Rewarding Learning

ADVANCED
General Certificate of Education
2023

Further Mathematics

Assessment Unit A2 1

assessing

Pure Mathematics

[AFM11]

FRIDAY 26 MAY, MORNING

MARK SCHEME

GCE ADVANCED/ADVANCED SUBSIDIARY (AS) FURTHER MATHEMATICS

Introduction

The mark scheme normally provides the most popular solution to each question. Other solutions given by candidates are evaluated and credit given as appropriate; these alternative methods are not usually illustrated in the published mark scheme.

The marks awarded for each question are shown in the right-hand column and they are prefixed by the letters **M**, **W** and **MW** as appropriate. The key to the mark scheme is given below:

M indicates marks for correct method.

W indicates marks for working.

MW indicates marks for combined method and working.

The solution to a question gains marks for correct method and marks for an accurate working based on this method. Where the method is not correct no marks can be given.

A later part of a question may require a candidate to use an answer obtained from an earlier part of the same question. A candidate who gets the wrong answer to the earlier part and goes on to the later part is naturally unaware that the wrong data is being used and is actually undertaking the solution of a parallel problem from the point at which the error occurred. If such a candidate continues to apply correct method, then the candidate's individual working must be followed through from the error. If no further errors are made, then the candidate is penalised only for the initial error. Solutions containing two or more working or transcription errors are treated in the same way. This process is usually referred to as "follow-through marking" and allows a candidate to gain credit for that part of a solution which follows a working or transcription error.

Positive marking:

It is our intention to reward candidates for any demonstration of relevant knowledge, skills or understanding. For this reason we adopt a policy of **following through** their answers, that is, having penalised a candidate for an error, we mark the succeeding parts of the question using the candidate's value or answers and award marks accordingly.

Some common examples of this occur in the following cases:

(a) a numerical error in one entry in a table of values might lead to several answers being incorrect, but these might not be essentially separate errors;

(b) readings taken from candidates' inaccurate graphs may not agree with the answers expected but might be consistent with the graphs drawn.

When the candidate misreads a question in such a way as to make the question easier only a proportion of the marks will be available (based on the professional judgement of the examining team).

1 Test $n = 1$ $\quad u_1 = 3(2^1) - 1$

$= 6 - 1$

$= 5$ MW1

Therefore, correct for $n = 1$

Assume $\quad u_k = 3(2^k) - 1$ M1

Then $\quad u_{k+1} = 2[3(2^k) - 1] + 1$ M1 W1

$= 6(2^k) - 2 + 1$

$= 3(2^{k+1}) - 1$ W1

Therefore, true for $n = k + 1$

Since true for $n = 1$, then by induction true for all positive integers. MW1 6

2 $A = \int_4^\infty \frac{1}{x+4}\,dx$

$= \lim_{t\to\infty} \int_4^t \frac{1}{x+4}\,dx$ MW1

$= \lim_{t\to\infty} [\ln|x+4|]_4^t$ MW1

$= \lim_{t\to\infty} [\ln|t+4| - \ln 8]$ MW1

But $\lim_{t\to\infty} \ln|t+4|$ does not exist and therefore A cannot be evaluated. MW1

$B = \int_4^\infty \frac{1}{x^2+16}\,dx$

$= \lim_{t\to\infty} \int_4^t \frac{1}{x^2+16}\,dx$

$= \lim_{t\to\infty} \left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right]_4^t$ MW1

$= \lim_{t\to\infty} \left[\frac{1}{4}\tan^{-1}\left(\frac{t}{4}\right) - \frac{1}{4}\tan^{-1}(1)\right]$ MW1

$= \frac{\pi}{8} - \frac{\pi}{16}$ W1

Therefore B has a value of $\frac{\pi}{16}$ W1 8

AVAILABLE MARKS

3 Auxiliary equation is $m^2 - 10m + 25 = 0$ M1

$(m-5)(m-5) = 0$

$\Rightarrow m = 5$ W1

Complementary function is $y = (A + Bx)e^{5x}$ M1 W1

Particular integral is $y = Ce^{2x}$ M1 W1

$$\frac{dy}{dx} = 2Ce^{2x}$$

$$\frac{d^2y}{dx^2} = 4Ce^{2x}$$ MW1

$\Rightarrow 4Ce^{2x} - 20Ce^{2x} + 25Ce^{2x} = e^{2x}$ M1

$9Ce^{2x} = e^{2x}$

$C = \frac{1}{9}$ W1

General solution is $y = (A + Bx)e^{5x} + \frac{1}{9}e^{2x}$ W1 10

4 $\sum_{r=1}^{n}(n + r - 1)(n + r)$

$= \sum_{r=1}^{n}(n^2 + nr + rn + r^2 - n - r)$ M1 W1

$= \sum_{r=1}^{n}n^2 + 2\sum_{r=1}^{n}nr + \sum_{r=1}^{n}r^2 - \sum_{r=1}^{n}n - \sum_{r=1}^{n}r$ M1 W1

$= n^2\sum_{r=1}^{n}1 + 2n\sum_{r=1}^{n}r + \sum_{r=1}^{n}r^2 - n\sum_{r=1}^{n}1 - \sum_{r=1}^{n}r$

$= n^3 + 2n \times \frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(2n+1) - n^2 - \frac{1}{2}n(n+1)$ MW5

$= \frac{1}{6}n[6n^2 + 6n(n+1) + (n+1)(2n+1) - 6n - 3(n+1)]$

$= \frac{1}{6}n[6n^2 + 6n^2 + 6n + 2n^2 + 3n + 1 - 6n - 3n - 3]$

$= \frac{1}{6}n[14n^2 - 2]$

$= \frac{1}{3}n[7n^2 - 1]$ W1 10

AVAILABLE MARKS

5 $\dfrac{9x^2 + 13x + 29}{(x+2)(x^2+9)} \equiv \dfrac{A}{x+2} + \dfrac{Bx + C}{x^2+9}$ M1 W1

AVAILABLE MARKS

$A(x^2 + 9) + (Bx + C)(x + 2) \equiv 9x^2 + 13x + 29$ M1

Let $x = -2$ Then $13A = 39$

$A = 3$ MW1

Coefficient of x^2 $A + B = 9$

$B = 6$ MW1

Constant $9A + 2C = 29$

$C = 1$ MW1

$$\int_0^3 \frac{9x^2 + 13x + 29}{(x+2)(x^2+9)}\,dx \equiv \int_0^3 \frac{3}{x+2}\,dx + \int_0^3 \frac{6x+1}{x^2+9}\,dx$$

$$\equiv \int_0^3 \frac{3}{x+2}\,dx + \int_0^3 \frac{6x}{x^2+9}\,dx + \int_0^3 \frac{1}{x^2+9}\,dx$$ M1W1

$$= \left[3\ln|x+2| + 3\ln|x^2+9| + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$ MW1 MW1

$$= \left[3\ln 5 + 3\ln 18 + \frac{1}{3}\tan^{-1}(1)\right] - \left[3\ln 2 + 3\ln 9 + \frac{1}{3}\tan^{-1}(0)\right]$$

$$= 3\ln 5 + \frac{\pi}{12}$$ MW1 11

6 **(i)** $9x^2 + 24x - 20 \equiv (3x + a)^2 - b$

$\Rightarrow 24 = 6a$ and $a^2 - b = -20$

$\Rightarrow a = 4$ and $b = 36$ MW1

$\Rightarrow 9x^2 + 24x - 20 \equiv (3x + 4)^2 - 36$

(ii) $\displaystyle\int_1^2 \frac{\mathrm{d}x}{\sqrt{9x^2 + 24x - 20}} = \int_1^2 \frac{\mathrm{d}x}{\sqrt{(3x + 4)^2 - 36}}$ M1

Let $X = 3x + 4 \Rightarrow \dfrac{\mathrm{d}X}{\mathrm{d}x} = 3$ M1W1

If $x = 1$, then $X = 7$ M1

If $x = 2$, then $X = 10$ W1

$\displaystyle\Rightarrow \int_7^{10} \frac{\frac{1}{3}\mathrm{d}X}{\sqrt{(X^2 - 6^2)}}$ MW1

$\displaystyle= \left[\frac{1}{3}\cosh^{-1}\left(\frac{X}{6}\right)\right]_7^{10}$ W1

$\displaystyle= \left[\frac{1}{3}\ln\left(X + \sqrt{X^2 - 6^2}\right)\right]_7^{10}$ M1

$\displaystyle= \frac{1}{3}\ln\left(10 + \sqrt{64}\right) - \frac{1}{3}\ln\left(7 + \sqrt{13}\right)$ W1

$\displaystyle= \frac{1}{3}\ln\left(\frac{18}{7 + \sqrt{13}}\right)$

$\displaystyle= \frac{1}{3}\ln\left(\frac{7 - \sqrt{13}}{2}\right)$ W1 11

AVAILABLE MARKS

					AVAILABLE MARKS
7	**(a)**	**(i)**	At A, $r = 3, \theta = \frac{\pi}{2}$	MW2	
			$3 = p\cos\left(\frac{\pi}{2}\right) + q$	M1	
			$\Rightarrow q = 3$	W1	
		(ii)	$r_{max} = (p\cos\theta + 3)_{max}$		
			Maximum value of $\cos\theta$ is 1 and $p > 0$	M1	
			At (8, 0), $r = 8$		
			$\Rightarrow 8 = p + 3$	M1	
			$p = 5$	W1	
	(b)		$r = 4\cos 2\theta$		
			$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2$		
			$r^2 = 16\cos^2 2\theta$	M1	
			$r^2 = 16(2\cos^2\theta - 1)^2$	MW1	
			$r^2 = 16\left(\frac{2x^2}{r^2} - 1\right)^2$	MW1	
			$r^6 = 16(2x^2 - r^2)^2$		
			$(x^2 + y^2)^3 = 16(2x^2 - x^2 - y^2)^2$	M1	
			$(x^2 + y^2)^3 = 16(x^2 - y^2)^2$	W1	12
8	**(i)**		$\frac{dP}{dt} = \frac{1}{100}P + 50$	MW1	
			$\frac{dP}{dt} - 0.01P = 50$		
	(ii)		Integrating factor $= e^{-\int 0.01 dt}$	M1	
			$= e^{-0.01t}$	W1	
			$\Rightarrow e^{-0.01t}\frac{dP}{dt} - 0.01Pe^{-0.01t} = 50e^{-0.01t}$	MW1	
			Integrate to give		
			$Pe^{-0.01t} = -5000e^{-0.01t} + c$	M1	
			$P = -5000 + ce^{0.01t}$	W1	
			$t = 0, \quad P = 40 \Rightarrow 40 = -5000 + c$	M1	
			$c = 5040$	W1	
			$t = 5 \Rightarrow P = -5000 + 5040e^{0.05}$		
			$P = 298$	MW1	9

9 **(a)** **(i)** $2\sinh(x-y)\ \sinh(x+y) \equiv 2 \times \frac{1}{2}(e^{x-y} - e^{-x+y}) \times \frac{1}{2}(e^{x+y} - e^{-x-y})$ M1 W1

$\equiv \frac{1}{2}(e^{2x} - e^{2y} - e^{-2y} + e^{-2x})$ W1

$\equiv \frac{1}{2}(e^{2x} + e^{-2x}) - \frac{1}{2}(e^{2y} + e^{-2y})$ W1

$\equiv \cosh 2x - \cosh 2y$ W1

(ii) $x - y = \theta$
$x + y = 3\theta$

$\Rightarrow 2x = 4\theta,\ 2y = 2\theta$

$\int_0^{0.5} \sinh\theta \sinh 3\theta\, d\theta = \frac{1}{2}\int_0^{0.5} (\cosh 4\theta - \cosh 2\theta)\, d\theta$ M1 W1

$= \left[\frac{1}{8}\sinh 4\theta - \frac{1}{4}\sinh 2\theta\right]_0^{0.5}$ M1 W1

$= \frac{1}{8}\sinh 2 - \frac{1}{4}\sinh 1$
$= 0.160$ (3s.f.) W1

(b) $\frac{dy}{dx} = 3\cosh 3x - e^x\ \cosh 2x - 2e^x\ \sinh 2x$ M2 W1

$\Rightarrow 3(e^{3x} + e^{-3x}) - e^x(e^{2x} + e^{-2x}) - 2e^x(e^{2x} - e^{-2x}) = 0$ M2 W1

$3e^{-3x} + e^{-x} = 0$
$e^{-3x}(3 + e^{2x}) = 0$
$e^{-3x} = 0$ or $e^{2x} = -3$ W1

Since neither of these have real solutions, there are no stationary points. MW1 18

AVAILABLE MARKS

10 (i) $f(x) = \ln(1+ \sin x)$

	AVAILABLE MARKS
$f(0) = 0$ MW1	
$f'(x) = \dfrac{\cos x}{1 + \sin x}$ M1 W1	
$f'(0) = 1$ MW1	
$f''(x) = \dfrac{(1 + \sin x)(- \sin x) - \cos x\,(\cos x)}{(1 + \sin x)^2}$ M1	
$f''(x) = \dfrac{- \sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$	
$f''(x) = \dfrac{-(1 + \sin x)}{(1 + \sin x)^2}$	
$f''(x) = \dfrac{-1}{1 + \sin x}$	
$f''(0) = -1$ MW1	
$f'''(x) = (1 + \sin x)^{-2} \times \cos x$	
$f'''(x) = 1$ MW1	
$\Rightarrow f(x) = 0 + 1x - \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3$ M1	
$f(x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \ldots$ W1	
(ii) $\ln(1 - \sin x) = \ln(1 + \sin(-x))$ M1	
$= -x - \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 + \ldots$ W1	
(iii) $\ln(\cos x) = \dfrac{1}{2}\ln(\cos^2 x)$ M1	
$= \dfrac{1}{2}\ln(1 - \sin^2 x)$ MW1	
$= \dfrac{1}{2}\ln[(1 - \sin x)(1 + \sin x)]$ W1	
$= \dfrac{1}{2}[\ln(1 - \sin x) + \ln(1 + \sin x)]$ W1	
$\approx -\dfrac{1}{2}x^2$ W1	16

		AVAILABLE MARKS
11 **(i)**	$I_n = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^n x \, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x \, dx$ M1W1	
	$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1)\tan^{n-2} x \, dx$ M1	
	$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x \, dx - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^{n-2} x \, dx$ W1	
	$= \left[\frac{1}{n-1}\tan^{n-1} x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - I_{n-2}$ M1W2	
	$= \frac{1}{n-1}(1)^{n-1} - \frac{1}{n-1}(-1)^{n-1} - I_{n-2}$ MW1	
	$I_n = \frac{2}{n-1} - I_{n-2}$ when n is even W1	
(ii)	Curve cuts x-axis when $1 - \tan^6 x = 0$ M1	
	$\Rightarrow \tan x = \pm 1$	
	$x = \pm\frac{\pi}{4}$ W1	
	Area $= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - \tan^6 x) \, dx$ M1 W1	
	$= [x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - I_6$	
	$= \frac{\pi}{2} - I_6$ W1	
	$I_6 = \frac{2}{5} - I_4$ and $I_4 = \frac{2}{3} - I_2$ MW1	
	$I_2 = 2 - I_0$	
	$= 2 - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 \, dx$ M1	
	$= 2 - [x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$	
	$= 2 - \frac{\pi}{2}$ W1	
	Alternative Solution for I_2	
	$I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \, dx$	
	$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1) \, dx$ M1	
	$= [\tan x - x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$	
	$= 2 - \frac{\pi}{2}$ W1	
	$\Rightarrow I_4 = \frac{\pi}{2} - \frac{4}{3}$ and $I_6 = \frac{26}{15} - \frac{\pi}{2}$ W1	
	$\Rightarrow$ Area $= \pi - \frac{26}{15}$ W1	19

12 (a) (i) $r\cos\theta = 16\sqrt{2}$
$r\sin\theta = 16\sqrt{2}$ M1

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$ W1

$r = 32$ W1

$z^5 = 32e^{\frac{i\pi}{4}}$
$z = 2e^{i\left(\frac{\pi}{20} + \frac{2n\pi}{5}\right)}$ MW1

A represents the root $2e^{\frac{i\pi}{20}}$ W1

(ii) Length of each "radius" of pentagon = 2
Angle between each "radius" of pentagon = $\frac{2\pi}{5}$

Area of pentagon = $5 \times \frac{1}{2} \times 2 \times 2 \times \sin\left(\frac{2\pi}{5}\right)$ M1 W1

= 9.51 square units W1

(b) (i) $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2$
$+ 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ M1W1

But $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ M1

$\Rightarrow \cos 5\theta = \cos^5\theta - 10\cos^3\theta\ \sin^2\theta + 5\cos\theta\ \sin^4\theta$ W1

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ W1

$\Rightarrow \tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\ \sin^2\theta + 5\cos\theta\ \sin^4\theta}$ MW1

Dividing numerator and denominator on RHS through by $\cos^5\theta$

$\Rightarrow \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ MW1

(ii) Let $\theta = \frac{\pi}{15}$, so $5\theta = \frac{\pi}{3}$ and $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\sqrt{3} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\left(\frac{\pi}{15}\right)$ M1 W1

$\Rightarrow \sqrt{3}(1 - 10t^2 + 5t^4) = 5t - 10t^3 + t^5$ M1

$t^5 - 5\sqrt{3}t^4 - 10t^3 + 10\sqrt{3}t^2 + 5t - \sqrt{3} = 0$ W1

Hence $\tan\left(\frac{\pi}{15}\right)$ is a root of the given equation. W1 20

Total **150**

AVAILABLE MARKS