Rewarding Learning

ADVANCED
General Certificate of Education
2022

Further Mathematics

StemUp Model Solution

Assessment Unit A2 1
assessing
Pure Mathematics

AFM11

[AFM11]
WEDNESDAY 25 MAY, MORNING

TIME

2 hours 15 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided.
Answer **all twelve** questions.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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Answer all twelve questions.

Show clearly the full development of your answers.

Answers should be given to three significant figures unless otherwise stated.

1 **(i)** Find

$$\sum_{r=1}^{n} r^2 (r+3)$$

in terms of n. [5]

(ii) Hence find the sum of the series

$$5^2 \times 8 + 6^2 \times 9 + \cdots + 40^2 \times 43$$ [3]

2 Using the principle of mathematical induction, prove that for all integers $n \geqslant 1$

$$5^{2n} - 1 \quad \text{is divisible by 8}$$ [7]

3 **(a)** Find

$$\int e^x \cos x \, dx$$ [8]

(b) Evaluate

$$\int_5^{\infty} \frac{1}{(x-3)^2} \, dx$$ [4]

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4 The graph of $r = 4 \sin 2\theta$ is shown in **Fig. 1** below.

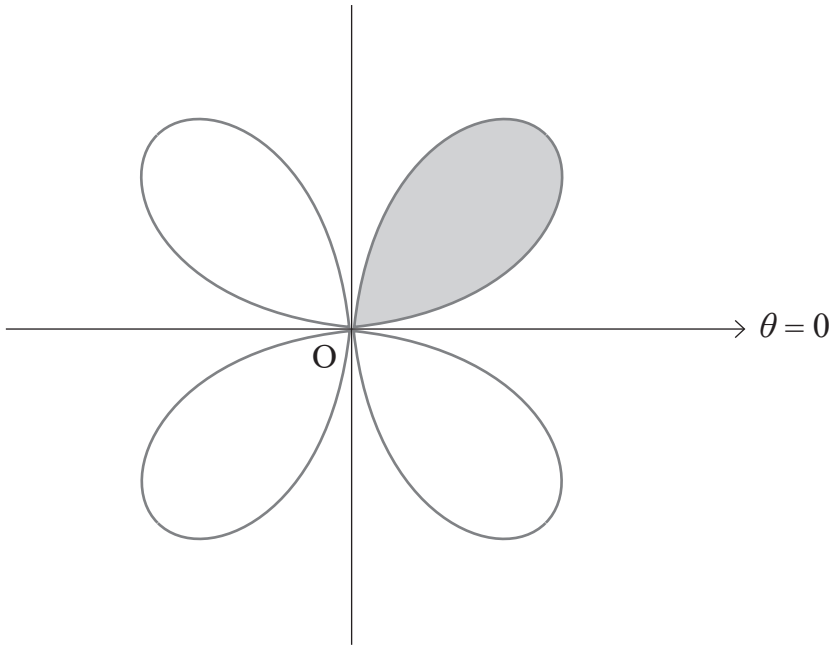


Fig. 1

Find the area of the shaded region. [8]

5 **(i)** Express

$$\frac{3r-1}{r^3-r}$$

in partial fractions. [6]

(ii) Hence, or otherwise, prove that

$$\sum_{r=2}^{n} \frac{3r-1}{r^3-r} = 2 - \frac{1}{n} - \frac{2}{n+1} \quad \text{for } n \geqslant 2$$ [5]

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6 **(i)** Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}x + \frac{x}{1+x^2}\right] = \frac{2}{(1+x^2)^2}$$ [5]

(ii) Hence find the exact value of

$$\int_3^4 \frac{1}{(x^2 - 6x + 10)^2}\,\mathrm{d}x$$ [6]

7 Solve the differential equation

$$(x^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy = 8x \quad \text{where } |x| > 1$$

given that $y = 5$ when $x = 2$ [10]

8 **(i)** If

$$\mathrm{f}(x) = (1 + \cos x)\,\mathrm{e}^x$$

show that

$$\mathrm{f}''(x) = (1 - 2\sin x)\,\mathrm{e}^x$$ [4]

(ii) Use Maclaurin's theorem to derive the series expansion for

$$\mathrm{f}(x) = (1 + \cos x)\,\mathrm{e}^x$$

up to and including the term in x^4 [7]

(iii) Hence, or otherwise, find the series expansion for

$$(1+ \cos x)\,\mathrm{e}^{-x}$$

up to and including the term in x^4 [4]

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9 **(i)** Given that

$$I_n = \int \sec^n x \, dx$$

show that for $n \geqslant 2$

$$(n-1)I_n = \sec^{n-2}x \tan x + (n-2)I_{n-2}$$ [8]

(ii) A vase can be modelled by rotating the curve

$$y = \sec^2 x$$

through 2π radians about the x-axis, between the ordinates $x = 0$ and $x = \frac{\pi}{3}$

Using part **(i)** find the volume of the vase. [6]

10 A particle moves so that at time t seconds its displacement x metres, from the origin O, is given by the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 52 \cos 2t$$

(i) Find the general solution of the differential equation. [13]

It is known that the particle's displacement remains finite as $t \to \infty$ and $x = 1$ when $t = 0$

(ii) Find the displacement x of the particle as a function of t. [3]

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11 **(a)** Find the equation of the tangent to the curve

$$y = \sinh^{-1}(2x - 1) + 4x$$

at the point where $x = \frac{1}{2}$ [7]

(b) **(i)** Show that

$$\sinh^{-1}x \equiv \ln\left(x + \sqrt{x^2 + 1}\right)$$ [5]

(ii) Hence find the exact solution of the equation

$$\sinh^{-1}x + \sinh^{-1}\left(\frac{4}{3}\right) = 2\sinh^{-1}\left(\frac{3}{4}\right)$$ [7]

12 **(a)** Find, in the form $re^{i\theta}$, the values of the 6 roots of the equation

$$z^6 + 64 = 0$$

and plot them on a carefully labelled Argand diagram. [8]

(b) Let $z = \cos\theta + i\sin\theta$

(i) Show that $z + \frac{1}{z} = 2\cos\theta$ [3]

(ii) Hence find an expression for $\cos^4\theta$ in the form

$$p\cos 4\theta + q\cos 2\theta + r$$ [6]

(iii) Hence find

$$\int \cos^4\theta \, d\theta$$ [2]

THIS IS THE END OF THE QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (i) $\sum_{r=1}^{n} r^2(r+3) = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2$

$= \frac{1}{4}n^2(n+1)^2 + 3 \times \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{4}n(n+1)[n(n+1) + 2(2n+1)]$

$= \frac{1}{4}n(n+1)(n^2+5n+2)$

(ii) $5^2 \times 8 + 6^2 \times 9 + \ldots + 40^2 \times 43 = \sum_{r=5}^{40} r^2(r+3)$

$= \sum_{r=1}^{40} r^2(r+3) - \sum_{r=1}^{4} r^2(r+3)$

$n = 40$: $\frac{1}{4}(40)(41)(1600+200+2) = 738820$

$n = 4$: $\frac{1}{4}(4)(5)(16+20+2) = 190$

Sum $= 738820 - 190 = 738630$

2 Let P(n) be the statement: $5^{2n} - 1$ is divisible by 8.

Basis: $n = 1$: $5^2 - 1 = 24 = 8 \times 3$, divisible by 8.

So P(1) is true.

Assume P(k) is true for some integer $k \geq 1$, i.e.

$5^{2k} - 1 = 8\alpha$ where α is an integer, so $5^{2k} = 8\alpha + 1$

Consider $n = k + 1$:

Question number

$5^{2(k+1)} - 1 = 25 \times 5^{2k} - 1$

$= 25(8\alpha + 1) - 1$

$= 200\alpha + 24$

$= 8(25\alpha + 3)$, and $25\alpha + 3$ is an integer

so $5^{2(k+1)} - 1$ is divisible by 8, i.e. $P(k + 1)$ is true.

$P(1)$ is true, and if $P(k)$ is true then $P(k + 1)$ is true.

Hence by mathematical induction, $5^{2n} - 1$ is divisible by 8

for all integers $n \geq 1$.

3 (a) Parts: $u = e^x$, $\frac{dv}{dx} = \cos x \Rightarrow \frac{du}{dx} = e^x$, $v = \sin x$

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

Parts again: $u = e^x$, $\frac{dv}{dx} = \sin x \Rightarrow \frac{du}{dx} = e^x$, $v = -\cos x$

$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

So $\int e^x \cos x \, dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x \, dx]$

$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + c$

$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$

(b) $\int_5^\infty \frac{1}{(x-3)^2} \, dx = \lim_{t \to \infty} \int_5^t (x-3)^{-2} \, dx$

$= \lim_{t \to \infty} [-(x-3)^{-1}]$ from 5 to t

$= \lim_{t \to \infty} \left(-\frac{1}{t-3} + \frac{1}{5-3}\right)$

As $t \to \infty$, $\frac{1}{t-3} \to 0$, so the integral $= 0 + \frac{1}{2} = \frac{1}{2}$

Question number

4

The shaded loop starts and ends at O, where $r = 0$:

$\sin 2\theta = 0 \Rightarrow \theta = 0, \pi/2, \ldots$ so the loop is $0 \leqslant \theta \leqslant \pi/2$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} (4\sin 2\theta)^2 \, d\theta = \int_0^{\pi/2} 8\sin^2 2\theta \, d\theta$$

Using $\cos 4\theta = 1 - 2\sin^2 2\theta$: $8\sin^2 2\theta = 4 - 4\cos 4\theta$

$$\text{Area} = \int_0^{\pi/2} (4 - 4\cos 4\theta) \, d\theta$$

$= [4\theta - \sin 4\theta]$ from 0 to $\pi/2$

$= (2\pi - \sin 2\pi) - (0 - 0)$

$= 2\pi \quad (= 6.28)$

5 (i)

$$\frac{3r-1}{r^3-r} = \frac{3r-1}{r(r-1)(r+1)} \equiv \frac{A}{r-1} + \frac{B}{r} + \frac{C}{r+1}$$

$3r - 1 \equiv Ar(r+1) + B(r-1)(r+1) + Cr(r-1)$

$r = 0$: $-1 = -B \Rightarrow B = 1$

$r = 1$: $2 = 2A \Rightarrow A = 1$

$r = -1$: $-4 = 2C \Rightarrow C = -2$

$$\frac{3r-1}{r^3-r} \equiv \frac{1}{r-1} + \frac{1}{r} - \frac{2}{r+1}$$

(ii)

$$\sum_{r=2}^{n} \frac{3r-1}{r^3-r} = \sum_{r=2}^{n} \left(\frac{1}{r-1} + \frac{1}{r} - \frac{2}{r+1}\right)$$

$r = 2$: $\frac{1}{1} + \frac{1}{2} - \frac{2}{3}$

$r = 3$: $\frac{1}{2} + \frac{1}{3} - \frac{2}{4}$

$r = 4$: $\frac{1}{3} + \frac{1}{4} - \frac{2}{5}$

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...

$r = n - 1$: $\frac{1}{n-2} + \frac{1}{n-1} - \frac{2}{n}$

$r = n$: $\frac{1}{n-1} + \frac{1}{n} - \frac{2}{n+1}$

Every term $1/k$ for $3 \leq k \leq n-1$ appears as $1 + 1 - 2 = 0$,

so the remaining terms are:

$\text{Sum} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} - \frac{2}{n+1}$

$= 2 - \frac{1}{n} - \frac{2}{n+1}$ for $n \geq 2$, as required

6 (i) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

$\text{Sum} = \frac{1+x^2}{(1+x^2)^2} + \frac{1-x^2}{(1+x^2)^2}$

$= \frac{2}{(1+x^2)^2}$ as required

(ii) $x^2 - 6x + 10 = (x-3)^2 + 1$

$\int_3^4 \frac{1}{(x^2-6x+10)^2}\,dx = \int_3^4 \frac{1}{(1+(x-3)^2)^2}\,dx$

By part (i), with x replaced by $x - 3$:

$= \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{1+(x-3)^2}\right]$ from 3 to 4

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$= \frac{1}{2}\left[\left(\tan^{-1} 1 + \frac{1}{2}\right) - \left(\tan^{-1} 0 + 0\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$

$= \frac{\pi}{8} + \frac{1}{4}$

7 Divide by $x^2 - 1$: $\frac{dy}{dx} + \frac{4x}{x^2 - 1}y = \frac{8x}{x^2 - 1}$

Integrating factor $= e^{\int \frac{4x}{x^2 - 1}\,dx} = e^{2\ln(x^2 - 1)} = (x^2 - 1)^2$

(valid since $|x| > 1$ so $x^2 - 1 > 0$)

Multiply through: $(x^2 - 1)^2\frac{dy}{dx} + 4x(x^2 - 1)y = 8x(x^2 - 1)$

$\frac{d}{dx}\left[(x^2 - 1)^2 y\right] = 8x(x^2 - 1)$

$(x^2 - 1)^2 y = \int 8x(x^2 - 1)\,dx = 2(x^2 - 1)^2 + c$

$y = 2 + \frac{c}{(x^2 - 1)^2}$

$x = 2, y = 5$: $5 = 2 + \frac{c}{9} \Rightarrow c = 27$

$y = 2 + \frac{27}{(x^2 - 1)^2}$

8 (i) $f(x) = (1 + \cos x)e^x$

$f'(x) = (1 + \cos x)e^x - e^x \sin x$

$f''(x) = (1 + \cos x)e^x - e^x \sin x - e^x \sin x - e^x \cos x$

$f''(x) = (1 - 2\sin x)e^x$ as required

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(ii) $f'''(x) = (1 - 2\sin x)e^x - 2e^x \cos x = (1 - 2\sin x - 2\cos x)e^x$

$f''''(x) = (1 - 2\sin x - 2\cos x)e^x + (-2\cos x + 2\sin x)e^x$

$f''''(x) = (1 - 4\cos x)e^x$

$f(0) = 2,\ f'(0) = 2,\ f''(0) = 1,\ f'''(0) = -1,\ f''''(0) = -3$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \ldots$

$(1 + \cos x)e^x = 2 + 2x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{3}{4!}x^4 + \ldots$

$(1 + \cos x)e^x = 2 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{8}x^4 + \ldots$

(iii) Replace x by $-x$, and use $\cos(-x) = \cos x$:

$(1 + \cos x)e^{-x} = 2 + 2(-x) + \frac{1}{2}(-x)^2 - \frac{1}{6}(-x)^3 - \frac{1}{8}(-x)^4 + \ldots$

$(1 + \cos x)e^{-x} = 2 - 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 + \ldots$

9 (i) $I_n = \int \sec^{n-2} x \sec^2 x \, dx$

Parts: $u = \sec^{n-2} x$, $\frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = (n - 2)\sec^{n-3} x \times \sec x \tan x = (n - 2)\sec^{n-2} x \tan x$, $v = \tan x$

$I_n = \sec^{n-2} x \tan x - \int (n - 2)\sec^{n-2} x \tan^2 x \, dx$

Using $\tan^2 x = \sec^2 x - 1$:

$I_n = \sec^{n-2} x \tan x - (n - 2)\int \sec^n x \, dx + (n - 2)\int \sec^{n-2} x \, dx$

$I_n = \sec^{n-2} x \tan x - (n - 2)I_n + (n - 2)I_{n-2}$

$I_n(1 + n - 2) = \sec^{n-2} x \tan x + (n - 2)I_{n-2}$

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$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$ as required

(ii) $V = \pi\int_0^{\pi/3} y^2\,dx = \pi\int_0^{\pi/3} \sec^4 x\,dx$

Using part (i) with $n = 4$ (definite integral from 0 to $\pi/3$):

$3\int_0^{\pi/3} \sec^4 x\,dx = [\sec^2 x \tan x]$ from 0 to $\pi/3 + 2\int_0^{\pi/3} \sec^2 x\,dx$

$\sec(\pi/3) = 2,\ \tan(\pi/3) = \sqrt{3}$

$= (4 \times \sqrt{3} - 0) + 2[\tan x]$ from 0 to $\pi/3$

$= 4\sqrt{3} + 2\sqrt{3} = 6\sqrt{3}$

$\int_0^{\pi/3} \sec^4 x\,dx = 2\sqrt{3}$

Volume $= 2\sqrt{3}\pi$ cubic units $(= 10.9)$

10 (i) Auxiliary equation: $m^2 - m - 6 = 0$

$(m-3)(m+2) = 0 \Rightarrow m = 3, -2$

Complementary function: $x = Ae^{3t} + Be^{-2t}$

Particular integral: try $x = C\cos 2t + D\sin 2t$

$\frac{dx}{dt} = -2C\sin 2t + 2D\cos 2t$

$\frac{d^2x}{dt^2} = -4C\cos 2t - 4D\sin 2t$

Substitute:

$-4C\cos 2t - 4D\sin 2t + 2C\sin 2t - 2D\cos 2t$

$\quad - 6C\cos 2t - 6D\sin 2t = 52\cos 2t$

$\cos 2t$: $-4C - 2D - 6C = 52 \Rightarrow -10C - 2D = 52$

$\sin 2t$: $-4D + 2C - 6D = 0 \Rightarrow 2C = 10D \Rightarrow C = 5D$

$-50D - 2D = 52 \Rightarrow D = -1,\ C = -5$

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General solution:

$x = Ae^{3t} + Be^{-2t} - 5\cos 2t - \sin 2t$

(ii) As $t \to \infty$, $e^{3t} \to \infty$, so for x to remain finite $A = 0$.

$t = 0, x = 1$: $1 = B - 5 \Rightarrow B = 6$

$x = 6e^{-2t} - 5\cos 2t - \sin 2t$

11 (a) $\frac{dy}{dx} = \frac{2}{\sqrt{(2x-1)^2 + 1}} + 4$

At $x = \frac{1}{2}$: $\frac{dy}{dx} = \frac{2}{\sqrt{0+1}} + 4 = 6$

At $x = \frac{1}{2}$: $y = \sinh^{-1}(0) + 2 = 2$

Tangent: $y - 2 = 6(x - \frac{1}{2})$

$y = 6x - 1$

(b)(i) Let $y = \sinh^{-1} x$, so $x = \sinh y$

$x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow e^y - e^{-y} = 2x$

Multiply by e^y: $e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$

But $x - \sqrt{(x^2 + 1)} < 0$ since $\sqrt{(x^2 + 1)} > x$, and $e^y > 0$,

so $e^y = x + \sqrt{(x^2 + 1)}$

$\sinh^{-1} x \equiv \ln(x + \sqrt{x^2 + 1})$ as required

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(ii) $\sinh^{-1}\left(\frac{4}{3}\right) = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right) = \ln\left(\frac{4}{3} + \frac{5}{3}\right) = \ln 3$

$\sinh^{-1}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$

So $\sinh^{-1} x + \ln 3 = 2 \ln 2$

$\sinh^{-1} x = 2 \ln 2 - \ln 3 = \ln \frac{4}{3}$

$\ln\left(x + \sqrt{x^2 + 1}\right) = \ln \frac{4}{3} \Rightarrow x + \sqrt{x^2 + 1} = \frac{4}{3}$

$\sqrt{x^2 + 1} = \frac{4}{3} - x$

$x^2 + 1 = \frac{16}{9} - \frac{8}{3}x + x^2$

$\frac{8}{3}x = \frac{7}{9}$

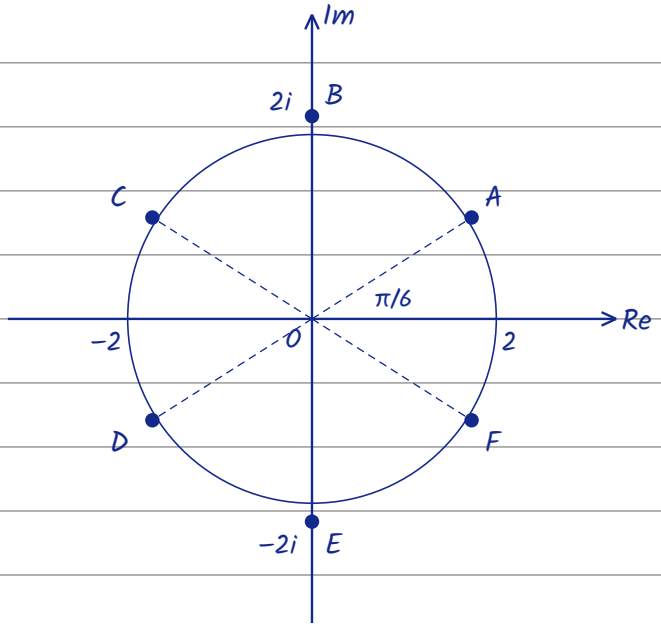
$x = \frac{7}{24}$

12 (a) $z^6 = -64 = 64e^{i(\pi + 2k\pi)}$

$z = 64^{\frac{1}{6}}\, e^{i\frac{(\pi + 2k\pi)}{6}} = 2e^{i\frac{(2k+1)\pi}{6}}, \quad k = 0, 1, 2, 3, 4, 5$

Roots: $2e^{i\frac{\pi}{6}}, \; 2e^{i\frac{\pi}{2}}, \; 2e^{i\frac{5\pi}{6}}, \; 2e^{i\frac{7\pi}{6}}, \; 2e^{i\frac{3\pi}{2}}, \; 2e^{i\frac{11\pi}{6}}$

Question number



All six roots lie on the circle $|z| = 2$, spaced $\pi/3$ apart:

$A = 2e^{i\pi/6}$, $B = 2e^{i\pi/2}$, $C = 2e^{i5\pi/6}$,

$D = 2e^{i7\pi/6}$, $E = 2e^{i3\pi/2}$, $F = 2e^{i11\pi/6}$

(b)(i) $\frac{1}{z} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$ (de Moivre)

$\frac{1}{z} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$ as required

(ii) Similarly $z^n + \frac{1}{z^n} = 2\cos n\theta$ (by de Moivre)

$(z + \frac{1}{z})^4 = (2\cos\theta)^4 = 16\cos^4\theta$

$(z + \frac{1}{z})^4 = z^4 + 4z^3(\frac{1}{z}) + 6z^2(\frac{1}{z^2}) + 4z(\frac{1}{z^3}) + \frac{1}{z^4}$

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$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\left(p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

(iii) $\int \cos^4\theta\, d\theta = \int \left(\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\right) d\theta$

$= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + c$

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