New Specification

Rewarding Learning

ADVANCED SUBSIDIARY (AS)

General Certificate of Education

Further Mathematics *StemUp Model Solution*

Assessment Unit AS 1

assessing

Pure Mathematics

SFM11

[SFM11]

Assessment

Assessment Level of Control:

Tick the relevant box (✓)

Controlled Conditions	
Other	

TIME

1 hour 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided.
Answer **all nine** questions.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

stemup.co.uk

Answer all nine questions.

Show clearly the full development of your answers.

Answers should be given to three significant figures unless otherwise stated.

1 Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(i) Show that $\mathbf{A}^2 = 7\mathbf{A} - 10\mathbf{I}$ [4]

(ii) Hence, or otherwise, express $\mathbf{A}^3$ in the form $m\mathbf{A} + n\mathbf{I}$, where m and n are integers. [4]

2 The roots of the equation

$$x^2 + px + q = 0$$

are α and β.

Form an equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ [6]

3 The complex numbers v and w are given by

$$v = 3 + 4i$$

$$w = 1 - 2i$$

Find:

(i) vw [2]

(ii) $\frac{v}{w}$ [3]

(iii) Describe, and sketch on a clearly labelled Argand diagram, the locus of the points z which satisfy

$$|z - v| = |z - w|$$ [3]

stemup.co.uk

4 **(a)** Describe fully the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ [2]

(b) A linear transformation maps the points A (1, 2) and B (0, 5) to the points A′ (9, 4) and B′ (20, 10) respectively.

(i) Find the matrix **M** which represents this transformation. [5]

The triangle OA′B′ is the image of the triangle OAB under the transformation represented by **M**

(ii) Find the area of the triangle OA′B′ [4]

5 One root of the equation

$$z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$$

is $1 + 2i$

(i) Write down another root of this equation. Justify your answer. [2]

(ii) Hence, or otherwise, solve the equation

$$z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$$ [7]

stemup.co.uk

6 **(i)** Show that the determinant of

$$\begin{pmatrix} 1 & 1 & c \\ c+1 & 6 & 9 \\ 5 & -1 & 4 \end{pmatrix}$$

is $-c^2 - 35c + 74$ [3]

Consider the system of linear equations

$$\begin{aligned} x + y + cz &= 0 \\ (c+1)x + 6y + 9z &= -7 \\ 5x - y + 4z &= 1 \end{aligned}$$

where x, y and z are real numbers.

(ii) In each of the following cases, determine the number of solutions for the above system of equations:

(a) $c = 1$

(b) $c = 2$ [7]

stemup.co.uk

7 The points Q, R and S are given by

$$Q(2, -1, 7)$$

$$R(3, 2, 5)$$

$$S(5, 8, 1)$$

(i) Show that Q, R and S are collinear. [5]

(ii) Find the vector equation of the line through Q and R. [2]

The line l is given by

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(-\mathbf{i} + p\mathbf{j} + p^2\mathbf{k})$$

(iii) Given that the lines QR and l are perpendicular, find the possible values of p. [4]

The point T has coordinates $(4, -1, 6)$

(iv) Find, in scalar product form, the equation of the plane which contains the points Q, R and T. [7]

8 The complex number z_1 is given by

$$z_1 = \sqrt{2} + \sqrt{2}\,\mathrm{i}$$

(i) Find the modulus and argument of z_1 [4]

The complex number z_2 has modulus 2 and argument $\frac{\pi}{6}$

(ii) Find z_2 in the form $a + \mathrm{i}b$ where a and b are real numbers. [2]

(iii) Sketch the points representing z_1, z_2 and $z_1 + z_2$ on a carefully labelled Argand diagram. [3]

(iv) Hence, or otherwise, show that

$$\tan\left(\frac{5\pi}{24}\right) = \frac{\sqrt{2} + 1}{\sqrt{2} + \sqrt{3}}$$ [4]

stemup.co.uk

9 The tetrahedron ABCD is shown in **Fig. 1** below.

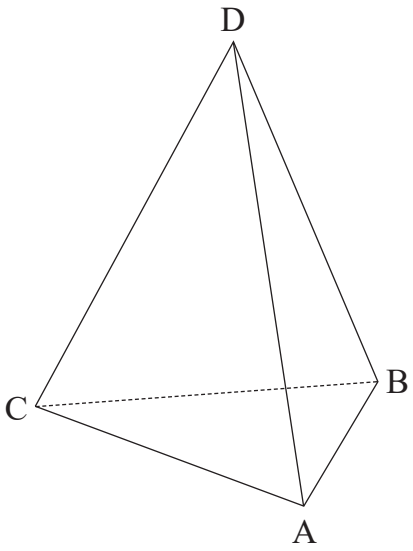


Fig. 1

The edges CA and CB have equations

$$\text{CA:} \quad \mathbf{r}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$$

$$\text{CB:} \quad \mathbf{r}_2 = -7\mathbf{j} + 4\mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

(i) Find the coordinates of C. [6]

The edge BD has equation

$$\mathbf{r}_3 = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k} + \gamma(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

and the face ACD has equation

$$6x + y = 8$$

(ii) Find the coordinates of D. [4]

The points A and B have coordinates (1, 2, 3) and (4, –1, 6) respectively.

(iii) Find the volume of the tetrahedron. [7]

THIS IS THE END OF THE QUESTION PAPER

stemup.co.uk

Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (i)

$$A^2 = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4+0 & 2+5 \\ 0+0 & 0+25 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 0 & 25 \end{pmatrix}$$

$$7A - 10I = \begin{pmatrix} 14 & 7 \\ 0 & 35 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 7 \\ 0 & 25 \end{pmatrix}$$

$\therefore A^2 = 7A - 10I$ as required

1 (ii)

$A^3 = A \times A^2 = A(7A - 10I)$

$= 7A^2 - 10A$

$= 7(7A - 10I) - 10A$

$= 49A - 70I - 10A$

$A^3 = 39A - 70I$ $\quad (m = 39, n = -70)$

2

$\alpha + \beta = -p, \quad \alpha\beta = q$

Sum of new roots: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{p}{q}$

Product of new roots: $\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{q}$

Equation: $x^2 - (\text{sum})x + (\text{product}) = 0$

$$x^2 + \frac{p}{q}x + \frac{1}{q} = 0$$

$$qx^2 + px + 1 = 0$$

Question number

3 (i) $vw = (3 + 4i)(1 - 2i)$

$= 3 - 6i + 4i - 8i^2$

$= 3 - 2i + 8 = 11 - 2i$

3 (ii) $\frac{v}{w} = \frac{3 + 4i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i}$

$= \frac{3 + 6i + 4i + 8i^2}{1 + 4}$

$= \frac{-5 + 10i}{5} = -1 + 2i$

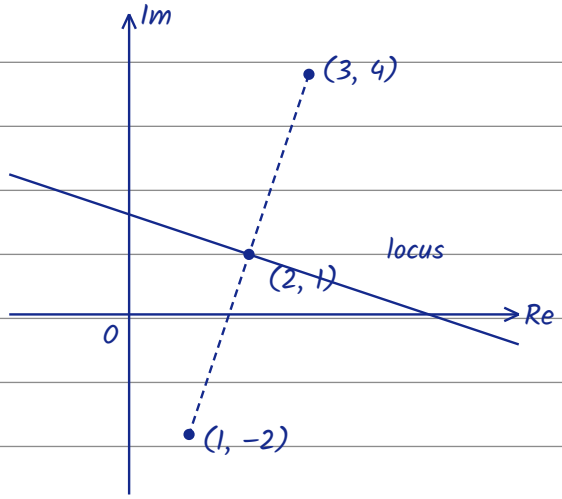
3 (iii) $|z - v| = |z - w|$: z is equidistant from v (3, 4) and w (1, −2).

The locus is the perpendicular bisector of the line joining

the points (3, 4) and (1, −2).

Midpoint = (2, 1); gradient of join $= \frac{4 - (-2)}{3 - 1} = 3$

so the bisector has gradient $-\frac{1}{3}$: $x + 3y = 5$



Question number

4 (a) Reflection in the y-axis (the line $x = 0$)

4 (b)(i) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $M\begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 20 \\ 4 & 10 \end{pmatrix}$

$\begin{pmatrix} a + 2b & 5b \\ c + 2d & 5d \end{pmatrix} = \begin{pmatrix} 9 & 20 \\ 4 & 10 \end{pmatrix}$

$5b = 20 \Rightarrow b = 4$; $\quad a + 2b = 9 \Rightarrow a = 1$

$5d = 10 \Rightarrow d = 2$; $\quad c + 2d = 4 \Rightarrow c = 0$

$M = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$

4 (b)(ii) Area of $OAB = \frac{1}{2} \times$ base $OB \times$ height $= \frac{1}{2} \times 5 \times 1 = 2.5$

$\det M = (1)(2) - (4)(0) = 2$

Area of $OA'B' = \det M \times$ area of OAB

$= 2 \times 2.5 = 5$ square units

5 (i) $1 - 2i$ is also a root.

The equation has real coefficients, so non-real roots occur

in complex conjugate pairs.

5 (ii) $[z - (1 + 2i)][z - (1 - 2i)] = [(z - 1) - 2i][(z - 1) + 2i]$

$= (z - 1)^2 - (2i)^2 = z^2 - 2z + 1 + 4$

$= z^2 - 2z + 5$

$z^4 - 2z^3 + 14z^2 - 18z + 45 = (z^2 - 2z + 5)(z^2 + pz + q)$

z^3: $p - 2 = -2 \Rightarrow p = 0$

constant: $5q = 45 \Rightarrow q = 9$

check z^2: $q - 2p + 5 = 14$ ✓ $\quad$ z: $5p - 2q = -18$ ✓

Question number

So $(z^2 - 2z + 5)(z^2 + 9) = 0$

$z^2 + 9 = 0 \Rightarrow z = \pm 3i$

Solutions: $z = 1 + 2i,\ 1 - 2i,\ 3i,\ -3i$

6 (i) $\det = 1(6 \times 4 - 9 \times (-1)) - 1((c + 1) \times 4 - 9 \times 5) + c((c + 1)(-1) - 6 \times 5)$

$= 1(24 + 9) - (4c + 4 - 45) + c(-c - 1 - 30)$

$= 33 - 4c + 41 - c^2 - 31c$

$= -c^2 - 35c + 74$ as required

6 (ii)(a) $c = 1$: $\det = -1 - 35 + 74 = 38$

$\det \neq 0$, so the matrix is non-singular and

the system has one unique solution.

6 (ii)(b) $c = 2$: $\det = -4 - 70 + 74 = 0$, so there is no unique solution.

$x + y + 2z = 0$ ①

$3x + 6y + 9z = -7$ ②

$5x - y + 4z = 1$ ③

① + ③: $6x + 6z = 1 \Rightarrow x + z = \frac{1}{6}$

② + 6 × ③: $33x + 33z = -1 \Rightarrow x + z = -\frac{1}{33}$

These are contradictory ($\frac{1}{6} \neq -\frac{1}{33}$), so the equations are

inconsistent: the system has no solutions.

7 (i) $\overrightarrow{QR} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Question number

$\overrightarrow{QS} = \begin{pmatrix}5\\8\\1\end{pmatrix} - \begin{pmatrix}2\\-1\\7\end{pmatrix} = \begin{pmatrix}3\\9\\-6\end{pmatrix} = 3\begin{pmatrix}1\\3\\-2\end{pmatrix}$

$\overrightarrow{QS} = 3\overrightarrow{QR}$, so QS and QR are parallel, and since they share the common point Q, the points Q, R and S are collinear.

7 (ii) $r = 2i - j + 7k + \lambda(i + 3j - 2k)$

7 (iii) Perpendicular $\Rightarrow (i + 3j - 2k)\cdot(-i + pj + p^2k) = 0$

$-1 + 3p - 2p^2 = 0$

$2p^2 - 3p + 1 = 0$

$(2p - 1)(p - 1) = 0$

$p = \frac{1}{2}$ or $p = 1$

7 (iv) $\overrightarrow{QT} = \begin{pmatrix}4\\-1\\6\end{pmatrix} - \begin{pmatrix}2\\-1\\7\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$

Normal: $\overrightarrow{QR} \times \overrightarrow{QT} = \begin{pmatrix}1\\3\\-2\end{pmatrix} \times \begin{pmatrix}2\\0\\-1\end{pmatrix}$

$= \begin{pmatrix}(3)(-1) - (-2)(0)\\(-2)(2) - (1)(-1)\\(1)(0) - (3)(2)\end{pmatrix} = \begin{pmatrix}-3\\-3\\-6\end{pmatrix}$

Take $n = i + j + 2k$. Plane: $r\cdot(i + j + 2k) = d$

Q on plane: $d = (2i - j + 7k)\cdot(i + j + 2k) = 2 - 1 + 14 = 15$

$r\cdot(i + j + 2k) = 15$

Question number

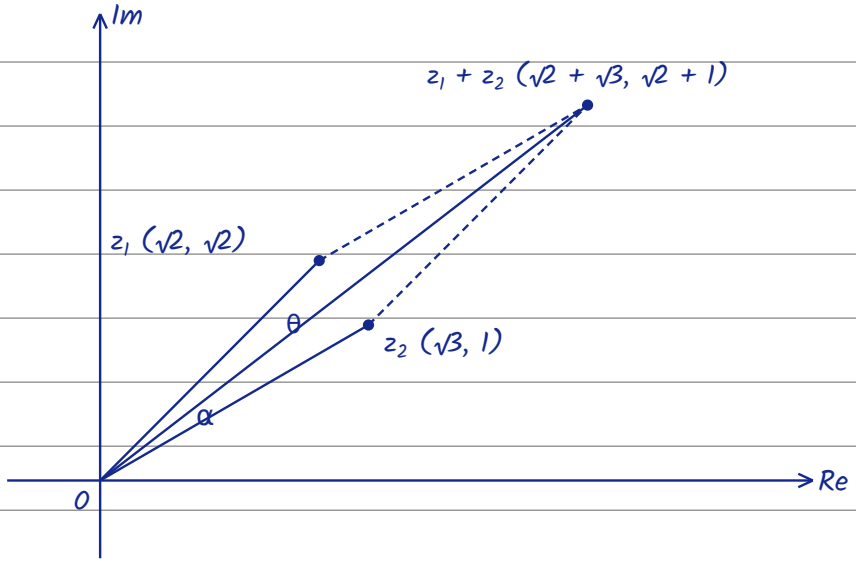
8 (i) $|z_1| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$

$\arg z_1 = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ (first quadrant)

8 (ii) $z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$z_2 = \sqrt{3} + i$

8 (iii) $z_1 + z_2 = (\sqrt{2} + \sqrt{3}) + (\sqrt{2} + 1)i$



8 (iv) $|z_1| = |z_2| = 2$, so O, z_1, $z_1 + z_2$, z_2 form a rhombus,

and the diagonal from O to $z_1 + z_2$ bisects the angle between z_1 and z_2.

Angle between z_1 and $z_2 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

$\theta = \frac{1}{2} \times \frac{\pi}{12} = \frac{\pi}{24}$

$\alpha = \arg(z_1 + z_2) = \frac{\pi}{6} + \frac{\pi}{24} = \frac{5\pi}{24}$

Question number

$\tan\alpha = \dfrac{\mathrm{Im}(z_1 + z_2)}{\mathrm{Re}(z_1 + z_2)}$

$\tan\left(\dfrac{5\pi}{24}\right) = \dfrac{\sqrt{2}+1}{\sqrt{2}+\sqrt{3}}$ as required

9 (i) At C: $\begin{pmatrix} 1+\lambda \\ 2-6\lambda \\ 3+2\lambda \end{pmatrix} = \begin{pmatrix} 2\mu \\ -7+3\mu \\ 4+\mu \end{pmatrix}$

i: $1 + \lambda = 2\mu$ ①

j: $2 - 6\lambda = -7 + 3\mu$ ②

k: $3 + 2\lambda = 4 + \mu$ ③

③ − 2 × ①: $1 = 4 - 3\mu \Rightarrow \mu = 1$

①: $1 + \lambda = 2 \Rightarrow \lambda = 1$

Check ②: $2 - 6 = -4$ and $-7 + 3 = -4$ ✓

$C = (2, -4, 5)$

9 (ii) On BD: $x = 4 + 2\gamma$, $y = -1 + 3\gamma$, $z = 6 - \gamma$

D lies on face ACD: $6x + y = 8$

$6(4 + 2\gamma) + (-1 + 3\gamma) = 8$

$24 + 12\gamma - 1 + 3\gamma = 8 \Rightarrow 15\gamma = -15 \Rightarrow \gamma = -1$

$D = (4 - 2, -1 - 3, 6 + 1) = (2, -4, 7)$

9 (iii) Volume $= \dfrac{1}{6}\left|(\overrightarrow{CA} \times \overrightarrow{CB}) \cdot \overrightarrow{CD}\right|$

$\overrightarrow{CA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ -2 \end{pmatrix}$

Question number

$$\overrightarrow{CB} = \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{CD} = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{pmatrix} (6)(1) - (-2)(3) \\ (-2)(2) - (-1)(1) \\ (-1)(3) - (6)(2) \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \\ -15 \end{pmatrix}$$

$$(\overrightarrow{CA} \times \overrightarrow{CB}) \cdot \overrightarrow{CD} = 0 + 0 + (-15)(2) = -30$$

$$\text{Volume} = \frac{1}{6} \times |-30| = 5 \text{ cubic units}$$

Permission to reproduce all copyright material has been applied for.
In some cases, efforts to contact copyright holders may have been unsuccessful and CCEA will be happy to rectify any omissions of acknowledgement in future if notified.

12338/4