W indicates marks for working.

MW indicates marks for combined method and working.

The solution to a question gains marks for correct method and marks for an accurate working based on this method. Where the method is not correct no marks can be given.

A later part of a question may require a candidate to use an answer obtained from an earlier part of the same question. A candidate who gets the wrong answer to the earlier part and goes on to the later part is naturally unaware that the wrong data is being used and is actually undertaking the solution of a parallel problem from the point at which the error occurred. If such a candidate continues to apply correct method, then the candidate's individual working must be followed through from the error. If no further errors are made, then the candidate is penalised only for the initial error. Solutions containing two or more working or transcription errors are treated in the same way. This process is usually referred to as "follow-through marking" and allows a candidate to gain credit for that part of a solution which follows a working or transcription error.

Positive marking:

It is our intention to reward candidates for any demonstration of relevant knowledge, skills or understanding. For this reason we adopt a policy of **following through** their answers, that is, having penalised a candidate for an error, we mark the succeeding parts of the question using the candidate's value or answers and award marks accordingly.

Some common examples of this occur in the following cases:

(a) a numerical error in one entry in a table of values might lead to several answers being incorrect, but these might not be essentially separate errors;

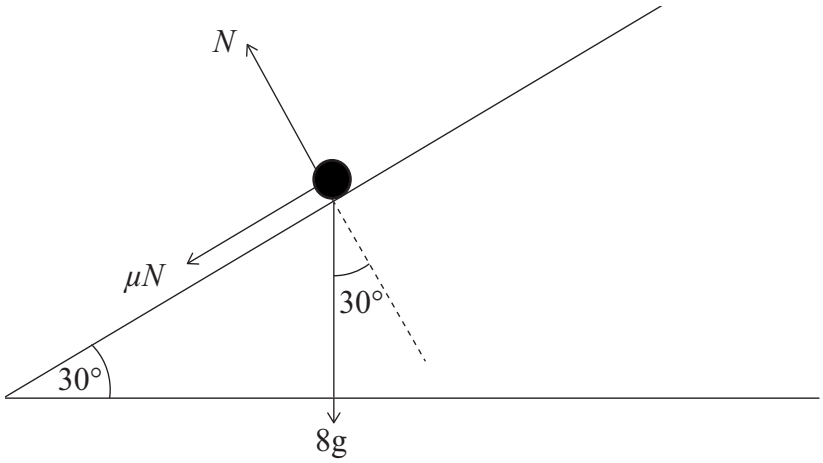
(b) readings taken from candidates' inaccurate graphs may not agree with the answers expected but might be consistent with the graphs drawn.

When the candidate misreads a question in such a way as to make the question easier only a proportion of the marks will be available (based on the professional judgement of the examining team).

SECTION A: Mechanics 1

AVAILABLE MARKS

1



Initial kinetic energy of P:	$\frac{1}{2} \times 8 \times u^2 = 4u^2$	M1 W1	
Resolving forces on P perp to plane:	$N = 8\,g\cos 30° = 4\sqrt{3}\,g$		
Friction is limiting:	$Fr = \mu N = \frac{\sqrt{3}}{6} \times 4\sqrt{3}\,g = 2\,g$	M1 W1	
Work done by friction:	$-2\,g \times 0.3 = -0.6\,g$	M1 W1	
Work done by normal reaction is zero			
Work done by gravity:	$-8\,g \times 0.3 \sin 30° = -1.2\,g$	MW1	
Work – Kinetic Energy equation:	$\frac{1}{2} \times 8 \times v^2 - \frac{1}{2} \times 8 \times u^2 = -0.6\,g - 1.2\,g$	M1	
	$4u^2 = 1.8\,g$		
	$u = 2.1\,\text{ms}^{-1}$	W1	9

2 **(a)** Work done $= \int H \, dx$

AVAILABLE MARKS

$$= \int_4^7 \left(4x - \frac{56}{x^2}\right) dx \qquad \text{M1 W1}$$

$$= \left[2x^2 + \frac{56}{x}\right]_4^7 \qquad \text{W1}$$

$$= (98 + 8) - (32 + 14)$$
$$= 60 \text{ J} \qquad \text{W1}$$

(b) **(i)** Total force is

$$\mathbf{F} + \mathbf{G} = \begin{pmatrix} 5 \\ 2 \\ 8 \end{pmatrix} + \begin{pmatrix} -1 + \lambda \\ 2 - 2\lambda \\ 3 + 3\lambda \end{pmatrix} = \begin{pmatrix} 4 + \lambda \\ 4 - 2\lambda \\ 11 + 3\lambda \end{pmatrix} \qquad \text{MW1}$$

Displacement is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} \qquad \text{MW1}$$

Work done =

$$(\mathbf{F} + \mathbf{G}) \bullet \left(\overrightarrow{OB} - \overrightarrow{OA}\right) \qquad \text{M1}$$

$$= \begin{pmatrix} 4 + \lambda \\ 4 - 2\lambda \\ 11 + 3\lambda \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$$

$$= 24 + 6\lambda + 16 - 8\lambda + 11 + 3\lambda$$

$$= (51 + \lambda) \text{ J} \qquad \text{MW1}$$

(ii) Thus

$$51 + \lambda = \frac{1}{2}v^2 - \frac{1}{2}u^2$$

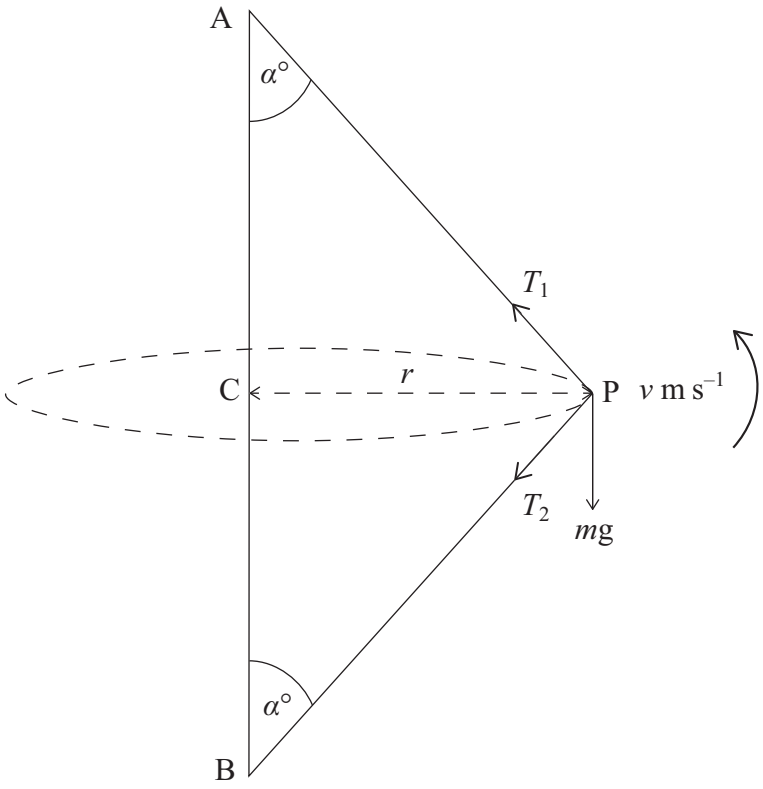
$$= \frac{1}{2}(225 - 121) \qquad \text{M1}$$

$$= 52$$

$$\lambda = 1 \qquad \text{W1} \qquad 10$$

			AVAILABLE MARKS
3	Cross section of pipe = $0.0001\ \pi\ m^2$	MW1	
	Volume of water per sec = $0.0001\ \pi \times 1.2 = 0.00012\pi\ m^3\ s^{-1}$	MW1	
	Mass of water per sec at 0.75 m = $0.12\ \pi$ kg s^{-1}	MW1	
	PE (at 0.75 m) per sec = $0.12\ \pi \times 9.8 \times 0.75 = 2.77088\ Js^{-1}$	M1 W1	
	KE (at 0.75 m) per sec = $0.06\ \pi \times 1.2^2 = 0.27143\ Js^{-1}$	MW1	
	Effective power of the pump = 3.04232 W → 3.04 W(3sf)	M1 W1	8

4



AVAILABLE MARKS

(i) Resolve vertically at P, equilibrium

$T_1 \cos\alpha = T_2 \cos\alpha + mg$ M1

$T_1 = T_2 + \dfrac{mg}{\cos\alpha}$ W1

(ii) Resolve horizontally at P, using Newton's 2nd Law:

$\dfrac{mv^2}{r} = T_1 \sin\alpha + T_2 \sin\alpha$ M1 M1

$T_1 + T_2 = \dfrac{mv^2}{r\sin\alpha}$ W1

(iii) Substituting **(i)** into **(ii)**

$T_2 + \dfrac{mg}{\cos\alpha} + T_2 = \dfrac{mv^2}{r\sin\alpha}$ M1

$T_2 = \dfrac{m}{2}\left(\dfrac{v^2}{r\sin\alpha} - \dfrac{g}{\cos\alpha}\right)$ W1

(iv) $T_2 \geqslant 0$ M1

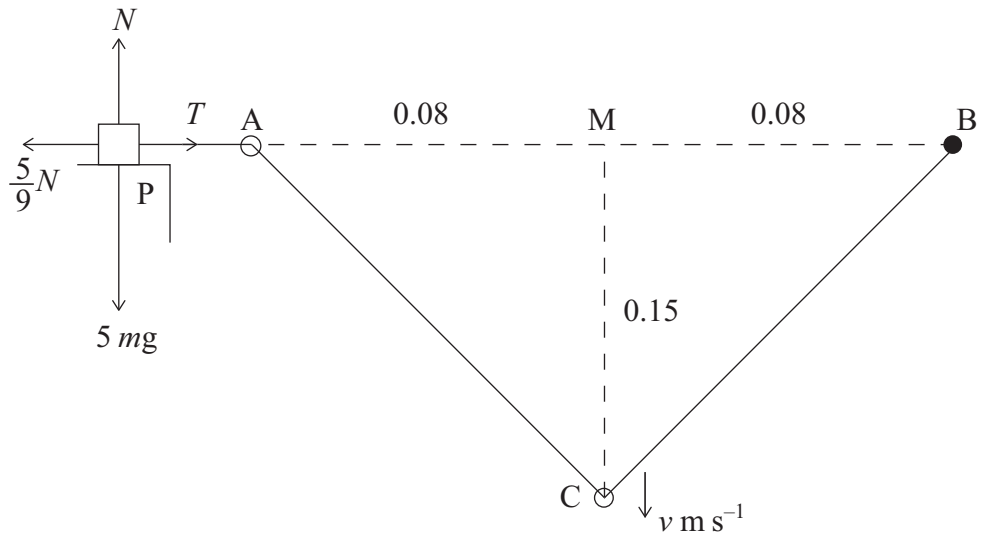
$\dfrac{m}{2}\left(\dfrac{v^2}{r\sin\alpha} - \dfrac{g}{\cos\alpha}\right) \geqslant 0$

$\dfrac{v^2}{r\sin\alpha} \geqslant \dfrac{g}{\cos\alpha}$

$v \geqslant \sqrt{rg\tan\alpha}$

v minimum $= \sqrt{rg\tan\alpha}$ W1 9

5 (i)



			AVAILABLE MARKS
Block at P resolve vertically:	$N = 5mg$		
Limiting friction:	$Fr = \mu N = \frac{1}{9} \times 5m \times g$	MW1	
Resolving horizontally:	$Fr = T$		
	$T = \frac{5}{9}mg$	MW1	
Calculate extension of the string:			
AM = 0.08, MC = 0.15 so AC =	$\sqrt{0.08^2 + 0.15^2} = 0.17$	MW1	
Extension = 0.02 + 0.17 + 0.17 – 0.18 = 0.18		MW1	
Hooke's law:	$T = \frac{\lambda x}{l} = \frac{\lambda \times 0.18}{0.18} = \lambda$	MW1	
Thus using horizontal equilibrium at P :	$T = \lambda = \frac{5}{9}mg$	W1	

(ii) Choose the level of A and B as the gravitational PE baseline.

Thus Initial Energy = 0		M1
Final Gravitational PE =	$-0.15mg$	MW1
Final Elastic PE =	$\frac{\lambda x^2}{2l} = \frac{\lambda \times 0.18^2}{0.36} = 0.09\lambda$	M1 W1
Final KE =	$\frac{1}{2}mv^2$	M1
By Conservation of Mechanical Energy		
$0 =$	$\frac{1}{2}mv^2 - 0.15mg + 0.09 \times \frac{5}{9}mg$	M1 W1
$v =$	1.4 m s^{-1}	W1

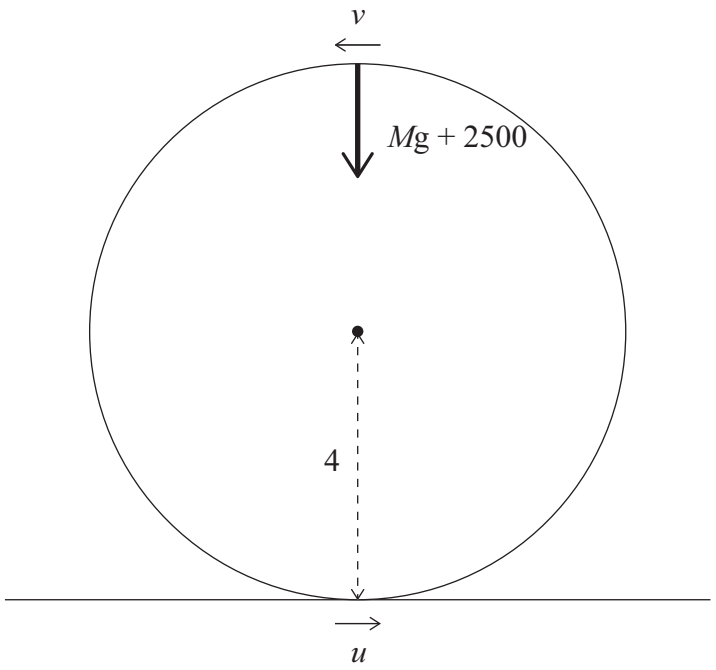
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Section A **50**

SECTION B: Mechanics 2

AVAILABLE MARKS

1



(i) At top of loop downward force = N + weight of car = 2500 + 1120 g

Central acceleration at top of loop = $\frac{1120v^2}{4}$ MW1

Newton's 2nd law vertically at top of loop: $\frac{1120v^2}{4} = 1120g + 2500$ M1 W1

$v = 6.94\,\text{m s}^{-1}$ (3sf) W1

(ii) By energy: $\frac{1}{2}Mu^2 = \frac{1}{2}Mv^2 + 8Mg$ M3

$u = 14.3\,\text{m s}^{-1}$ (3sf) W1 8

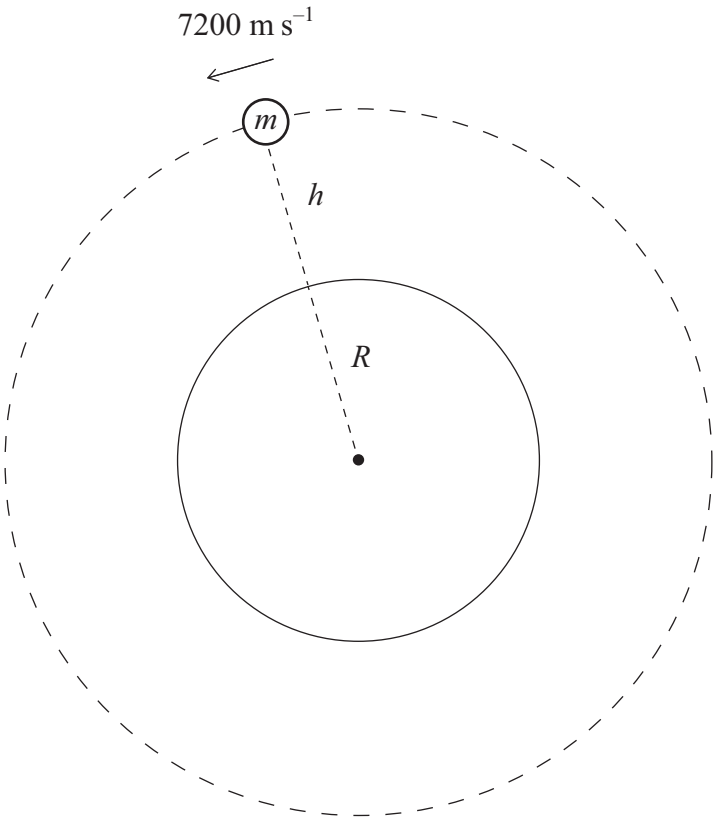
2 **(i)** Newton's Law of Gravitation for satellite on the earth's surface:

AVAILABLE MARKS

$\frac{GMm}{R^2} = mg$ M1

Therefore $G = \frac{gR^2}{M}$ W1

(ii)



Central acceleration $= \frac{7200^2}{R+h}$ MW1

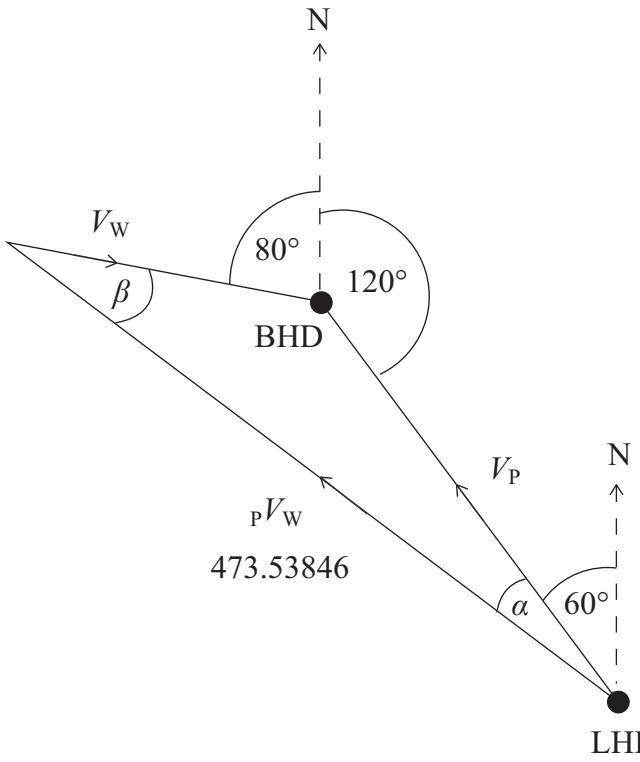
Newton's second law : $m\left(\frac{7200^2}{R+h}\right) = \frac{GMm}{(R+h)^2}$ M1 W1

So $\frac{7200^2}{R+h} = \frac{gR^2}{(R+h)^2}$

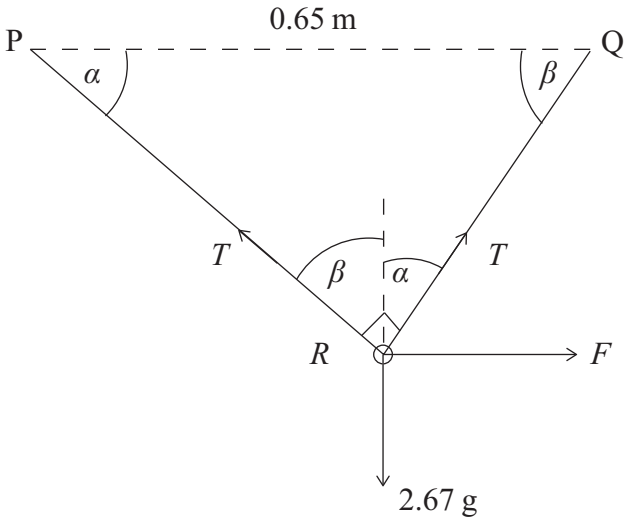
$h + 6.37 \times 10^6 = \frac{9.8 \times (6.37 \times 10^6)^2}{7200^2}$

$h = 1300.7 \rightarrow 1300\,\text{km}$ (3sf) W1 6

			AVAILABLE MARKS
3	**(i)** $[g] = LT^{-2}$ and $[a] = L$	W1	
	$L = \dfrac{[S]}{ML^{-3} \times LT^{-2} \times L}$	MW1	
	$[S] = MT^{-2}$	W1	
	(ii) $y = k\rho^{\alpha} g^{\beta} S^{\gamma}$ so $[y] = [\rho]^{\alpha}[g]^{\beta}[S]^{\gamma}$	MW1	
	$L = (ML^{-3})^{\alpha}(LT^{-2})^{\beta}(MT^{-2})^{\gamma}$	M1 W1	
	Compare indices:	M1	
	M: $0 = \alpha + \gamma$		
	L: $1 = -3\alpha + \beta$		
	T: $0 = -2\beta - 2\gamma$	W1	
	So $\alpha = -\frac{1}{2}$ $\beta = -\frac{1}{2}$ $\gamma = +\frac{1}{2}$	MW1 W1	
	Thus $y = k\sqrt{\frac{S}{\rho g}}$		
	(iii) $y_o = k\sqrt{\frac{S_0}{\rho_0 g}}$	MW1	
	$S \rightarrow \frac{5}{8}S_0$ and $\rho \rightarrow \frac{2}{5}\rho_0$		
	So $y_1 \rightarrow k\sqrt{\frac{\frac{5}{8}S_0}{\frac{2}{5}\rho_0 g}} = \sqrt{\frac{5}{8} \times \frac{5}{2}}\, y_0 = \frac{5}{4}y_0$	M1 W1	13

			AVAILABLE MARKS
4	**(i)**	Speed of aeroplane in still air = $513 \div 1\frac{1}{12} = 473.53846 = 474\ \text{km h}^{-1}$ M1 W1	
	(ii)	 MW2	
		$\frac{\sin\alpha}{41} = \frac{\sin 160°}{473.53846}$ M1 W1	
		$\alpha = 1.69694°$ W1	
		$\beta = 180 - 1.69694 - 160 = 18.30306°$ MW1	
		$V_p^2 = 41^2 + 473.53846^2 - 2 \times 41 \times 473.53846 \cos 18.30306° = 189053.98707$ M1	
		[Alternative for above mark $\frac{\sin\beta}{V_p} = \frac{\sin 160°}{473.53846}$]	
		$V_p = 434.803\ \text{km h}^{-1}$ W1	
		return time = $513 \div 434.803$ = 1.18 hr (3sf) = 1 hr 11 min W1	11

5 (i)



		AVAILABLE MARKS
If PR = 0.89 – x and RQ = x then $(0.89 - x)^2 + x^2 = 0.65^2$	M1	
$0.7921 - 1.78x + 2x^2 = 0.4225$		
$x^2 - 0.89x + 0.1848 = 0$	W1	
$x = 0.33$ m or $x = 0.56$ m	MW1	
As RQ is shorter, RQ = 0.33 m	W1	
(ii) Both tensions are equal	M1	
Resolving vertically: $T\cos\alpha + T\cos\beta = 2.67\,g$	M1 W1	
$T = \frac{2.67 \times 9.8}{1} \times \frac{65}{89} = 19.11$	W1	
Hooke's law: $T = \frac{0.29}{0.6}\lambda$	M1	
$\lambda = 39.538 \rightarrow 39.5$ N (3sf)	W1	
(iii) Resolving horizontally:		
$T\sin\beta = T\sin\alpha + F$	MW1	
$F = T\sin\beta - T\sin\alpha = 19.11 \times \left(\frac{56}{65} - \frac{33}{65}\right) = 6.76$ N (3sf)	W1	12
Section B		**50**

SECTION C: Statistics

			AVAILABLE MARKS
1	**(a)**	$E(X) = 15$ $\quad$ $Var(X) = 6$ $E(Y) = 25$ $\quad$ $Var(Y) = 13.5$	
		$Var(Y) = Var(aX + b) = a^2Var(X)$ M1 $13.5 = 6a^2$ MW1 $a = 1.5$ W1	
		$E(Y) = E(aX + b) = aE(X) + b$ $25 = 15a + b$ MW1 $b = 25 - 15(1.5) = 2.5$ W1	
	(b)	**(i)** IQ score W1	
		(ii) This regression line should not be used for this estimate. The regression line of head circumference on IQ score would be required for this purpose. MW1	
		(iii) Does not provide sufficient evidence. W1 Correlation does not necessarily imply causation (or equivalent). MW1	9
2	**(i)**	$X \sim \text{Geo}(0.2)$ $\quad p = 0.2$ $\quad q = 0.8$ M1	
		$P(X = 1) = p = 0.2$ MW1 $P(X = 2) = qp = 0.8 \times 0.2 = 0.16$ MW1 $P(X = 3) = q^2p = 0.8^2 \times 0.2 = 0.128$ MW1 $P(X \geqslant 4) = 1 - 0.488 = 0.512$ MW1	
	(ii)	$Var(X) = \dfrac{1-p}{p^2} = \dfrac{0.8}{0.2^2} = 20$ M1W1	7
3	**(i)**	$6! = 720$ MW1	
	(ii)	$\dfrac{1}{{}^6P_3} = \dfrac{1}{120}$ M1W1	
		Alternative solution for part (ii)	
		$\dfrac{3!}{6!} = \dfrac{6}{720} = \dfrac{1}{120}$ M1W1	
	(iii)	$\dfrac{1}{{}^6C_3} = \dfrac{1}{20}$ M1W1	
		Alternative solution for part (iii)	
		$\dfrac{3!}{{}^6P_3} = \dfrac{6}{120} = \dfrac{1}{20}$ M1W1	
	(iv)	$(6 - 1)! = 5! = 120$ M1W1	
	(v)	$(4 - 1)! \times 3! = 3! \times 3! = 36$ M2 W1	10

4 **(i)** X is the random variable 'number of meteors detected per hour'

AVAILABLE MARKS

$X \sim \text{Po}(20)$

$\text{P}(X = 15) = \text{e}^{-20}\dfrac{20^{15}}{15!}$ M1

$\text{P}(X = 15) = 0.0516$ (3sf) W1

(ii) Y is the random variable 'number of meteors detected per 30-minute period'

$Y \sim \text{Po}(10)$ MW1

$\text{P}(Y > 12) = 1 - \text{P}(Y \leqslant 12)$ M1

$= 1 - 0.7916$

$= 0.208$ (3sf) MW1

(iii) $W \sim \text{Po}(\lambda)$

$\text{P}(W > 10) = 0.3$

$\text{P}(W \leqslant 10) = 0.7$ MW1

From tables $\text{P}(W \leqslant 10) = 0.7060$ when $\lambda = 9.0$ M1

$\lambda = 9$ meteors per hour W1 8

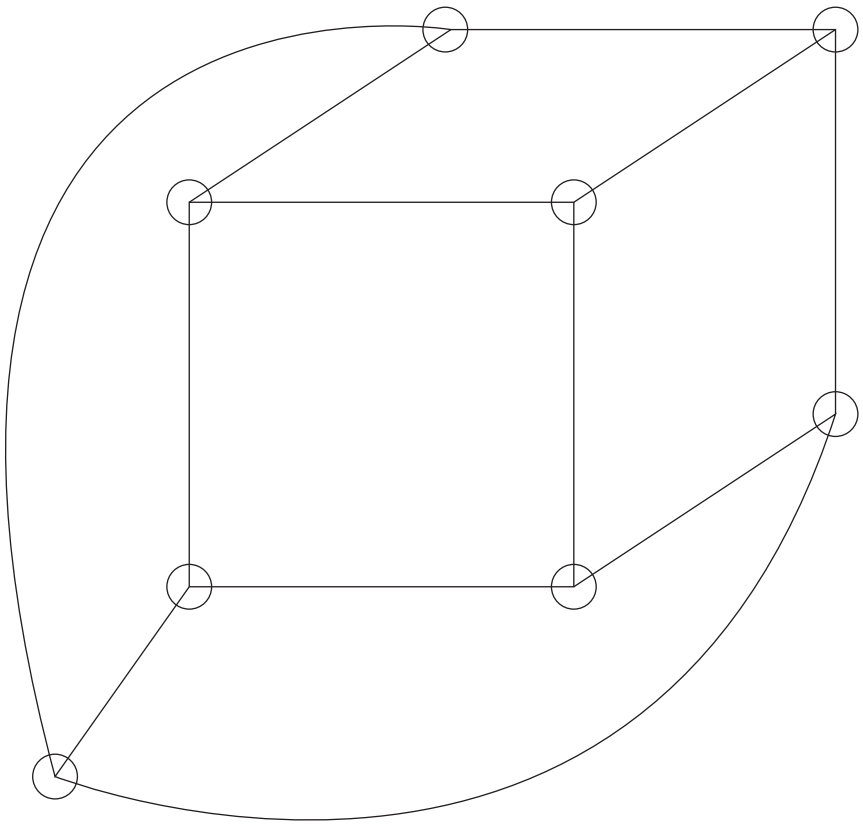
			AVAILABLE MARKS
5	**(i)** $\int_0^2 \left(ax^2 + bx + \frac{1}{6}\right)\mathrm{d}x = 1$	MW1W1	
	$\left[\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + \frac{1}{6}x\right]_0^2 = 1$	MW1	
	$\frac{8}{3}a + 2b = \frac{2}{3}$ ①	MW1	
	$\int_0^2 xf(x)\mathrm{d}x = \frac{13}{9}$		
	$\int_0^2 \left(ax^3 + bx^2 + \frac{1}{6}x\right)\mathrm{d}x = \frac{13}{9}$	M1MW1	
	$\left[\frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{12}x^2\right]_0^2 = \frac{13}{9}$	MW1	
	$4a + \frac{8}{3}b = \frac{10}{9}$ ②	MW1	
	Solving ① and ② simultaneously gives $a = \frac{1}{2}$ and $b = -\frac{1}{3}$	M1W1	
	(ii) $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}$ is a quadratic function with a minimum point.		
	Therefore, the maximum value of f(x) in the interval $0 \leqslant x \leqslant 2$ will occur at one of the endpoints of the interval.		
	f(0) = $\frac{1}{6}$ and f(2) = $\frac{3}{2}$	M1	
	Therefore the mode is $x = 2$	W1	
	(iii) $\int_0^M \left(\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}\right)\mathrm{d}x = 0.5$	MW1M1	
	$\left[\frac{1}{6}x^3 - \frac{1}{6}x^2 + \frac{1}{6}x\right]_0^M = 0.5$	MW1	
	$\frac{1}{6}M^3 - \frac{1}{6}M^2 + \frac{1}{6}M = 0.5$		
	$M^3 - M^2 + M - 3 = 0$ as required	W1	16
		Section C	**50**

Section D Discrete and Decision Mathematics

AVAILABLE MARKS

1 (a) (i) G_C is planar, $K_{3,3}$ is not planar M2

(ii) G_C is planar so



M2

(b)

Graph	Has an Eulerian trail	Number of edges that must be removed to leave an Eulerian circuit
R	yes	1
S	no	4

MW4 8

2 **(i)** Consider generation $n + 2$ – number of locusts = u_{n+2}
Number of locusts in parent and grandparent generations are u_{n+1} and u_n
Thus $u_{n+2} = 2u_{n+1} + 3u_n$ MW1

(ii) Substitute $u_n = kx^n$ M1
Obtaining $kx^{n+2} = 2kx^{n+1} + 3kx^n$
$kx^n(x^2 - 2x - 3) = 0$
So $x = 0$ or $x^2 - 2x - 3 = 0$ W1
$(x - 3)(x + 1) = 0$
$x = 3$ or $x = -1$ MW1
$u_n = 3^nA + (-1)^nB$ M1
$n = 0 \quad 1 = A + B$
$n = 1 \quad 1 = 3A - B$
so $A = \frac{1}{2}$ and $B = \frac{1}{2}$ MW1
$u_n = \frac{1}{2}(3^n + (-1)^n)$ W1

(iii) $n = 10$
$u_{10} = \frac{1}{2}(3^{10} + (-1)^{10}) = 29525$ M1
so 29 525 000 locusts W1

AVAILABLE MARKS: 9

3

p	q	r	$\sim p$	$\sim q$	$\sim r$	p and $\sim q$	(p and q) or $\sim r$	$\sim p$ or q	($\sim p$ or q) and r	$\sim$[($\sim p$ or q) and r]
T	T	T	F	F	F	F	F	T	T	F
T	T	F	F	F	T	F	T	T	F	T
T	F	T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	T	T	F	F	T
F	T	T	T	F	F	F	F	T	T	F
F	T	F	T	F	T	F	T	T	F	T
F	F	T	T	T	F	F	F	T	T	F
F	F	F	T	T	T	F	T	T	F	T
a	a	a	a	a	a	b	c	d	e	f

a M1 W1
b,c,d,e,f MW5
conclusion c = f MW1

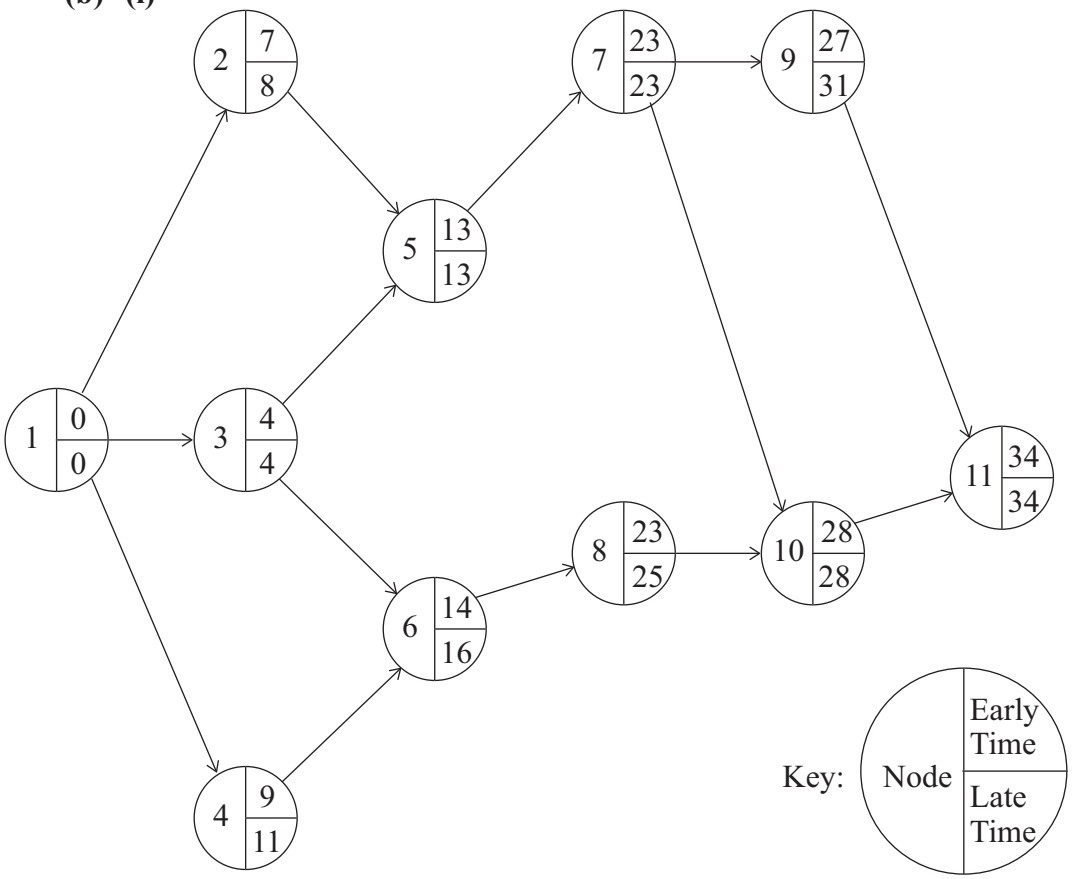
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4 (a) (i) Method of selecting the next shortest edge linking the already chosen vertices with the rest. M1

[AB] + [BC] + [CG] + [GI] + W1
[GF] + [FH] + W1
[AD] + [DE] W1

(ii) Distance of spanning tree = 3 + 2 + 4 + 2 + 4 + 4 + 5 + 2 = 26 km MW1

(b) (i)



Early times for nodes 1 to 4 MW1
Early times for nodes 5 to 11 MW1
Late times for nodes 8 to 11 MW1
Late times for nodes 1 to 7 MW1

(ii) Critical path: B, E, H, K, N MW1

(iii) Time of critical path = 4 + 9 + 10 + 5 + 6 = 34 days MW1

11

AVAILABLE MARKS

5 **(i)**

$\times_{40}$	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

[1] for each of 3 correct entries up to [3] MW3

(ii) Identity = 25 MW1

(iii)

Element	5	15	25	35
Period	2	2	1	2

M1 W1

(iv) Subsets of G which form subgroups under $\times_{40}$ are:
{25} and G MW1
{25, 5}, {25, 15}, {25, 35} MW2

(v)

$\circ$	p	q	r	s
p	p	q	r	s
q	q	p	s	r
r	r	s	q	p
s	s	r	p	q

[1] for each of 3 correct entries up to [3] MW3

(vi) (H, $\circ$) is isomorphic to (K, $\times$) M1

As the periods of the elements of (K, $\times$) are

Element	p	q	r	s
Period	1	2	4	4

So an isomorphism between (H, $\circ$) and (K, $\times$)
could be: $e \leftrightarrow p$ $b \leftrightarrow q$ $a \leftrightarrow r$ $c \leftrightarrow s$ W1

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Section D **50**

Total **100**