Rewarding Learning

ADVANCED

General Certificate of Education

2022

Further Mathematics *StemUp Model Solution*

AFM21

Assessment Unit A2 2

assessing

Applied Mathematics

[AFM21]

TUESDAY 7 JUNE, AFTERNOON

TIME

2 hours 15 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided and on the Section D Supplementary Answer Booklet (if applicable).
You must answer **all** questions from sections A and B **or** A and C **or** A and D **or** C and D.
You should spend equal time on each of the two sections.
Candidates taking Section D should use the insert provided and attach to your Answer Booklet using the treasury tag provided.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150.
The total mark for each section of this paper is 75.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
Answers should include diagrams where appropriate and marks may be awarded for them.
Take $g = 9.8\,\text{m}\,\text{s}^{-2}$, unless specified otherwise.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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SECTION A Mechanics 1

Answer all five questions in this section.

1 The letter E is shown in **Fig. 1** below.

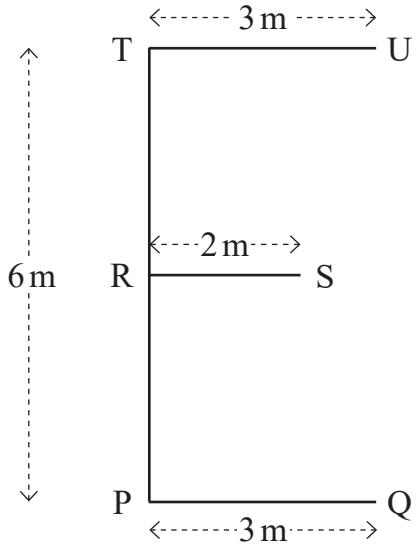


Fig. 1

It is formed by joining four uniform rods PT, PQ, RS and TU, each of mass ρ kg per m.

PQ = TU = 3 m

RS = 2 m

PT = 6 m

PQ, RS and TU are each perpendicular to PT and R is the midpoint of PT.

(i) Write down the distance of the centre of mass of the letter E from PQ. [1]

(ii) Find the distance of the centre of mass of the letter E from PT. [8]

The letter E is freely suspended from the corner T.

(iii) Find the angle which PT makes with the vertical when the letter is in equilibrium. [3]

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2 **(a)** A particle moves so that at time t seconds its displacement from a fixed point is x metres.

Its motion is modelled by the differential equation

$$p\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + q\frac{\mathrm{d}x}{\mathrm{d}t} + rx = 0$$

where p, q and r are constants.

State the condition which must hold if the motion is

(i) critically damped; [2]

(ii) over damped. [1]

(b) The pointer on a set of scales initially oscillates and then gradually settles down to a final position O.

At time t seconds the displacement of the pointer from O is x metres.

The displacement is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 0$$

(i) Find a general equation for the displacement x of the particle in terms of t. [6]

(ii) State a limitation of this model. [1]

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3 A car of mass m kg is travelling in a horizontal circle of radius r metres on a road banked at $\alpha°$ to the horizontal as shown in **Fig. 2** below.

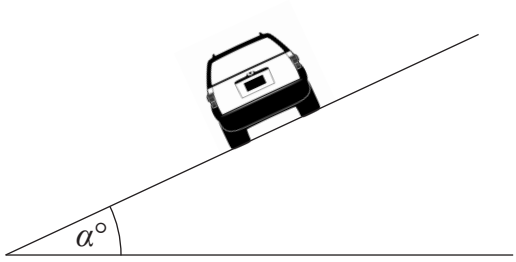


Fig. 2

The coefficient of friction between the tyres of the car and the road surface is μ.
When the car is travelling at u m s^{-1} it is just about to slide down the slope.

(i) By modelling the car as a particle, show that

$$u^2 = \frac{rg(\tan\alpha - \mu)}{1 + \mu\tan\alpha}$$ [9]

When the car is travelling at v m s^{-1} it is just about to slide up the slope.

(ii) Find an expression for v^2 in terms of r, g, α and μ. [3]

(iii) Given that $\alpha = 45$ and $\mu = 0.6$, show that

$$v = 4u$$ [5]

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4 A spring has natural length 2 m and modulus of elasticity 32 N.

One end is attached to a fixed point A on a smooth horizontal table.
The other end is attached to a particle P of mass 4 kg.

The spring has extension x metres at time t seconds as shown in **Fig. 3** below.

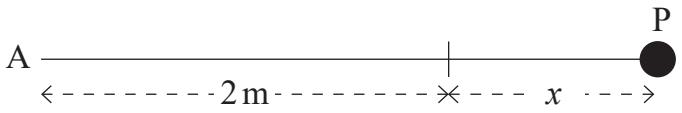


Fig. 3

(i) Show that the Simple Harmonic Motion of the particle can be described by the equation

$$\ddot{x} = -\omega^2 x$$

and find the value of ω. [5]

The particle is held on the table at a point B, which is a distance of 3 m from A.
The particle is then released from rest.

The point C is 0.4 m from B.

(ii) Find the speed of the particle when it reaches C. [4]

(iii) Find the time that elapses from when the particle is first at C until it next returns to C. [6]

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5 A framework ABCDEF consists of nine light pin-jointed rods held in a vertical plane as shown in **Fig. 4** below.

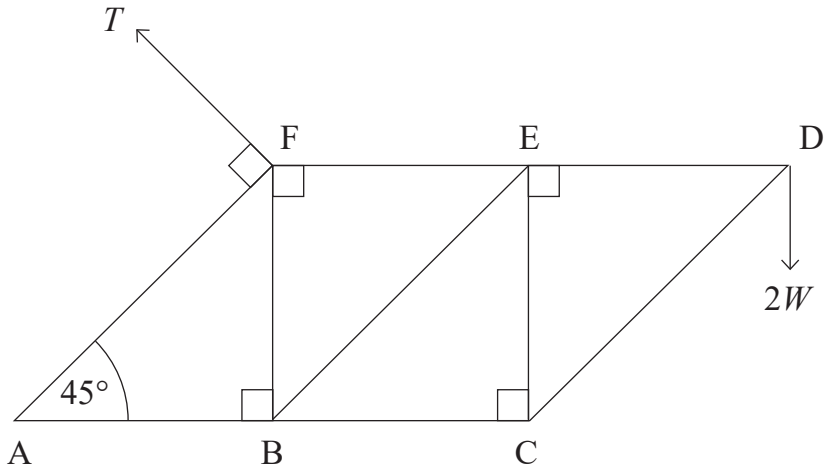


Fig. 4

The framework is freely hinged at A to a fixed point and carries a load of $2W$ newtons at the point D.

A cable, in the same plane, is attached at the point F and is perpendicular to the rod AF.

The cable holds the framework in equilibrium with ABC horizontal.

AB = BC = DE = EF = BF = CE = d metres

$\angle$FAB = 45°

(i) Find in terms of W the tension T in the cable. [5]

(ii) Find the magnitude and direction of the reaction at A. [9]

(iii) Find the forces in the rods AB, AF and BF and state which of these forces is a thrust. [7]

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SECTION B Mechanics 2

Answer all five questions in this section.

1 **Fig. 1** below shows four forces acting on a regular hexagon ABCDEF with centre O and sides of length $2l$ metres.

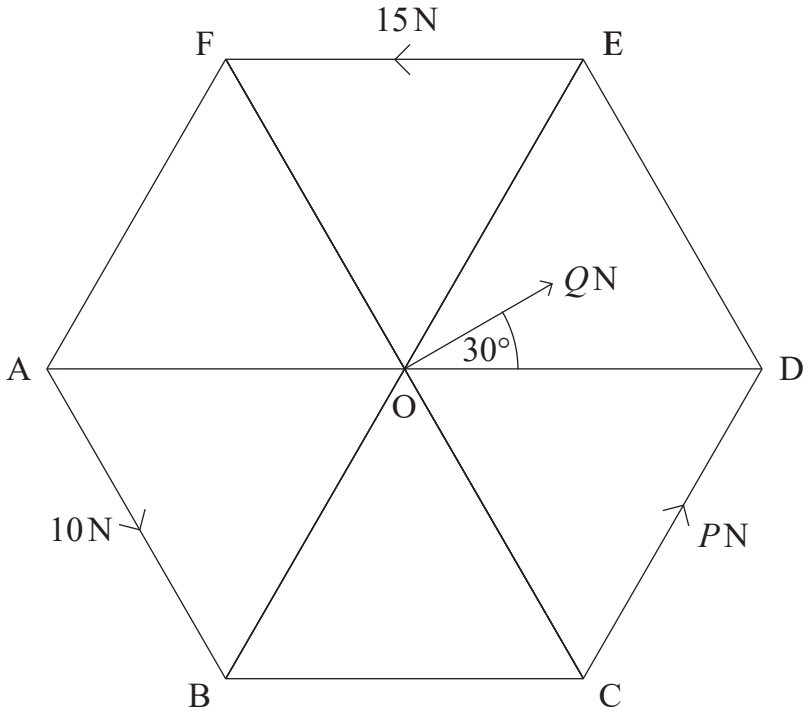


Fig. 1

The forces acting are:

10 N along AB

P N along CD

15 N along EF

Q N at 30° to OD

The system of forces reduces to a couple S.

(i) Find the values of P and Q. [8]

(ii) Find the magnitude and direction of S. [5]

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2 The acceleration of a particle P at time t seconds is given by

$$\mathbf{a} = (5 \sin t\,\mathbf{i} + 2t\mathbf{j} + 5 \cos t\,\mathbf{k})\ \text{m s}^{-2}$$

where **i**, **j** and **k** are unit vectors in the directions of the x, y and z Cartesian axes respectively.

When $t = 0$ the velocity of P is $(-5\mathbf{i} + 8\mathbf{j})\ \text{m s}^{-1}$

(i) Find an expression for the velocity of P at time t. [6]

(ii) Find the time when P has a speed of $13\ \text{m s}^{-1}$ [3]

When $t = 0$ the displacement of P from the origin O is $(-\mathbf{i} - 3\mathbf{k})$ m.

(iii) Find an expression for the displacement of P from O at time t. [6]

(iv) Show that the path of P never intersects the x axis. [4]

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3 **Fig. 2** below shows a particle A of mass m kg which is moving with constant velocity u m s^{-1} along a straight horizontal surface.



Fig. 2

Particle A moves directly towards a particle B of mass km where k is a constant.

Particle B is initially at rest.

The coefficient of restitution between A and B is e.

(i) The velocities of A and B after the impact are v_A and v_B respectively.

Show that

$$v_A = \frac{u(1 - ek)}{k + 1}$$

and find an expression for v_B [7]

B then hits a wall perpendicular to its path and rebounds.
The coefficient of restitution between B and the wall is also e.

(ii) Find the velocity of B after this impact. [2]

(iii) Given that $k = 4$ and $e = 0.5$, determine whether A and B will collide for a second time. [4]

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4 A lorry of mass 3 tonnes travels on a straight horizontal road with its engine running at a constant rate of 54 kW.

The total resistance to motion is proportional to the square of its speed, with constant of proportionality k.

The maximum speed of the lorry is $30\,\text{m}\,\text{s}^{-1}$

(i) Show that $k = 2$ [5]

(ii) Hence find the distance travelled by the lorry in accelerating from $18\,\text{m}\,\text{s}^{-1}$ to $24\,\text{m}\,\text{s}^{-1}$ [8]

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5 **Fig. 3** below shows the cross-section, through a plane of symmetry, of a garden ornament.

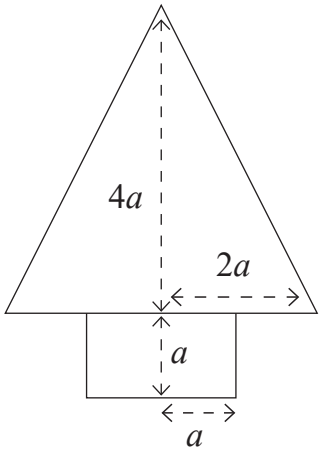


Fig. 3

The ornament consists of a solid cylinder of radius a and height a joined to a solid cone of base radius $2a$ and height $4a$. All dimensions are in metres.

The ornament is symmetrical and has uniform density ρ kg m^{-3}

(i) Show that the centre of mass of the ornament lies on its axis of symmetry at a distance of $\frac{67a}{38}$ above its base. [9]

A wooden trolley is used to transport the ornament to a new position in the garden.

Fig. 4 below shows the ornament resting on the trolley which is inclined at an angle of $\theta°$ to the horizontal.

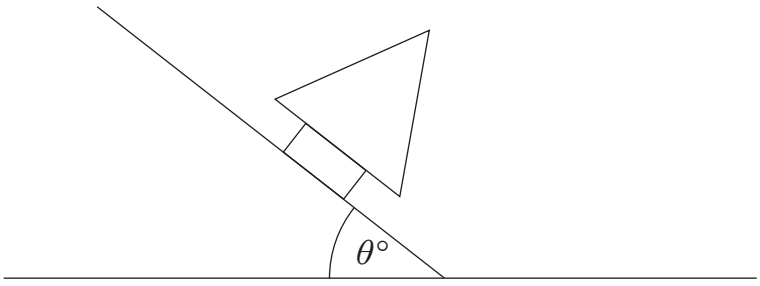


Fig. 4

The coefficient of friction between the ornament and the trolley is 0.3

(ii) If θ is slowly increased from zero show that the ornament slides before it topples. [8]

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SECTION C Statistics

Answer all five questions in this section.

1 **(a)** Explain what is meant by a "99% confidence interval for the mean". [1]

(b) A research team is investigating the microplastic contamination in one-litre bottles of water produced by the Pure Hydro manufacturing company.

A random sample of 50 bottles is selected.
The team records the number of microplastic particles x in each bottle.
The standard deviation of x for the sample was found to be 350 particles.

(i) Show that the unbiased estimator for the population variance is $S^2 = 125\,000$ [2]

The team used the results to calculate the upper confidence limit for the mean.
At the 99% significance level this was found to be 361 particles.

(ii) Find a 95% confidence interval for the mean number of microplastic particles in a one-litre bottle of water. [7]

(iii) Find the smallest sample size required to produce a 95% confidence interval of width no greater than 150 particles. [3]

(iv) State one assumption you have made in part **(ii)** and your justification for using it. [2]

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2 An upgrade to the motorway between Lisburn and Belfast has just been completed.

The government believes this will reduce commuting times between the two cities.

To test this assertion, the commuting times of 10 randomly selected commuters are recorded on a Monday morning before the upgrade is put into operation.

After the upgrade is opened, the commuting times of the same 10 commuters are again recorded on a particular Monday morning.

The results are summarised in **Table 1** below.

Table 1

Commuter	A	B	C	D	E	F	G	H	I	J
Time before upgrade (mins)	18	24	55	50	38	48	32	62	44	40
Time after upgrade (mins)	15	27	57	49	36	46	28	59	40	42

(i) Test the government's claim, at a 10% level of significance, that the motorway upgrade will reduce commuting times. [16]

(ii) State the assumption you have made regarding the distribution of the time differences. [1]

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3 An electrical company manufactures two types of power bank charging devices: Standard and Ultimate.

The charging capacity S of its Standard charger is Normally distributed with mean 3000 mAh and standard deviation 100 mAh.

(i) Find the probability that the sum of the capacities of four Standard charging devices exceeds 11 900 mAh. [6]

The charging capacity U of its Ultimate charger is Normally distributed with mean 11 900 mAh and standard deviation 150 mAh.

(ii) Find the probability that an Ultimate charger will provide at least four times the capacity of a Standard charger. [9]

(iii) State at which point in part **(ii)** you used the assumption that the variables S and U were independent. [1]

4 A random sample of 12 observations of a Normal variable X produced the following summary statistics.

$$\sum x = 516 \qquad \sum (x - \bar{x})^2 = 1342$$

It is thought that the sample was drawn from a Normal distribution with a mean of 37.273

To verify this claim a suitable hypothesis test was conducted.

It was found that the value of the test statistic was located on the boundary between the acceptance and critical regions.

Find the significance level of the test. [11]

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5 A footballer takes four practice penalty kicks at each training session during the year. The number of successful penalty kicks x at each session was recorded.

The results are shown in **Table 2** below.

Table 2

x	0	1	2	3	4
Number of sessions	3	10	12	23	2

It is suggested that a Binomial distribution, with parameter $p = 0.6$, might provide a suitable model for the data.

Using this model, the expected frequencies were calculated, to two decimal places, and are shown in **Table 3** below.

Table 3

x	0	1	2	3	4
Expected number of sessions	1.28	7.68	17.28	17.28	a

(i) Show that $a = 6.48$ [1]

(ii) Carry out a suitable test, at a 5% level of significance, to determine if these results are consistent with the assumption of a Binomial distribution with parameter $p = 0.6$ [11]

Instead of using $p = 0.6$, it is thought that the model could be improved by estimating the value of p from the observed results in **Table 2**.

(iii) Calculate this estimate for the value of p. [3]

(iv) State how using the value of p calculated in part **(iii)**, as the parameter for the Binomial distribution, would affect the number of restrictions placed upon the degrees of freedom. [1]

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SECTION D Discrete and Decision Mathematics

Answer all six questions in this section.

Note: There is a Supplementary Answer Booklet for questions **1(a)** and **2(ii)**
Please attach this to your answer booklet using the treasury tag provided.

1 **(a)** Mark a minimum vertex colouring on the separate graphs in **Fig. 1** and **Fig. 2** below using the copies in the Supplementary Answer Booklet.
In each case use the minimum number of colours necessary.
(You may denote colours by the integers 1, 2, etc.)

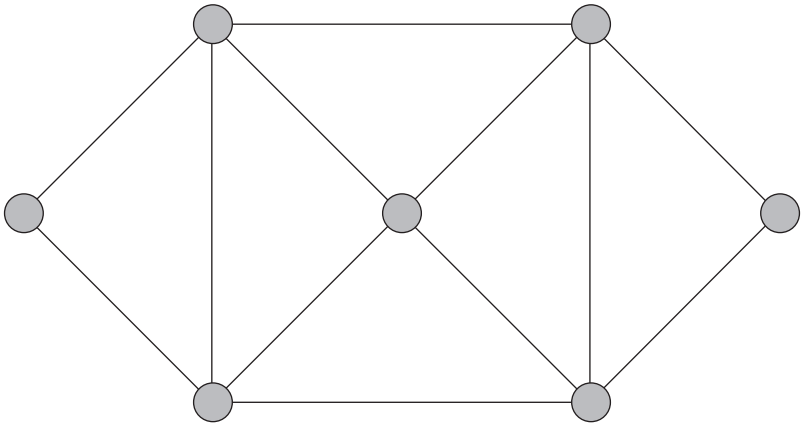


Fig. 1

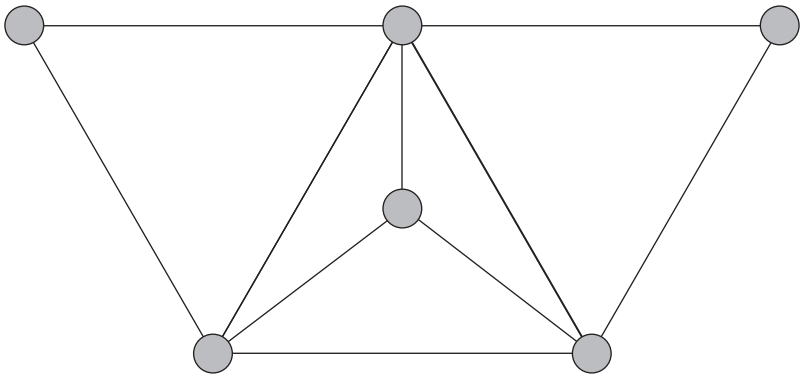


Fig. 2 [3]

(b) Find the minimum number of colours necessary to colour the vertices of $K_{5,7}$ so that no two adjacent vertices are identically coloured. [2]

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(c) A party has been organised for seven children.
Seven presents have been bought, one for each child.
Each child has expressed the list of presents they would be happy to take home.

This is illustrated in **Fig. 3** below.

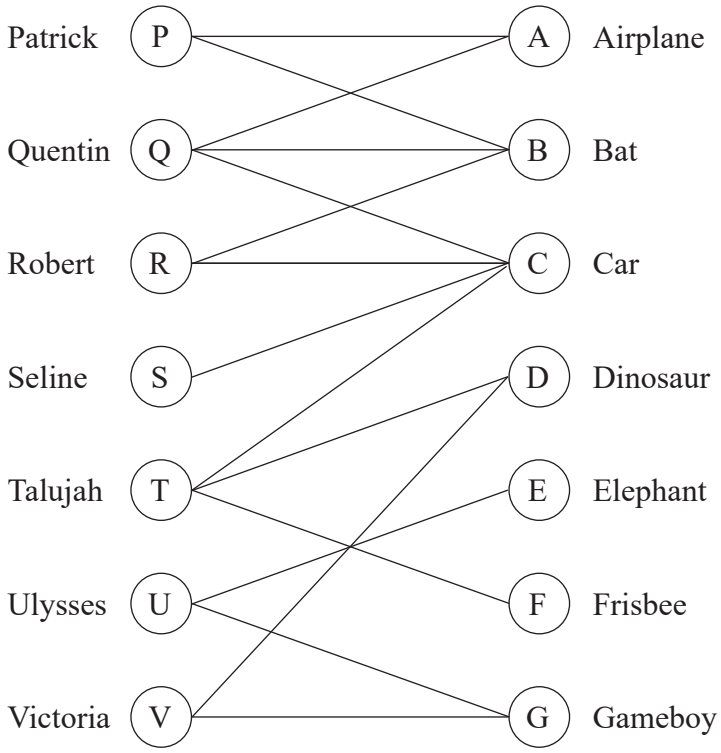


Fig. 3

(i) Using Hall's Marriage theorem, show that it is impossible to satisfy all seven children. [2]

The organiser has a bright idea and brings out a Hoverboard, with which any of the children would now be satisfied.

(ii) Write down a complete matching of the seven children to the eight presents (including the Hoverboard). [2]

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2 A precedence table is shown in **Fig. 4** below for the activities of a building project.

The numbers are the times in days for each activity to be completed.

Activity	Predecessor	Optimal time	Normal time	Pessimistic time	Expected time
A	–	2	4	9	4.5
B	A	1	4	7	4
C	A	4	8	9	7.5
D	B	7	8	9	
E	A, B, C	9	13	17	
F	C	3	7	14	
G	D, E, F	6	7	11	

Fig. 4

(i) Calculate the expected times for activities D, E, F and G. [1]

(ii) In the Supplementary Answer Booklet, complete the PERT chart for this project. [4]

(iii) List the critical path for the project. [1]

(iv) Write down the value and explain the meaning of the float time for activity D. [2]

(v) Calculate the time by which one can be 90% confident that the project will be completed. [7]

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3 A linear programming problem is stated as:

Maximise $P = 4x + 7y$

subject to the two constraints $5x + 3y \leqslant 31.2$

$2x + 5y \leqslant 21.6$

where x and y are positive.

(i) Restate the problem in terms of equalities in preparation for the simplex method. [2]

(ii) Hence, copy the grid below and fill out an initial simplex tableau for this problem.

[3]

(iii) Perform one iteration of the simplex method, carefully identifying the pivot element and rewriting the tableau. [4]

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4 **Fig. 5** below shows a board.

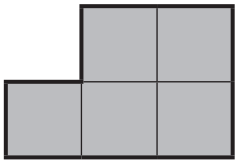


Fig. 5

(i) Show that the rook polynomial for the board above is

$$1 + 5x + 4x^2$$

[2]

Fig. 6 below shows a compound board.

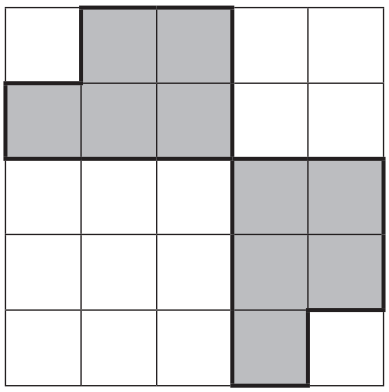


Fig. 6

(ii) Show that the rook polynomial for this compound board is

$$1 + 10x + 33x^2 + 40x^3 + 16x^4$$ [5]

(iii) A Head of Year is assigning pupils Anna, Beatrice, Ciara, Donna and Emily to five classes, subject to the following rules:

Overall	no two pupils can be in the same class
Anna	cannot be placed in classes 2 or 3
Beatrice	cannot be placed in classes 1, 2 or 3
Ciara	cannot be placed in classes 4 or 5
Donna	cannot be placed in classes 4 or 5
Emily	cannot be placed in class 4

Using part **(ii)** and the Rook Inclusion Exclusion Theorem, show that this can only be achieved in 14 different ways. [7]

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5 **(a)** The cycle index for a regular polygon with p vertices, where p is prime, is

$$P_G = \frac{1}{2p}\left(x_1^p + (p-1)x_p^1 + px_1^1 x_2^{\frac{p-1}{2}}\right)$$

(i) Write down the cycle index for the vertices of a regular 13-sided polygon. [1]

The mayor of a city commissions a ceremonial neck chain with 13 equally spaced oval discs along its length.
The chain may be worn with either side facing outwards.
Each disc may be either platinum or bronze.

(ii) How many different chains are possible? [3]

(b) The cycle index for the edge symmetries of a cube is:

$$P_G = \frac{1}{24}\left(x_1^{12} + 3x_2^6 + 6x_1^2 x_2^5 + 6x_4^3 + 8x_3^4\right)$$

(i) Calculate the pattern inventory for the colouring of the edges of a cube using two colours, purple (P) and green (G).

(You need not fully expand the expression.) [3]

A toy designer wishes to colour the edges of a cubical framework purple and green.

(ii) How many different possible colourings are there with 4 purple edges and 8 green edges? [6]

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6 **(i)** Using the factorial definition of $\binom{n}{r}$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

show that

$$\binom{n}{r} = \binom{n}{n-r}$$ [2]

(ii) Verify that

$$\binom{8}{4} = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$ [2]

(iii) Using the generating function for binomial coefficients

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

and considering the product

$$(1+x)^{2n} = (1+x)^n \times (1+x)^n$$

prove that

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0}$$ [5]

(iv) Show that this is equivalent to

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$$ [2]

(v) Using the generating function for binomial coefficients and by differentiating, prove that

$$n \times 2^{n-1} = 1 \times \binom{n}{1} + 2 \times \binom{n}{2} + \ldots + n \times \binom{n}{n}$$ [4]

THIS IS THE END OF THE QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

Section A Mechanics 1

A1 (i) By symmetry about RS, the centre of mass is 3 m from PQ.

(ii) Take moments about PT (mass of each rod = ρ × length):

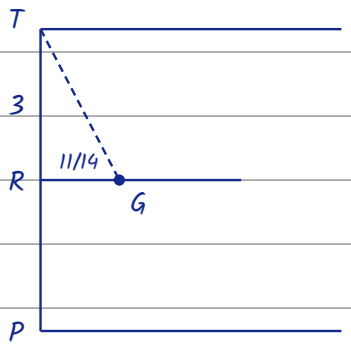
Rod	mass	distance from PT	moment
PQ	3ρ	1.5	4.5ρ
RS	2ρ	1	2ρ
TU	3ρ	1.5	4.5ρ
PT	6ρ	0	0
E	14ρ	$\bar{x}$	14ρ$\bar{x}$

$14\rho\bar{x} = 4.5\rho + 2\rho + 4.5\rho + 0 = 11\rho$

$\bar{x} = \frac{11}{14}$

Centre of mass is $\frac{11}{14}$ m (0.786 m) from PT.

(iii)



When freely suspended from T, TG is vertical.

G is 3 m below T along PT and 11/14 m from PT, so

Question number

$\tan\theta = \frac{11/14}{3} = \frac{11}{42}$

$\theta = 14.7°$ (3 s.f.), the angle between PT and the vertical.

A2 (a)(i) Critically damped: $q^2 = 4pr$

(ii) Over damped: $q^2 > 4pr$

(b)(i) Auxiliary equation: $m^2 + 6m + 10 = 0$

$m = \frac{-6 \pm \sqrt{36 - 40}}{2} = \frac{-6 \pm 2i}{2}$

$m = -3 \pm i$

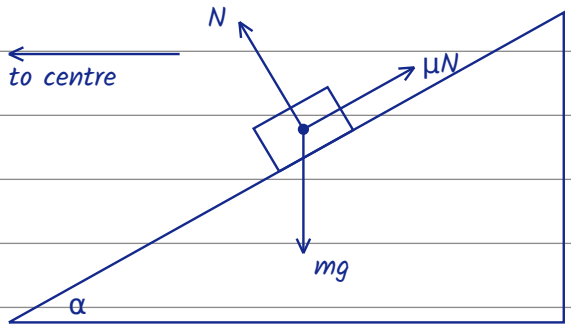
Complex roots so the general solution is

$x = e^{-3t}(A \cos t + B \sin t)$

(ii) The model predicts the pointer oscillates about 0 forever

(it never actually comes to rest at 0), which is unrealistic.

A3 (i)



About to slide down, so friction $F = \mu N$ acts up the slope.

Horizontally (towards centre): $N \sin\alpha - \mu N \cos\alpha = \frac{mu^2}{r}$

Question number

Vertically: $N\cos\alpha + \mu N\sin\alpha = mg$

Divide the first equation by the second:

$$\frac{N\sin\alpha - \mu N\cos\alpha}{N\cos\alpha + \mu N\sin\alpha} = \frac{u^2}{rg}$$

Divide top and bottom by $N\cos\alpha$:

$$\frac{\tan\alpha - \mu}{1 + \mu\tan\alpha} = \frac{u^2}{rg}$$

$$u^2 = \frac{rg(\tan\alpha - \mu)}{1 + \mu\tan\alpha} \quad \text{as required}$$

(ii) About to slide up, so friction μN now acts down the slope,

i.e. replace μ by $-\mu$ in the working above:

$$N\sin\alpha + \mu N\cos\alpha = \frac{mv^2}{r}, \quad N\cos\alpha - \mu N\sin\alpha = mg$$

$$v^2 = \frac{rg(\tan\alpha + \mu)}{1 - \mu\tan\alpha}$$

(iii) $\alpha = 45°$ so $\tan\alpha = 1$, $\mu = 0.6$

$$u^2 = \frac{rg(1 - 0.6)}{1 + 0.6} = \frac{0.4rg}{1.6} = \frac{1}{4}rg$$

$$v^2 = \frac{rg(1 + 0.6)}{1 - 0.6} = \frac{1.6rg}{0.4} = 4rg$$

$v^2 = 16 \times (\frac{1}{4}rg) = 16u^2$

$v = 4u$ as required

A4 (i) Hooke: $T = \frac{\lambda x}{l} = \frac{32x}{2} = 16x$ (towards A)

Newton II along AP: $4\ddot{x} = -16x$

Question number

$\ddot{x} = -4x$

This is of the form $\ddot{x} = -\omega^2 x$, so the motion is SHM about the point where the spring has its natural length, with $\omega = 2$.

(ii) At B the extension is $3 - 2 = 1$ m and P is at rest, so the amplitude is $a = 1$ m.

At C, $AC = 3 - 0.4 = 2.6$ m so $x = 0.6$ m.

$v^2 = \omega^2(a^2 - x^2) = 4(1^2 - 0.6^2) = 4 \times 0.64 = 2.56$

$v = 1.6 \text{ m s}^{-1}$

(iii) Take $t = 0$ at B: $x = a \cos \omega t = \cos 2t$

At C: $0.6 = \cos 2t \Rightarrow t = \frac{1}{2} \cos^{-1}(0.6) = 0.4636$ s

$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ s

P goes from C through B and back to C in 2×0.4636 s, so the time from C round to C again (via the other side) is

$\text{time} = \pi - 2 \times \frac{1}{2} \cos^{-1}(0.6) = \pi - 0.9273$

time = 2.21 s (3 s.f.)

AS (i) AF = $d\sqrt{2}$ (diagonal of a square of side d).

Take moments about A (T is perpendicular to AF):

$T \times d\sqrt{2} = 2W \times 3d$

$T = \frac{6W}{\sqrt{2}} = 3\sqrt{2}W$ N

(ii) Let the reaction at A have components X (→) and Y (↑).

T acts at 45° to the horizontal, up and to the left.

(↑): $Y + T \cos 45° = 2W \Rightarrow Y + 3W = 2W \Rightarrow Y = -W$

Question number

$(\rightarrow)$: $X - T\cos 45° = 0 \Rightarrow X = 3W$

$R = \sqrt{((3W)^2 + W^2)} = \sqrt{10}\,W = 3.16W$ N

$\tan\theta = \frac{W}{3W} = \frac{1}{3} \Rightarrow \theta = 18.4°$

Reaction at A is $\sqrt{10}\,W$ N at 18.4° below the horizontal,

acting in the direction from A towards B (to the right).

(iii) Take all rod forces as tensions.

At A: forces $3W$ $(\rightarrow)$, W $(\downarrow)$, T_{AF} along AF, T_{AB} along AB

$(\uparrow)$: $T_{AF}\cos 45° - W = 0 \Rightarrow T_{AF} = \sqrt{2}\,W$

$(\rightarrow)$: $3W + T_{AB} + T_{AF}\cos 45° = 0$

$3W + T_{AB} + W = 0 \Rightarrow T_{AB} = -4W$

At F $(\uparrow)$: $T\cos 45° - T_{AF}\cos 45° - T_{BF} = 0$

$3W - W - T_{BF} = 0 \Rightarrow T_{BF} = 2W$

AB: $4W$ N thrust, AF: $\sqrt{2}\,W$ N tension, BF: $2W$ N tension

So the force in AB is a thrust.

Section B Mechanics 2

B1 (i) AB and CD each make 60° with AD (horizontal).

Directions: AB down-right at 60°, CD up-right at 60°,

EF to the left, Q at 30° above OD.

Reduces to a couple, so the resultant force is zero.

$(\uparrow)$: $P\sin 60° + Q\sin 30° - 10\sin 60° = 0$

$\Rightarrow \sqrt{3}P + Q = 10\sqrt{3}$ ... (1)

$(\rightarrow)$: $10\cos 60° + P\cos 60° + Q\cos 30° - 15 = 0$

$\Rightarrow P + \sqrt{3}Q = 20$ ... (2)

Question number	
	(1) × √3: $3P + \sqrt{3}Q = 30$
	subtract (2): $2P = 10 \Rightarrow P = 5$
	$Q = 10\sqrt{3} - 5\sqrt{3} = 5\sqrt{3}$ $(= 8.66)$
(ii)	Perpendicular distance from O to each side $= 2l \sin 60° = \sqrt{3}\, l$
	Q passes through O so has no moment about O.
	Moments about O (anticlockwise):
	$S = 10(\sqrt{3}\, l) + 5(\sqrt{3}\, l) + 15(\sqrt{3}\, l) = 30\sqrt{3}\, l$
	$S = 30\sqrt{3}\, l$ N m $(= 52.01$ N m), anticlockwise
B2 (i)	$v = \int (5 \sin t\, i + 2t\, j + 5 \cos t\, k)\, dt$
	$v = -5 \cos t\, i + t^2 j + 5 \sin t\, k + c$
	$t = 0$: $-5i + 8j = -5i + c \Rightarrow c = 8j$
	$v = -5 \cos t\, i + (t^2 + 8) j + 5 \sin t\, k$
(ii)	Speed$^2 = 25 \cos^2 t + (t^2 + 8)^2 + 25 \sin^2 t = 169$
	$25 + (t^2 + 8)^2 = 169$
	$(t^2 + 8)^2 = 144 \Rightarrow t^2 + 8 = 12$ $(t^2 + 8 > 0)$
	$t^2 = 4 \Rightarrow t = 2$ s
(iii)	$r = \int v\, dt = -5 \sin t\, i + (\frac{1}{3}t^3 + 8t) j - 5 \cos t\, k + d$
	$t = 0$: $-i - 3k = -5k + d \Rightarrow d = -i + 2k$
	$r = (-5 \sin t - 1) i + (\frac{1}{3}t^3 + 8t) j + (2 - 5 \cos t) k$
(iv)	To meet the x axis, the j and k components must both be zero
	at the same time.
	j: $\frac{1}{3}t^3 + 8t = 0 \Rightarrow t(\frac{1}{3}t^2 + 8) = 0$
	$\frac{1}{3}t^2 + 8 > 0$ for all t, so $t = 0$ only.

Question number

k at $t = 0$: $2 - 5\cos 0 = -3 \neq 0$

So the j and k components are never zero together,

hence the path of P never intersects the x axis.

B3 (i) Conservation of momentum: $mu = mv_A + kmv_B$

$\Rightarrow u = v_A + kv_B$ ... (1)

Newton's law of restitution: $v_B - v_A = eu$ ... (2)

$k \times$ (2): $kv_B - kv_A = keu$

(1) − this: $u - keu = v_A + kv_A$

$v_A = \frac{u(1 - ek)}{k + 1}$ as required

(1) + (2): $u + eu = (k + 1)v_B$

$v_B = \frac{u(1 + e)}{k + 1}$

(ii) B rebounds with speed $e \times \frac{u(1 + e)}{k + 1}$, so its velocity is

$v = -\frac{eu(1 + e)}{k + 1}$ (i.e. away from the wall)

(iii) $k = 4$, $e = 0.5$: $v_A = \frac{u(1 - 2)}{5} = -0.2u$

B after wall: $-\frac{0.5u(1.5)}{5} = -0.15u$

Both A and B now move away from the wall in the same

direction. A moves at 0.2u, faster than B at 0.15u,

so B can never catch up with A.

A and B will not collide a second time.

Question number

B4 (i) Driving force $= \frac{P}{v} = \frac{54000}{v}$

Newton II: $\frac{54000}{v} - kv^2 = 3000a$

At maximum speed $a = 0$ and $v = 30$:

$\frac{54000}{30} - k(30)^2 = 0$

$1800 = 900k \Rightarrow k = 2$ as required

(ii) $\frac{54000}{v} - 2v^2 = 3000v\frac{dv}{ds}$

$\frac{54000 - 2v^3}{v} = 3000v\frac{dv}{ds}$

$\int_0^x ds = \int_{18}^{24} \frac{3000v^2}{54000 - 2v^3}\, dv$

$x = [-500 \ln(54000 - 2v^3)]$ from 18 to 24

$x = 500 \ln(54000 - 2(18)^3) - 500 \ln(54000 - 2(24)^3)$

$x = 500 \ln\frac{42336}{26352}$

$x = 237$ m (3 s.f.)

B5 (i) Measure distances from the base along the axis.

Part	mass	CM from base	moment
Cylinder	$\pi a^3\rho$	$\frac{1}{2}a$	$\frac{1}{2}\pi a^4\rho$
Cone	$(16/3)\pi a^3\rho$	$a + \frac{1}{4}(4a) = 2a$	$(32/3)\pi a^4\rho$
Total	$(19/3)\pi a^3\rho$	$\bar{y}$	$(19/3)\pi a^3\rho\bar{y}$

Question number

Cone mass $= \frac{1}{3}\pi(2a)^2(4a)\rho = \frac{16}{3}\pi a^3\rho$

$\frac{19}{3}\pi a^3\rho\bar{y} = \frac{1}{2}\pi a^4\rho + \frac{32}{3}\pi a^4\rho = \frac{67}{6}\pi a^4\rho$

$\bar{y} = \frac{67}{6} \times \frac{3}{19}a = \frac{67a}{38}$

By symmetry the CM lies on the axis, 67a/38 above the base.

(ii) Sliding: on the point of slipping at angle θ_1

$N = W\cos\theta_1$, $F = W\sin\theta_1$, $F = 0.3N$

$W\sin\theta_1 = 0.3W\cos\theta_1 \Rightarrow \tan\theta_1 = 0.3$

$\theta_1 = 16.7°$

Toppling: about to topple about the lowest point of the base

when G is vertically above it. Base radius is a, so

$\tan\theta_2 = \frac{a}{\bar{y}} = \frac{38}{67}$

$\theta_2 = 29.6°$

$16.7° < 29.6°$, so as θ increases the ornament slides

before it topples, as required.

Section C Statistics

C1 (a) If many samples were taken and a 99% interval found from

each, 99% of these intervals would contain the population

mean μ.

Question number

(b)(i) $S^2 = \frac{n}{n-1}s^2 = \frac{50}{49} \times 350^2 = 125000$ as required

(ii) S.E. $= \sqrt{(125000 \div 50)} = \sqrt{2500} = 50$

99% upper limit: $\bar{x} + 2.576 \times 50 = 361$

$\bar{x} = 361 - 128.8 = 232.2$

95% CI: $232.2 \pm 1.96 \times 50 = 232.2 \pm 98$

95% CI = (134, 330) particles (3 s.f.)

(iii) Width $= 2 \times 1.96 \times \sqrt{(125000 \div n)} \leq 150$

$\sqrt{n} \geq 3.92 \times \sqrt{125000} \div 150 = 9.24$

$n \geq 85.4$

Smallest sample size is 86 bottles.

(iv) Assumption: the sample mean $\bar{X}$ is (approximately) Normal.

Justification: $n = 50$ is large (> 30), so by the Central

Limit Theorem $\bar{X}$ is approximately Normally distributed.

C2 (i) Paired data. Let D = time before − time after.

H_0: $\mu_D = 0$, H_1: $\mu_D > 0$ (upgrade reduces times)

Small sample, σ unknown: one-tailed paired t-test, $\nu = 9$

	A	B	C	D	E	F	G	H	I	J
d	3	−3	−2	1	2	2	4	3	4	−2
d^2	9	9	4	1	4	4	16	9	16	4

$\Sigma d = 12$, $\Sigma d^2 = 76$, $\bar{d} = \frac{12}{10} = 1.2$

$s_d^2 = \frac{1}{9}\left(76 - \frac{12^2}{10}\right) = \frac{61.6}{9} = \frac{308}{45} = 6.844$

Question number

$t = (\bar{d} - 0) \div \sqrt{(s_d^2 \div n)} = 1.2 \div \sqrt{(6.844 \div 10)} = 1.450$

Critical value t_9(10%, one tail) = 1.383

1.450 > 1.383, so reject H_0.

There is sufficient evidence at the 10% level to support the claim that the upgrade reduces commuting times.

(ii) The time differences D are Normally distributed.

C3 (i) Let $V = S_1 + S_2 + S_3 + S_4$ (independent observations of S)

$E(V) = 4 \times 3000 = 12000$

$Var(V) = 4 \times 100^2 = 40000$, so $V \sim N(12000, 200^2)$

$$P(V > 11900) = P\left(Z > \frac{11900 - 12000}{200}\right) = P(Z > -0.5)$$

$= \Phi(0.5) = 0.691$

(ii) Let $W = U - 4S$. Need $P(U \geq 4S) = P(W \geq 0)$.

$E(W) = 11900 - 4 \times 3000 = -100$

$Var(W) = Var(U) + 4^2 Var(S) = 150^2 + 16 \times 100^2 = 182500$

$W \sim N(-100, 182500)$

$$P(W \geq 0) = P\left(Z \geq \frac{0 - (-100)}{\sqrt{182500}}\right) = P(Z \geq 0.234)$$

$= 1 - 0.5925 = 0.407$

(iii) When finding $Var(W) = Var(U) + 16Var(S)$: adding the variances is only valid because S and U are independent.

C4 H_0: $\mu = 37.273$, H_1: $\mu \neq 37.273$ (two-tailed)

Small sample from a Normal population, σ unknown: t-test, $\nu = 11$

Question number

$\bar{x} = \frac{516}{12} = 43$

$s^2 = \frac{\Sigma(x - \bar{x})^2}{n - 1} = \frac{1342}{11} = 122$

$t = (\bar{x} - \mu) \div \sqrt{(s^2 \div n)} = (43 - 37.273) \div \sqrt{(122 \div 12)}$

$t = 5.727 \div 3.1885 = 1.796$

The test statistic lies on the boundary, so the critical value is $t_{11} = 1.796$.

From tables, $t_{11} = 1.796$ cuts off 5% in the upper tail.

Two-tailed test, so 5% in each tail:

significance level = 2 × 5% = 10%

C5 (i)

Total sessions = 3 + 10 + 12 + 23 + 2 = 50

$a = 50 \times P(X = 4) = 50 \times 0.6^4 = 50 \times 0.1296 = 6.48$

(or $a = 50 - 43.52 = 6.48$)

(ii)

H_0: $X \sim B(4, 0.6)$ is a suitable model

H_1: $X \sim B(4, 0.6)$ is not a suitable model

$E(x = 0) = 1.28 < 5$, so combine $x = 0$ and $x = 1$.

x	O	E	$(O - E)^2/E$
0 or 1	13	8.96	1.8216
2	12	17.28	1.6133
3	23	17.28	1.8934
4	2	6.48	3.0973

$\chi^2 = \Sigma(O - E)^2/E = 8.43$

$\nu = 4 - 1 = 3$ (4 classes, total fixed)

Critical value $\chi^2_3(5\%) = 7.815$

$8.43 > 7.815$, so reject H_0.

Question number

There is sufficient evidence at the 5% level that the results are not consistent with a B(4, 0.6) distribution.

(iii) $\Sigma fx = 0 + 10 + 24 + 69 + 8 = 111$

$\bar{x} = \frac{111}{50} = 2.22$

$np = \bar{x} \Rightarrow p = \frac{2.22}{4} = 0.555$

(iv) Estimating p from the data adds one more restriction, so the number of restrictions increases by 1 (ν decreases by 1).

Section D Discrete and Decision Mathematics

D1 (a) See the colourings marked on Fig. 1 and Fig. 2 in the Supplementary Answer Booklet.

Fig. 1 contains triangles so needs at least 3 colours: 3 used.

Fig. 2 contains a K_4 (the centre vertex, the top middle vertex and the two bottom vertices), so needs 4 colours: 4 used.

(b) $K_{5,7}$ is bipartite: each of the 5 vertices is joined to each of the 7, but no two vertices in the same set are joined.

Colour the set of 5 with colour 1 and the set of 7 with colour 2.

Minimum number of colours = 2

(c)(i) Hall: a complete matching needs every set of k children to be happy with at least k presents between them.

Question number

Patrick, Quentin, Robert and Seline (4 children) only want presents from {A, B, C}, which is only 3 presents.

$3 < 4$, so Hall's condition fails: not all seven can be satisfied.

(ii) Patrick – Airplane, Quentin – Bat, Robert – Car, Seline – Hoverboard, Talujah – Frisbee, Ulysses – Elephant, Victoria – Gameboy

D2 (i) Expected time $= \frac{a + 4m + b}{6}$

D: $\frac{7 + 32 + 9}{6} = 8$ E: $\frac{9 + 52 + 17}{6} = 13$

F: $\frac{3 + 28 + 14}{6} = 7.5$ G: $\frac{6 + 28 + 11}{6} = 7.5$

(ii) See the completed PERT chart in the Supplementary Answer Booklet.

(iii) Critical path: A → C → E → G (32.5 days)

(iv) Float for D = 17 – 8.5 = 8.5 days.

Activity D can start up to 8.5 days after its earliest start time (or overrun by 8.5 days) without delaying the project.

(v) Expected project time = 4.5 + 7.5 + 13 + 7.5 = 32.5 days

Variance of each activity = $(b - a)^2 \div 36$

Critical path: A: $\frac{49}{36}$, C: $\frac{25}{36}$, E: $\frac{64}{36}$, G: $\frac{25}{36}$

Question number

Project variance $= \frac{163}{36} = 4.528$

$T \sim N(32.5, 4.528)$

$P(T < x) = 0.9 \Rightarrow \frac{x - 32.5}{\sqrt{4.528}} = 1.2816$

$x = 32.5 + 1.2816 \times 2.128 = 35.2$ days

D3 (i) Introduce slack variables $s, t \geq 0$:

Maximise P where $P - 4x - 7y = 0$

subject to $5x + 3y + s = 31.2$

$2x + 5y + t = 21.6$

with $x, y, s, t \geq 0$

(ii)

	x	y	s	t	P	Value
s	5	3	1	0	0	31.2
t	2	5	0	1	0	21.6
P	−4	−7	0	0	1	0

(iii) Pivot column: y (most negative entry, −7, in the P row)

Ratios: $31.2 \div 3 = 10.4$, $21.6 \div 5 = 4.32$ (smallest)

Pivot element: 5 in the t row, y column

Divide pivot row by 5: 0.4, 1, 0, 0.2, 0, 4.32

R1 − 3 × new R2, R3 + 7 × new R2:

	x	y	s	t	P	Value
s	3.8	0	1	−0.6	0	18.24
y	0.4	1	0	0.2	0	4.32
P	−1.2	0	0	1.4	1	30.24

Question number

D4 (i)

0 rooks: 1 way

1 rook: 5 ways (5 squares)

2 rooks: each top square pairs with the 2 bottom squares not in its column: 2 + 2 = 4 ways

3 rooks: impossible (only 2 rows)

Rook polynomial = $1 + 5x + 4x^2$ as required

(ii)

The two shaded parts of Fig. 6 share no rows and no columns, so they are non-interfering boards.

The lower part (2 + 2 + 1 squares in 3 rows, 2 columns) is the board of Fig. 5 reflected, so its polynomial is $1 + 5x + 4x^2$.

Rook polynomial = $(1 + 5x + 4x^2)(1 + 5x + 4x^2)$

$= 1 + 5x + 4x^2 + 5x + 25x^2 + 20x^3 + 4x^2 + 20x^3 + 16x^4$

$= 1 + 10x + 33x^2 + 40x^3 + 16x^4$ as required

(iii)

Rows A, B, C, D, E (pupils), columns 1 to 5 (classes).

Forbidden squares: A2, A3, B1, B2, B3, C4, C5, D4, D5, E4.

These form exactly the compound board of Fig. 6, so

$r_1 = 10$, $r_2 = 33$, $r_3 = 40$, $r_4 = 16$, $r_5 = 0$.

Rook inclusion exclusion theorem:

ways = $5! - r_1\,4! + r_2\,3! - r_3\,2! + r_4\,1! - r_5\,0!$

$= 120 - 10(24) + 33(6) - 40(2) + 16(1) - 0$

$= 120 - 240 + 198 - 80 + 16$

$= 14$ ways as required

D5 (a)(i)

$p = 13$: $P_G = \frac{1}{26}(x_1^{13} + 12x_{13} + 13x_1x_2^6)$

Question number

(ii) Two colours, so replace every x_i by 2:

$$\frac{1}{26}(2^{13} + 12 \times 2 + 13 \times 2 \times 2^6)$$

$$= \frac{1}{26}(8192 + 24 + 1664) = \frac{9880}{26}$$

$= 380$ different chains

(b)(i) Replace x_i by $P^i + G^i$:

$$PI = \frac{1}{24}[(P+G)^{12} + 3(P^2+G^2)^6 + 6(P+G)^2(P^2+G^2)^5 + 6(P^4+G^4)^3 + 8(P^3+G^3)^4]$$

(ii) Coefficient of P^4G^8 in each term:

$(P+G)^{12}$: ${}^{12}C_4 = 495$

$3(P^2+G^2)^6$: $3 \times {}^6C_2 = 3 \times 15 = 45$

$6(P^2+2PG+G^2)(P^2+G^2)^5$:

$P^2 \times (5P^2G^8) + G^2 \times (10P^4G^6)$ gives $6(5+10) = 90$

$6(P^4+G^4)^3$: $6 \times {}^3C_1 = 18$

$8(P^3+G^3)^4$: powers of P are multiples of 3, so 0

Number of colourings $= \frac{1}{24}(495 + 45 + 90 + 18 + 0)$

$= \frac{648}{24} = 27$

D6 (i) Here ${}^nC_r$ means the binomial coefficient (n over r).

Using the definition with r replaced by $n - r$:

$$RHS = \frac{n!}{(n-r)!(n-(n-r))!}$$

Question number

$= \dfrac{n!}{(n-r)!r!}$

$= \dfrac{n!}{r!(n-r)!}$

$= LHS$ as required

(ii)

$LHS: {}^{8}C_{4} = \dfrac{8!}{4!4!} = 70$

$RHS: 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70$

$LHS = RHS$, verified.

(iii)

Equate coefficients of x^n on both sides of

$(1 + x)^{2n} = (1 + x)^n \times (1 + x)^n$

LHS: coefficient of x^n in $(1 + x)^{2n}$ is ${}^{2n}C_n$

RHS: $[{}^nC_0 + {}^nC_1x + \ldots + {}^nC_nx^n] \times [{}^nC_0 + {}^nC_1x + \ldots + {}^nC_nx^n]$

An x^n term comes from x^r in the first bracket times x^{n-r}

in the second, for $r = 0, 1, \ldots, n$, so the coefficient is

${}^nC_0{}^nC_n + {}^nC_1{}^nC_{n-1} + \ldots + {}^nC_n{}^nC_0$

Hence ${}^{2n}C_n = {}^nC_0{}^nC_n + {}^nC_1{}^nC_{n-1} + \ldots + {}^nC_n{}^nC_0$

(iv)

By part (i), ${}^nC_{n-r} = {}^nC_r$, so each product

${}^nC_r \times {}^nC_{n-r} = ({}^nC_r)^2$

$\Rightarrow {}^{2n}C_n = ({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_n)^2$

(v)

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$

Differentiate both sides with respect to x:

$n(1 + x)^{n-1} = 1 \times {}^nC_1 + 2 \times {}^nC_2x + \ldots + n \times {}^nC_nx^{n-1}$

Put $x = 1$:

Question number

$n \times 2^{n-1} = 1 \times {}^{n}C_1 + 2 \times {}^{n}C_2 + \ldots + n \times {}^{n}C_n$

as required.

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Further Mathematics *StemUp Model Solution*

Assessment Unit A2 2
assessing
Applied Mathematics

[AFM21]
TUESDAY 7 JUNE, AFTERNOON

SECTION D

SUPPLEMENTARY ANSWER BOOKLET

stemup.co.uk

1 **(a)** Mark a minimum vertex colouring on the separate graphs in **Fig. 1** and **Fig. 2** below. In each case use the minimum number of colours necessary.
(You may denote colours by the integers 1, 2, etc.)

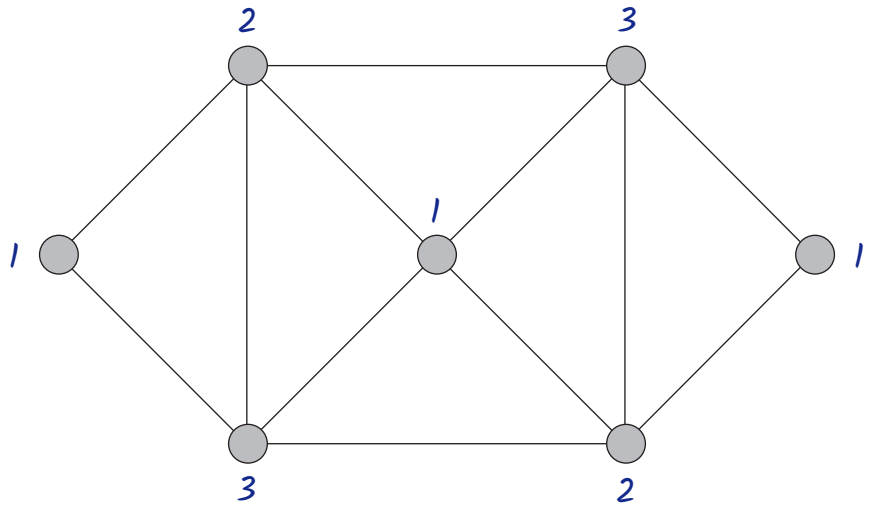


Fig. 1

3 colours needed

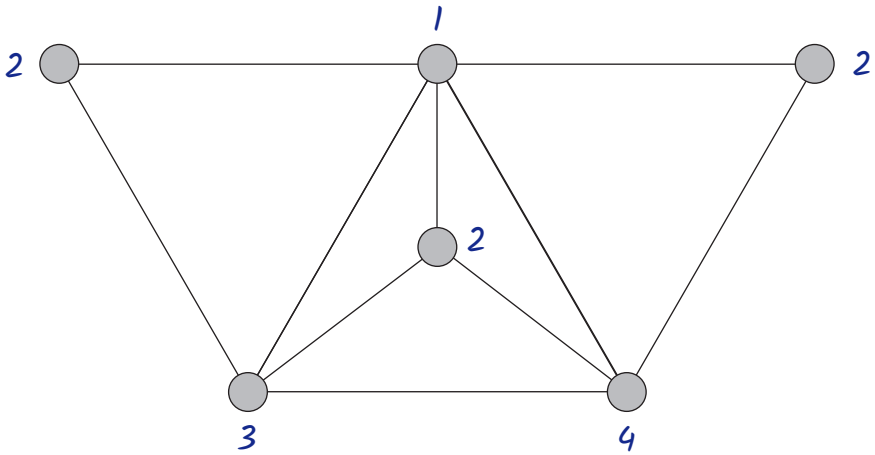
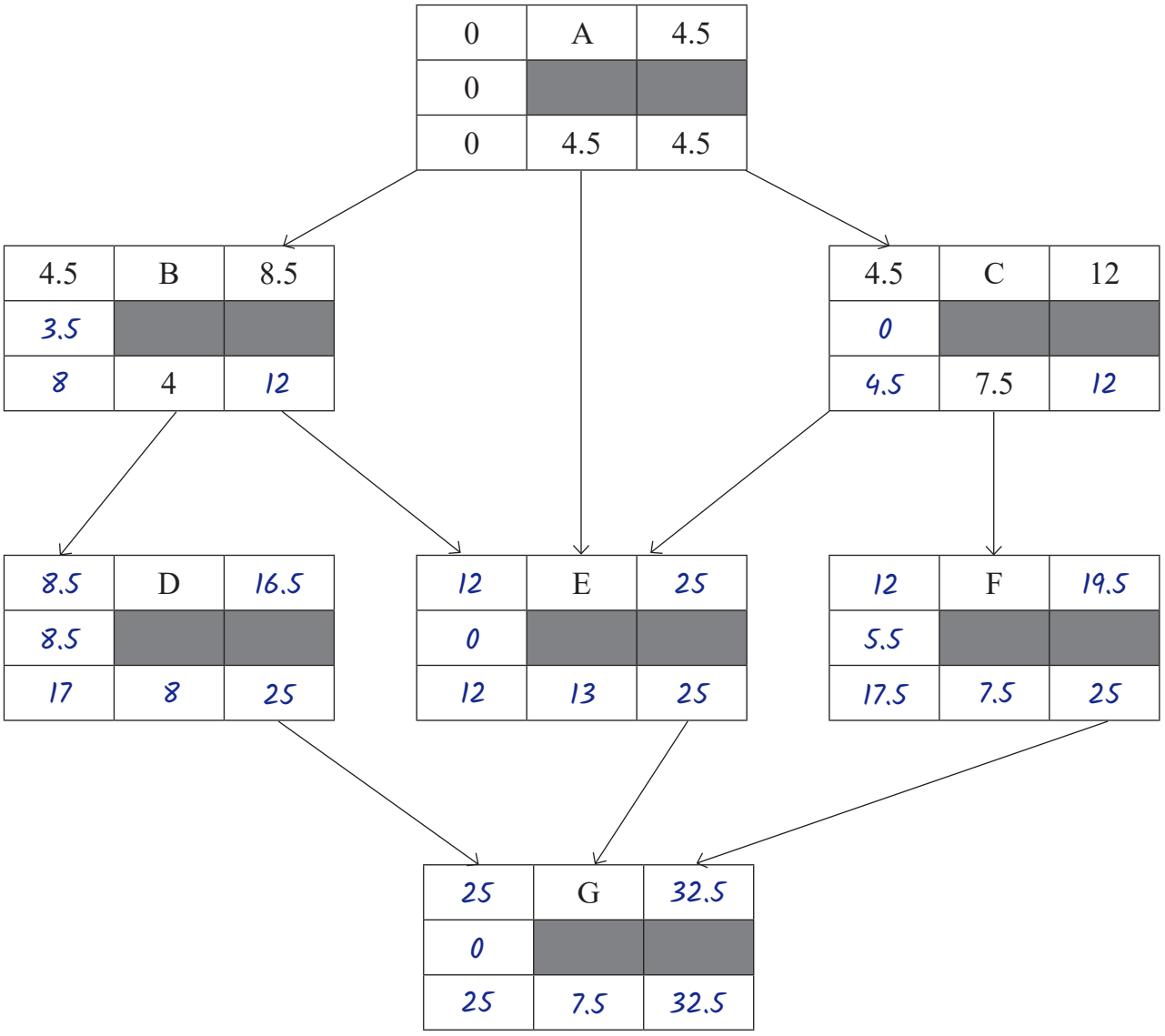


Fig. 2

4 colours needed

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2 (ii) Complete the PERT chart for this building project.



Key:

ES	Node	EF
Float		
LS	Expected time	LF

ES = Early start time
LS = Late start time
EF = Early finish time
LF = Late finish time

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