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ADVANCED SUBSIDIARY (AS)
General Certificate of Education
2022

Further Mathematics *StemUp Model Solution*

Assessment Unit AS 1
assessing
Pure Mathematics

SFM11

[SFM11]
MONDAY 16 MAY, AFTERNOON

TIME

1 hour 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided.
Answer **all eight** questions.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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Answer all eight questions.

Show clearly the full development of your answers.

Answers should be given to three significant figures unless otherwise stated.

1 Matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & 0 \\ 3 & -2 & p \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \\ 1 & p+1 \end{pmatrix}$$

(i) Calculate the matrix product **AB**, leaving your answer in terms of p. [2]

(ii) Given that **AB** is singular, find the possible values of p. [4]

2 The complex numbers z_1 and z_2 are given by

$$z_1 = 2 + 3\mathrm{i} \quad \text{and} \quad z_2 = 1 + q\mathrm{i}$$

where q is a real number.

(i) Find the modulus and argument of z_1 [3]

(ii) Given that $z_1 + 3z_2$ is a real number, find the value of q. [2]

(iii) Hence simplify

$$\frac{z_1}{z_2}$$

giving your answer in the form $a + b\mathrm{i}$, where a and b are real numbers. [4]

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3 The roots of the quadratic equation

$$2x^2 - 4x + 5 = 0$$

are α and β.

(i) Write down the values of $(\alpha + \beta)$ and $\alpha\beta$. [3]

(ii) By considering $(\alpha + \beta)^3$, find the value of $\alpha^3 + \beta^3$ [4]

(iii) Find a quadratic equation with integer coefficients whose roots are

$$\frac{1}{\alpha + 1} \quad \text{and} \quad \frac{1}{\beta + 1}$$ [6]

4 **(i)** The matrix $\mathbf{S} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Describe fully the transformation represented by **S** [2]

(ii) The set of points which form the curve

$$x^2 + 10y^2 + 6xy - 25 = 0$$

is transformed by the matrix **S**

Show that the curve formed by the image points is a circle and state its centre and radius. [6]

(iii) The matrix $\mathbf{N} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

Find the single matrix that represents the combined effect of the transformation represented by **S** followed by that represented by **N** [2]

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5 Two lines are given by the equations

$$\frac{x+2}{4} = \frac{y-1}{2} = \frac{z+4}{3} \quad \text{and} \quad x-3 = \frac{y-2}{-1} = \frac{z+3}{-2}$$

(i) Show that the two lines intersect and find their point of intersection. [9]

(ii) Find the Cartesian equation of the plane which contains both these lines. [6]

(iii) Find the coordinates of the point at which this plane cuts the y-axis. [1]

6 A matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & p \\ 1 & -2 & -2 \\ 0 & p+1 & 2 \end{pmatrix}$$

(i) Show that the determinant of **M** is given by

$$p^2 + 5p - 6$$ [3]

A system of linear equations is given by

$$\begin{aligned} 2x + y + pz &= 3 \\ x - 2y - 2z &= 6 \\ (p+1)y + 2z &= -7 \end{aligned}$$

(ii) Find the possible values of p for which this system has a unique solution. [3]

(iii) If $p = 2$ find, without the use of a calculator, the inverse of **M** [5]

(iv) Hence for $p = 2$, find the unique solution of the system of equations. [4]

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7 **(a)** **(i)** Find the real values of a and b such that

$$(a + b\mathrm{i})^2 = 12 - 16\mathrm{i}$$

[6]

(ii) Hence write down the solutions to the equation

$$z^2 = 12 - 16\mathrm{i}$$ [2]

(b) Sketch, on a carefully labelled Argand diagram, the region consisting of points which represent a complex number, z, satisfying both

$$|z - 2\mathrm{i}| \leqslant 4 \quad \text{and} \quad |z + 4 - \mathrm{i}| \leqslant |z + \mathrm{i}|$$ [7]

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8 ABCDE is a rectangular-based pyramid with apex E, as shown in **Fig. 1** below.

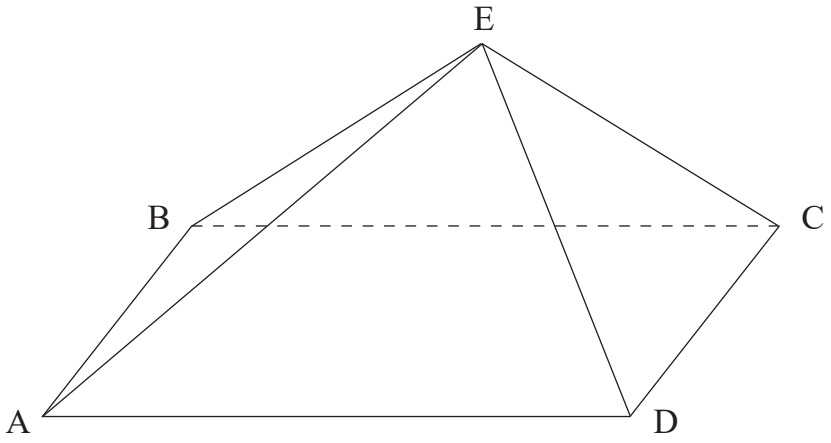


Fig. 1

The vertices A, B, D and E have coordinates (−2, −2, 1), (1, 2, 6), (−3, 0, 0) and (2, −2, 2) respectively.

(i) Find the coordinates of vertex C. [3]

(ii) Find the volume of the pyramid. [7]

The line l has the vector equation

$$\mathbf{r} = \begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

(iii) Find the shortest distance between the point E and the line l. [6]

THIS IS THE END OF THE QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (i)

$$AB = \begin{pmatrix} 4 & 1 & 0 \\ 3 & -2 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 2 \\ 1 & p+1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-2+0 & 0+2+0 \\ 3+4+p & 0-4+p(p+1) \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 2 \\ 7+p & p^2+p-4 \end{pmatrix}$$

(ii)

AB singular $\Rightarrow$ $\det AB = 0$

$2(p^2 + p - 4) - 2(7 + p) = 0$

$2p^2 + 2p - 8 - 14 - 2p = 0$

$2p^2 - 22 = 0$

$p^2 = 11$

$p = \pm\sqrt{11}$ $(p = \pm 3.32$ to 3 s.f.$)$

2 (i)

$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.61$ (3 s.f.)

z_1 is in the first quadrant:

$\arg z_1 = \tan^{-1}\left(\frac{3}{2}\right) = 0.983$ radians (3 s.f.)

(ii)

$z_1 + 3z_2 = 2 + 3i + 3(1 + qi)$

$= 5 + (3 + 3q)i$

Real, so imaginary part $= 0$: $3 + 3q = 0$

$q = -1$

Question number

(iii) $z_2 = 1 - i$

$$\frac{z_1}{z_2} = \frac{2 + 3i}{1 - i} \times \frac{1 + i}{1 + i}$$

$$= \frac{2 + 2i + 3i + 3i^2}{1 - i^2}$$

$$= \frac{2 + 5i - 3}{2} = \frac{-1 + 5i}{2}$$

$$\frac{z_1}{z_2} = -\frac{1}{2} + \frac{5}{2}i$$

3 (i) $\alpha + \beta = -\frac{b}{a} = -\frac{(-4)}{2} = 2$

$\alpha\beta = \frac{c}{a} = \frac{5}{2}$

(ii) $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= 2^3 - 3(\frac{5}{2})(2)$

$= 8 - 15$

$\alpha^3 + \beta^3 = -7$

(iii) Sum of new roots $= \frac{1}{\alpha + 1} + \frac{1}{\beta + 1} = \frac{(\beta + 1) + (\alpha + 1)}{(\alpha + 1)(\beta + 1)}$

$$= \frac{(\alpha + \beta) + 2}{\alpha\beta + (\alpha + \beta) + 1}$$

Question number

$= \dfrac{2+2}{\frac{5}{2}+2+1} = \dfrac{4}{\frac{11}{2}} = \dfrac{8}{11}$

Product of new roots $= \dfrac{1}{\alpha+1} \times \dfrac{1}{\beta+1} = \dfrac{1}{\alpha\beta+(\alpha+\beta)+1}$

$= \dfrac{1}{\frac{11}{2}} = \dfrac{2}{11}$

Equation: $x^2 - (\text{sum})x + (\text{product}) = 0$

$x^2 - \dfrac{8}{11}x + \dfrac{2}{11} = 0$

$11x^2 - 8x + 2 = 0$

4 (i) S is a shear with the x-axis invariant (parallel to the x-axis),

shear factor 3: $(x, y) \to (x + 3y, y)$

(ii) $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$X = x + 3y, \quad Y = y$

so $y = Y$ and $x = X - 3Y$

Substitute into $x^2 + 10y^2 + 6xy - 25 = 0$:

$(X - 3Y)^2 + 10Y^2 + 6(X - 3Y)Y - 25 = 0$

$X^2 - 6XY + 9Y^2 + 10Y^2 + 6XY - 18Y^2 - 25 = 0$

$X^2 + Y^2 - 25 = 0$

$X^2 + Y^2 = 25$, which is a circle

Centre $(0, 0)$, radius 5

(iii) S first, then N: matrix = NS

Question number

$NS = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3+0 & 9+2 \\ 1+0 & 3+4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 11 \\ 1 & 7 \end{pmatrix}$

5 (i) $l_1: \ r = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \qquad l_2: \ r = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

If they meet:

i: $-2 + 4\lambda = 3 + \mu$ ①

j: $1 + 2\lambda = 2 - \mu$ ②

k: $-4 + 3\lambda = -3 - 2\mu$ ③

① + ②: $-1 + 6\lambda = 5$

$6\lambda = 6 \Rightarrow \lambda = 1$

Sub into ②: $1 + 2 = 2 - \mu \Rightarrow \mu = -1$

Check ③: l_1: $-4 + 3(1) = -1$

l_2: $-3 - 2(-1) = -1$

Both give -1, so the equations are consistent and the lines intersect.

$\lambda = 1$: $r = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Point of intersection $(2, 3, -1)$

(ii) Normal $n = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

Question number

$= i((2)(-2) - (3)(-1)) - j((4)(-2) - (3)(1)) + k((4)(-1) - (2)(1))$

$= i(-4 + 3) - j(-8 - 3) + k(-4 - 2)$

$= -i + 11j - 6k$

$r \cdot n = a \cdot n$ using the point $(2, 3, -1)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 11 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 11 \\ -6 \end{pmatrix}$$

$-x + 11y - 6z = -2 + 33 + 6$

$-x + 11y - 6z = 37$

(iii) On the y-axis $x = 0$ and $z = 0$:

$11y = 37 \Rightarrow y = \frac{37}{11}$

Point $(0, \frac{37}{11}, 0)$

6 (i) Expand along the top row:

$\det M = 2((-2)(2) - (-2)(p + 1)) - 1((1)(2) - (-2)(0))$

$+ p((1)(p + 1) - (-2)(0))$

$= 2(-4 + 2(p + 1)) - 1(2 - 0) + p(p + 1 - 0)$

$= 2(2p - 2) - 2 + p^2 + p$

$= 4p - 4 - 2 + p^2 + p$

$= p^2 + 5p - 6$ as required

(ii) Unique solution $\Leftrightarrow$ $\det M \neq 0$

$p^2 + 5p - 6 = 0$

$(p + 6)(p - 1) = 0$

$p = -6$ or $p = 1$

Question number

Unique solution for all p except $p = -6$ and $p = 1$

(i.e. $p \neq -6$ and $p \neq 1$)

(iii) $p = 2$: $M = \begin{pmatrix} 2 & 1 & 2 \\ 1 & -2 & -2 \\ 0 & 3 & 2 \end{pmatrix}$

$\det M = 2^2 + 5(2) - 6 = 8$

Matrix of minors $= \begin{pmatrix} 2 & 2 & 3 \\ -4 & 4 & 6 \\ 2 & -6 & -5 \end{pmatrix}$

Matrix of cofactors $= \begin{pmatrix} 2 & -2 & 3 \\ 4 & 4 & -6 \\ 2 & 6 & -5 \end{pmatrix}$

Adjoint (transpose) $= \begin{pmatrix} 2 & 4 & 2 \\ -2 & 4 & 6 \\ 3 & -6 & -5 \end{pmatrix}$

$M^{-1} = \frac{1}{8}\begin{pmatrix} 2 & 4 & 2 \\ -2 & 4 & 6 \\ 3 & -6 & -5 \end{pmatrix}$

(iv) $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix}$

$= \frac{1}{8}\begin{pmatrix} 2 & 4 & 2 \\ -2 & 4 & 6 \\ 3 & -6 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix}$

Question number

$= \frac{1}{8}\begin{pmatrix} 6 + 24 - 14 \\ -6 + 24 - 42 \\ 9 - 36 + 35 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 16 \\ -24 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$x = 2, \quad y = -3, \quad z = 1$

7 (a)(i) $(a + bi)^2 = a^2 + 2abi + b^2i^2 = (a^2 - b^2) + 2abi$

Compare with $12 - 16i$:

Real: $a^2 - b^2 = 12$ Imaginary: $2ab = -16 \Rightarrow b = -\frac{8}{a}$

$a^2 - (-\frac{8}{a})^2 = 12$

$a^2 - \frac{64}{a^2} = 12$

$a^4 - 12a^2 - 64 = 0$

$(a^2 - 16)(a^2 + 4) = 0$

$a^2 = 16$ ($a^2 = -4$ has no real solution)

$a = 4, \ b = -2$ or $a = -4, \ b = 2$

(ii) $z = 4 - 2i$ or $z = -4 + 2i$

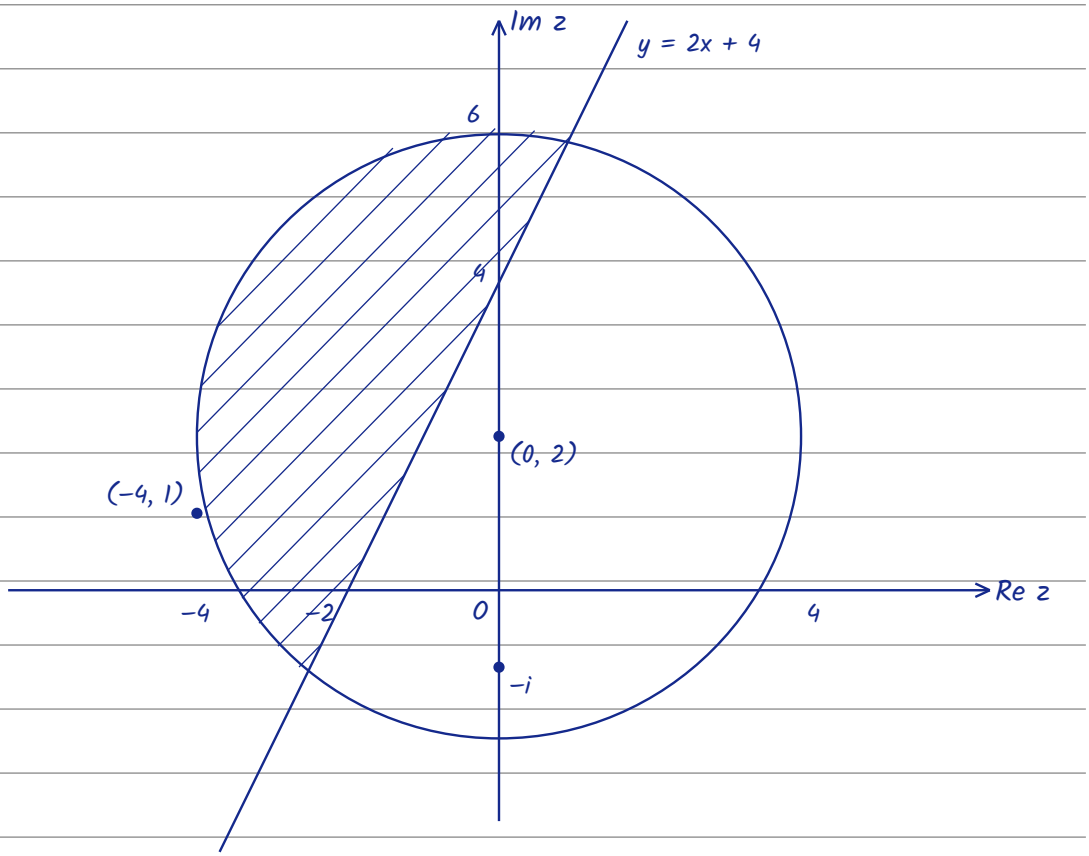
(b) $|z - 2i| \leq 4$: on or inside the circle centre $(0, 2)$, radius 4

$|z + 4 - i| \leq |z + i|$: z is at least as close to $-4 + i$ as to $-i$,

so on the side of the perpendicular bisector of $(-4, 1)$ and $(0, -1)$

containing $(-4, 1)$. Bisector: midpoint $(-2, 0)$, gradient 2: $y = 2x + 4$

Question number



The region required is shaded: inside or on the circle and on the side of the line $y = 2x + 4$ containing $(-4, 1)$, boundaries included.

8 (i)

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$ABCD$ is a rectangle, so $\overrightarrow{DC} = \overrightarrow{AB}$

$$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{AB} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$$

$C(0, 4, 5)$

Question number

(ii)

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{AE} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

Volume $= \frac{1}{3} \times$ base area $\times$ height $= \frac{1}{3} \left| \overrightarrow{AE} \cdot (\overrightarrow{AD} \times \overrightarrow{AB}) \right|$

$\overrightarrow{AD} \times \overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$= i((2)(5) - (-1)(4)) - j((-1)(5) - (-1)(3)) + k((-1)(4) - (2)(3))$

$= 14i + 2j - 10k$

Volume $= \frac{1}{3} \left| \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 2 \\ -10 \end{pmatrix} \right|$

$= \frac{1}{3} \left| 56 + 0 - 10 \right|$

$= \frac{46}{3} = 15.3$ cubic units (3 s.f.)

(iii)

Let P be the foot of the perpendicular from E to l:

$\overrightarrow{OP} = \begin{pmatrix} -4 + 3\lambda \\ 2 - \lambda \\ -1 + 2\lambda \end{pmatrix}$

$\overrightarrow{PE} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 + 3\lambda \\ 2 - \lambda \\ -1 + 2\lambda \end{pmatrix} = \begin{pmatrix} 6 - 3\lambda \\ -4 + \lambda \\ 3 - 2\lambda \end{pmatrix}$

Question number

$\overrightarrow{PE}$ is perpendicular to l: $\begin{pmatrix} 6-3\lambda \\ -4+\lambda \\ 3-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$

$18 - 9\lambda + 4 - \lambda + 6 - 4\lambda = 0$

$28 - 14\lambda = 0 \Rightarrow \lambda = 2$

$\overrightarrow{PE} = \begin{pmatrix} 6-6 \\ -4+2 \\ 3-4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$

$|\overrightarrow{PE}| = \sqrt{0^2 + (-2)^2 + (-1)^2} = \sqrt{5}$

Shortest distance $= \sqrt{5} = 2.24$ (3 s.f.)

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