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# Further Mathematics *StemUp Model Solution*

## Assessment Unit A2 2

*assessing*

Applied Mathematics

AFM21

## [AFM21]

## Assessment

**Assessment Level of Control:**
**Tick the relevant box (✓)**

| | |
|---|---|
| Controlled Conditions | |
| Other | |

**TIME**

2 hours 15 minutes.

**INSTRUCTIONS TO CANDIDATES**

Write your Centre Number and Candidate Number on the Answer Booklet provided and on the Section D Supplementary Answer Booklet (if applicable).
You must answer **all** questions from sections A and B **or** A and C **or** A and D **or** C and D.
You should spend equal time on each of the two sections.
Candidates taking Section D should use the Supplementary Answer Booklet provided and attach to your Answer Booklet using the treasury tag provided.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

**INFORMATION FOR CANDIDATES**

The total mark for this paper is 150.
The total mark for each section of this paper is 75.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
Answers should include diagrams where appropriate and marks may be awarded for them.
Take $g = 9.8\,\text{ms}^{-2}$, unless specified otherwise.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$



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## SECTION A Mechanics 1

**Answer all five questions in this section.**

**1** A particle P moves with Simple Harmonic Motion along a horizontal line between the points A and B as shown in **Fig. 1** below.

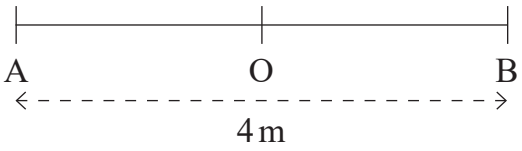



**Fig. 1**

The centre of the motion is O.

The end points of the motion A and B are 4 m apart.

**(i)** Write down the amplitude of the motion. [1]

The maximum acceleration of P is 6 $ms^{-2}$

**(ii)** Find the period of the motion. [3]

The point C is 0.5 m from O, as shown in **Fig. 2** below.

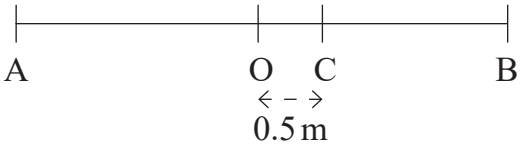



**Fig. 2**

The particle is moving from O towards C.

**(iii)** Find the speed of P at C. [3]

**(iv)** Find the least time taken for P to travel from O to C. [3]



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**2** A sign in a toy shop is shown in **Fig. 3** below.

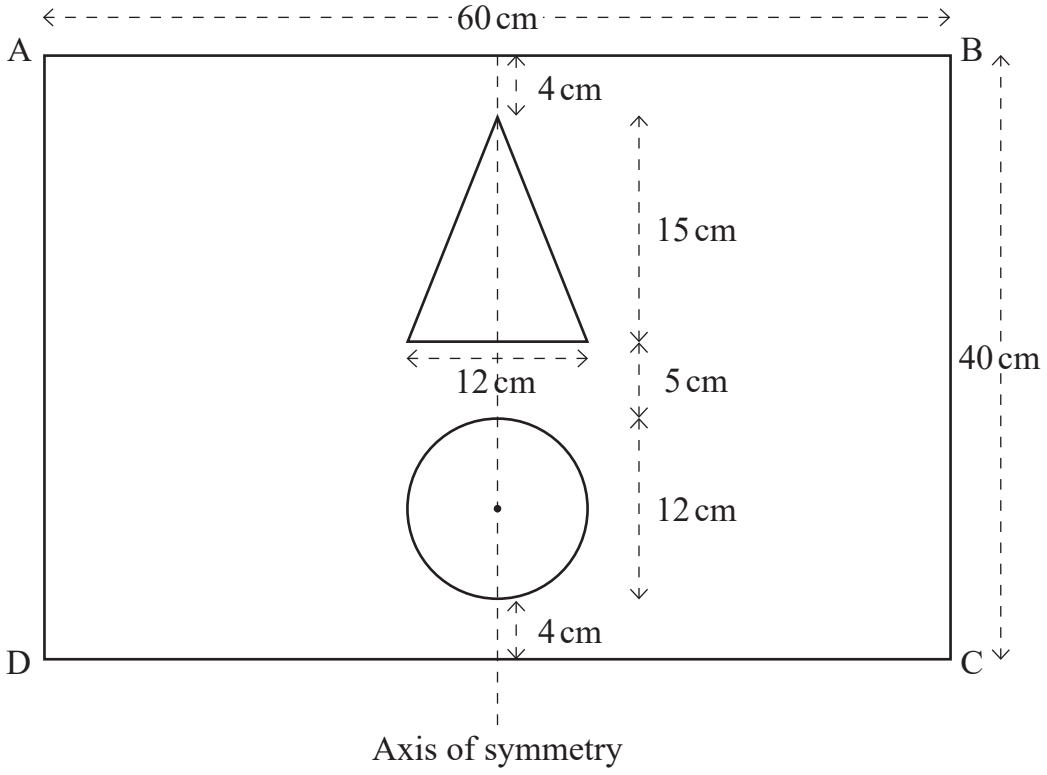



**Fig. 3**

The sign is modelled as a uniform rectangular lamina from which an isosceles triangle and a circle have been removed.

The axes of symmetry of the triangle and the centre of the circle both lie on the axis of symmetry of the rectangle.

The sign has dimensions:

| | | |
|---|---|---|
| Rectangle | AB = 60 cm | CB = 40 cm |
| Triangle | Base = 12 cm | Perpendicular height = 15 cm |
| Circle | Diameter = 12 cm | |

Let $m$ be the mass per unit area, in g cm$^{-2}$, of the lamina.

**(i)** Find the distance of the centre of mass of the sign from DC. [10]

The sign is freely suspended from the point A.

**(ii)** Find the angle which AD makes with the vertical when the sign is in equilibrium. [4]



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**3** A quad bike of mass $m$ kg is travelling on a rough road which is banked at an angle of 30° to the horizontal as shown in **Fig. 4** below.

The wheels A and B of the bike are 0.8 m apart and the centre of gravity G of the bike lies on the axis of symmetry at a height of 1 m above the road.

$R_A$ and $R_B$ are the normal reactions at A and B respectively.

$Fr$ is the total frictional force between the wheels and the road.

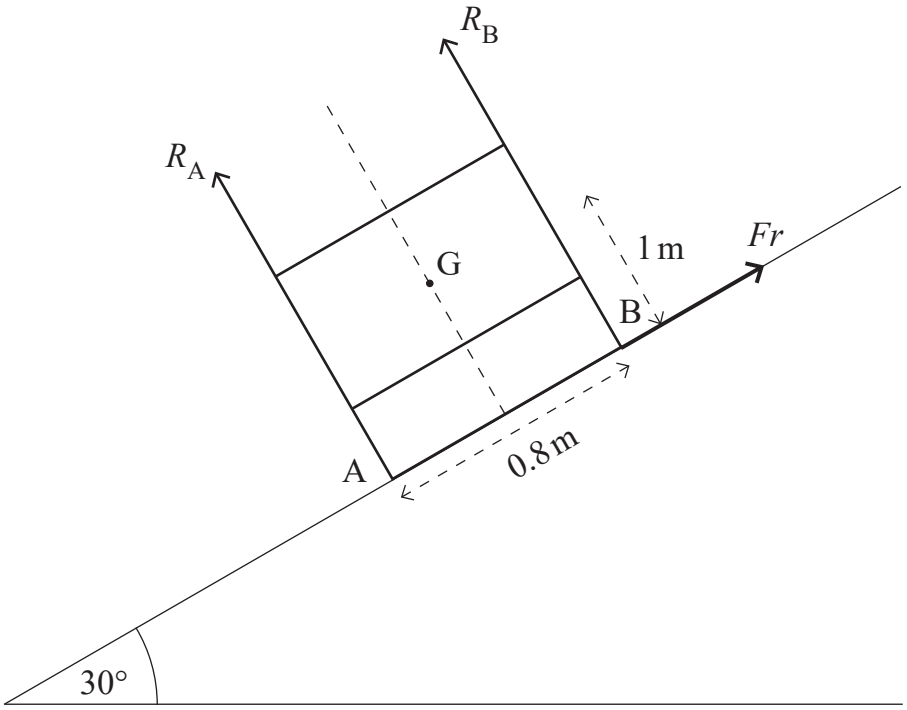



**Fig. 4**

The bike moves in a horizontal circle of radius 20 m on this road.
When it is travelling at $v$ ms$^{-1}$ it is on the point of toppling inwards.

**(i)** State the value of $R_B$ at this instant. [1]

**(ii)** Find the value of $v$.
(You may assume that the road is sufficiently rough to prevent sliding.) [13]


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**4** A toy car of mass 0.6 kg moves freely along a smooth horizontal track and collides with a buffer at the end of the track.

The buffer can be modelled as a perfect spring of natural length 25 cm and modulus of elasticity $\lambda$ newtons and a perfect dashpot with damping constant 3 $\text{Nm}^{-1}\text{s}$.

At time $t$ seconds the compression of the buffer is $x$ metres.

**(i)** Show that the equation of the motion of the truck is given by

$$\ddot{x} + 5\dot{x} + \frac{20}{3}\lambda x = 0 \qquad [7]$$

**(ii)** Find the range of values of $\lambda$ for which the system is under-damped. [4]

**(iii)** If $\lambda = 0.9$, find a general expression for $x$ in terms of $t$. [6]



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**5** A framework of 11 light pin-jointed rods is shown in **Fig. 5** below.

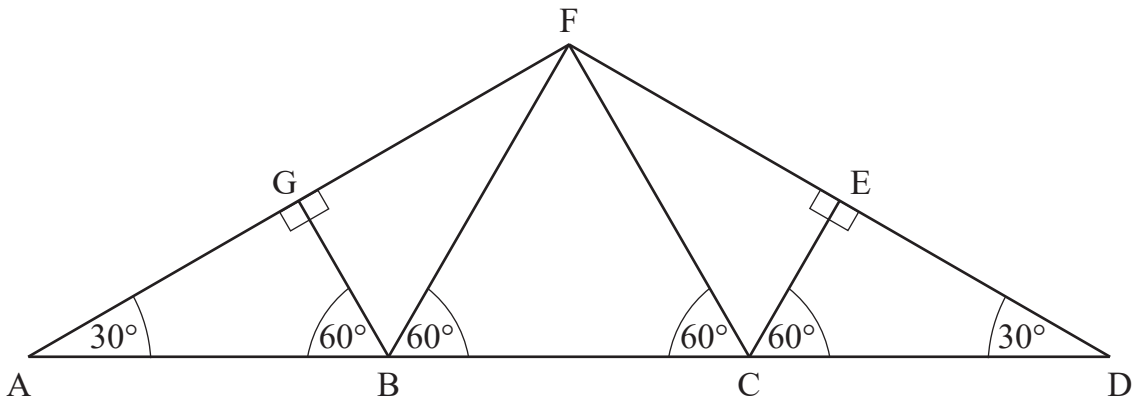



**Fig. 5**

A weight of 90 N is suspended from B.

The framework is symmetrical and is supported by vertical forces at A and D.

ABCD is horizontal and the rods all lie in the same vertical plane.

$\angle GAB = \angle EDC = 30°$
$\angle GBA = \angle FBC = \angle FCB = \angle ECD = 60°$
$\angle CED = \angle CEF = \angle AGB = \angle FGB = 90°$

**(i)** Given that BC has length 2 m show that AB also has length 2 m. [2]

**(ii)** Find the magnitudes of the supporting forces at A and D. [4]

**(iii)** Show that the forces in each of the rods CE and FC are 0 N. [5]

**(iv)** Find the forces in the rods DE, CD and BC and determine whether they are tensions or thrusts. [9]


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## SECTION B Mechanics 2

**Answer all five questions in this section.**

**1** **Fig. 1** below shows a system of forces acting along the sides of a rectangle ABCD.

The forces are:

8 N along AB
6 N along BC
6 N along CD
12 N along AC

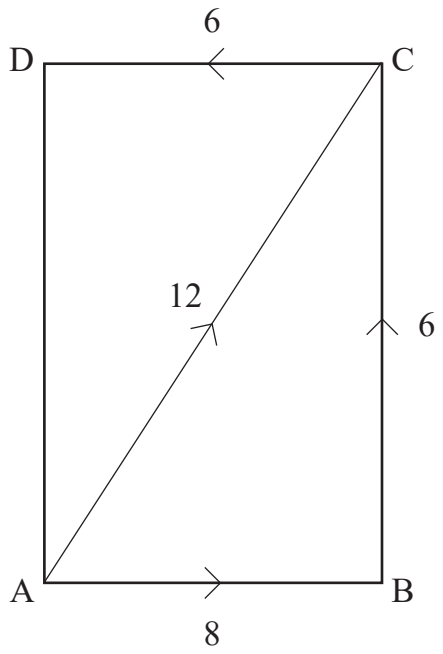



**Fig. 1**

AB = 1 m

BC = $\sqrt{3}$ m

The system of forces is equivalent to a single force $R$.

**(i)** Find the magnitude and direction of $R$. [7]

**(ii)** Show that $R$ acts through B. [3]



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**2** At time $t$ seconds a particle has a velocity of

$$\mathbf{v} = (-3t^2 + 12)\mathbf{i} + (4t - 1)\mathbf{j} + (t^3 - 8)\mathbf{k} \text{ ms}^{-1}$$

**(i)** Find the acceleration of the particle at time $t$. [4]

When $t = 0$, the particle passes through the point with position vector $\mathbf{i} + 2\mathbf{j}$

**(ii)** Find the distance of the particle from the origin O at time $t = 4$ [8]

**(iii)** Find the time at which the particle is moving parallel to the vector $\mathbf{j}$ [4]

**3** Three spheres A, B and C of equal radius and masses $6m$, $4m$ and $m$ respectively are at rest in that order in a straight line on a smooth horizontal surface.

A is projected towards B with speed $u$ ms$^{-1}$ and collides directly with B.

The coefficient of restitution between A and B is 0.25

**(i)** Find the velocities of A and B after this collision (in terms of $u$). [7]

A direct collision now takes place between B and C.
The coefficient of restitution between B and C is $e$.

**(ii)** Find the velocities of B and C, in terms of $e$ and $u$, after this collision. [4]

**(iii)** Find the range of values of $e$ for which A and B will collide again. [3]

**4** A cyclist is travelling along a straight horizontal road.
Model the cyclist and bicycle as a particle of mass 100 kg.

The cyclist produces a driving force of $10e^{-0.02t}$ newtons where $t$ is the time in seconds.
She experiences wind resistance of $3v$ newtons where $v$ is her speed at time $t$.

Given that the cyclist has an initial speed of 9 ms$^{-1}$, find her speed when $t = 30$ [14]


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**5** A solid hemisphere has radius $r$, density $\rho$ and volume $\frac{2\pi r^3}{3}$

Its centre of mass lies a distance $d$ from its base along the axis of symmetry.

**(i)** Show that $d = \frac{3}{8}r$ [8]

A child's toy is shown below in **Fig. 2**

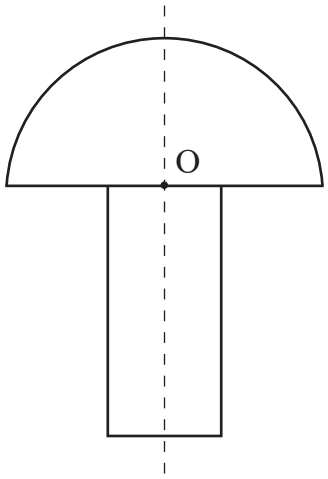



**Fig. 2**

The toy consists of a solid hemisphere of radius $3a$ joined to a solid cylinder of radius $a$ and height $3a$.
Each part of the toy is made from the same material.
The toy is symmetrical and O is the centre of the plane face of the hemisphere.

**(ii)** Find the distance, in terms of $a$, of the toy's centre of mass from O. [9]

The toy is placed on a rough plane inclined at an angle $\theta°$ to the horizontal as shown in **Fig. 3** below.

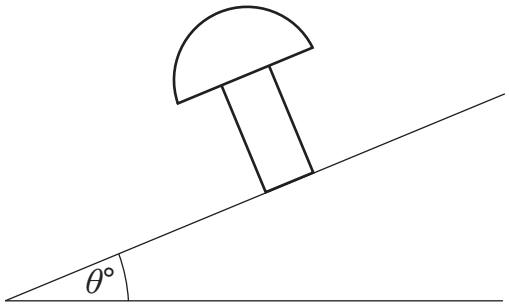



**Fig. 3**

**(iii)** Find the maximum value of $\theta$ for which the toy does not topple.
(You may assume that the plane is sufficiently rough to prevent sliding.) [4]



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## SECTION C Statistics

**Answer all five questions in this section.**

**1** $W$ and $Y$ are independent random variables.

$E(W) = 5$ and $Var(W) = 0.5$
$E(Y) = 4$ and $Var(Y) = 4$

**(i)** Given that $X = 2W - 0.5Y$, show that

$$E(X) = 8 \text{ and } Var(X) = 3 \quad [6]$$

Forty random observations of the variable $X$ are recorded.

**(ii)** Describe the main features of the central limit theorem. [2]

**(iii)** Using the central limit theorem, find the probability that the mean of the 40 observations is greater than 8.5 [6]

**2** A school categorises pupil attendance as good, satisfactory or poor.
The attendance levels of 80 junior and 60 senior pupils are recorded in **Table 1** below.

**Table 1**

| | | Level of attendance | | |
|---|---|---|---|---|
| | | Good | Satisfactory | Poor |
| School age group | Junior | 46 | 22 | 12 |
| | Senior | 24 | 20 | 16 |

Carry out a suitable test, at a 5% level of significance, to determine whether the school age group and level of attendance are independent. [16]


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**3** The mass $X$ of a standard size Venus chocolate bar may be regarded as being a Normal random variable with mean 55 grams and standard deviation 4 grams.
The company that makes Venus chocolate bars has decided to sell them in the following two formats:

- A packet of three standard size bars.
- A king size bar with mass 3.5 times the mass of a standard size bar.

Let the difference between the masses of the two formats be modelled by the random variable $D$, where $D = 3.5X - (X_1 + X_2 + X_3)$.
$X_1$, $X_2$ and $X_3$ are independent random variables each with the same distribution as $X$.

**(i)** Using this model, calculate the probability that the mass of a king size bar will be greater than the mass of a packet of three standard size bars.
(You may assume that the mass of any packaging is negligible.) [10]

**(ii)** State one assumption that is made about the variance of the mass of the king size bar.
Comment on your answer. [2]

**4** A mobile phone company has created a new keyboard format for text messaging, which it believes enables faster texting.
To test this theory, the same message was typed by 16 mobile phone users, divided at random into two groups A and B.

Group A has 7 of the users and used the new keyboard format.
Group B has the other 9 users and used the old keyboard format.

The time $x$, in seconds, for each user to complete the message was recorded.

The summary statistics are shown in **Table 2** below.

**Table 2**

| Group | $n$ | $\sum x$ | $\sum x^2$ |
|---|---|---|---|
| A | 7 | 520.1 | 38 740.85 |
| B | 9 | 703.8 | 55 161.96 |

**(i)** Show that the pooled variance estimator is $S_P^2 = 15.873$ (to three decimal places). [8]

**(ii)** Assume the times are Normally distributed.
Test, at a 5% level of significance, the claim that the new keyboard format enables faster texting. [10]



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**5** A computer company is investigating the mean start-up time for its new model of laptop.

The company engineer records the start-up times $x$, in seconds, of a random sample of 40 laptops.

The summary statistics are shown below.

$$\sum x = 900 \qquad \sum x^2 = 20\,330$$

**(i)** Find a 95% confidence interval for the mean start-up time of the new laptops. [9]

For advertising purposes, the company would like to be able to claim that the width of the confidence interval is three-quarters of a second.

**(ii)** Find the percentage confidence that would produce this new interval width. Give your answer to the nearest whole number. [6]


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**(Questions continue overleaf)**



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## SECTION D Discrete and Decision Mathematics

**Answer all six questions in this section.**

Note: There is a Supplementary Answer Booklet for questions **1(b)(ii)** and **2(ii)**
Please attach to your answer booklet using the treasury tag provided.

**1** The complete graph on six vertices $K_6$ is shown in **Fig. 1** below.

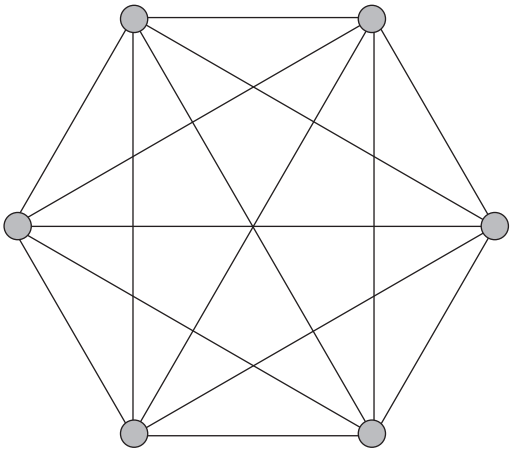


**Fig. 1**

**(a)** What is the minimum number of colours necessary to colour the vertices of $K_6$ so that no two adjacent vertices are identically coloured? [2]

**(b)** The Petersen graph on 10 vertices is shown in **Fig. 2** below.

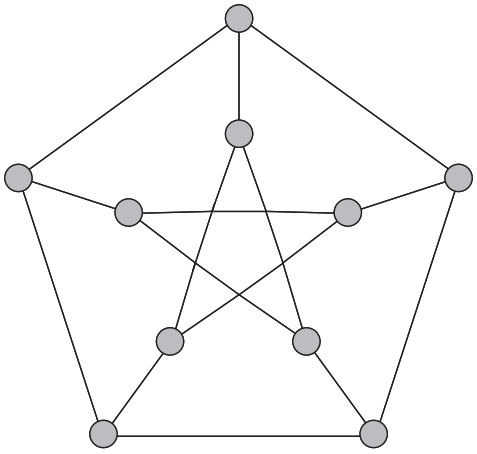


**Fig. 2**

**(i)** How many edges must be added to the Petersen graph to form the graph $K_{10}$? [3]

**(ii)** Use the template of the Petersen graph in the Supplementary Booklet.
Mark on it a minimum vertex colouring, using the least number of colours so that no two adjacent vertices are identically coloured.
(You may denote colours by integers.) [2]


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**(c)** Six people have expressed their preferences for six jobs.

These are listed in the bipartite graph in **Fig. 3** below.

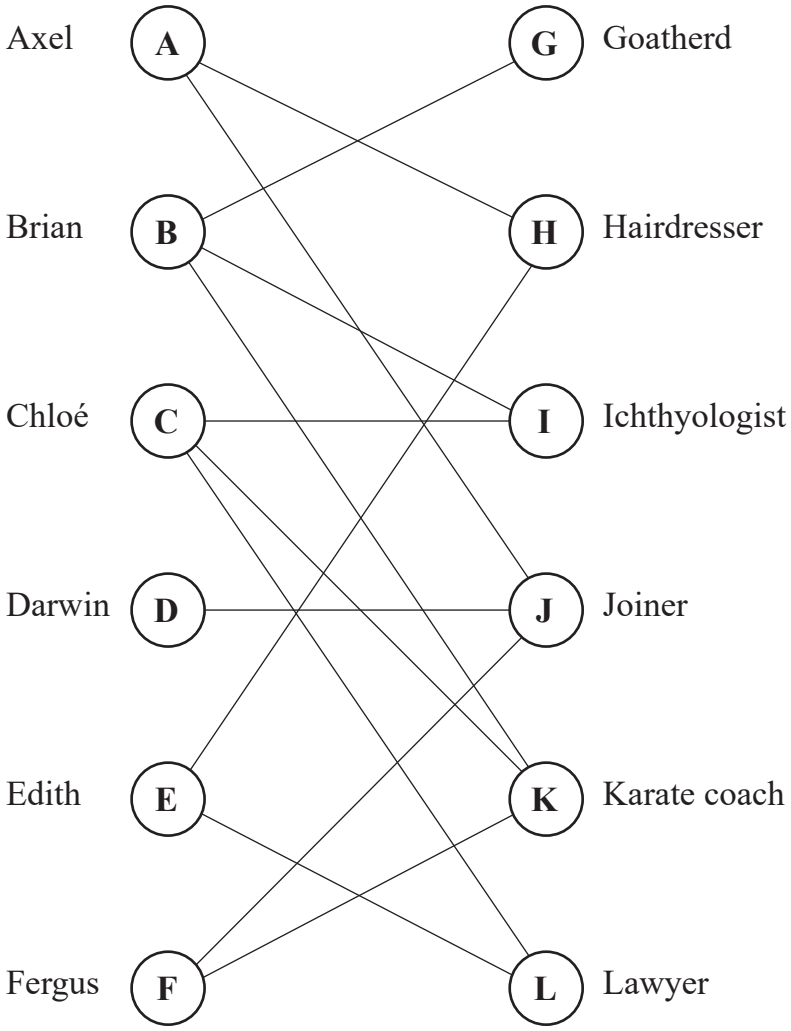



**Fig. 3**

List a possible complete matching for this graph. [2]



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**2** A precedence table is shown in **Fig. 4** below for the activities of a business project. The numbers are times in weeks.

| Activity | Predecessor | Optimal time | Normal time | Pessimistic time | Expected time |
|---|---|---|---|---|---|
| A | – | 4 | 6 | 8 | 6 |
| B | A | 1.5 | 3 | 4.5 | 3 |
| C | A | 2 | 4.25 | 5 | 4 |
| D | A | 2.4 | 3.8 | 4.6 | 3.7 |
| E | B | 1 | 4 | 7 | |
| F | B, C | 2 | 3 | 5.5 | |
| G | C, D | 2 | 4.5 | 7 | |
| H | E, F, G | 5.5 | 7 | 8.5 | |

**Fig. 4**

**(i)** Find the expected times for activities E, F, G and H. [1]

**(ii)** In the Supplementary Booklet, complete the PERT chart for this business project. [4]

**(iii)** List the critical path. [1]

**(iv)** Calculate the probability that the project will take longer than 23 weeks. [7]


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**3** The sequence $a_0,\ a_1,\ a_2,\ a_3,\ a_4, \ldots$ has the generating function

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots$$

**(i)** Write down the sequence which has $\frac{d}{dt}\left[f(t)\right]$ as its generating function. [1]

The series expansion for $\frac{1}{1-t}$ is given by

$$1 + t + t^2 + t^3 + t^4 + \ldots$$

**(ii)** Find the series expansion for $g(t) = \frac{1}{(1-t)^2}$ [2]

**(iii)** Write down the sequence for which $g(t)$ is the generating function. [1]

**(iv)** Hence sum the series

$$1 + \frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \ldots$$

[2]

**(v)** Sum the series

$$\frac{2}{3} + \frac{6}{9} + \frac{12}{27} + \frac{20}{81} + \ldots$$

[4]



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**4** **Fig. 5** below shows a board.

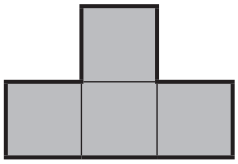


**Fig. 5**

**(i)** Show that the rook polynomial for the board above is

$$1 + 4x + 2x^2$$ [2]

**Fig. 6** below shows a compound board.

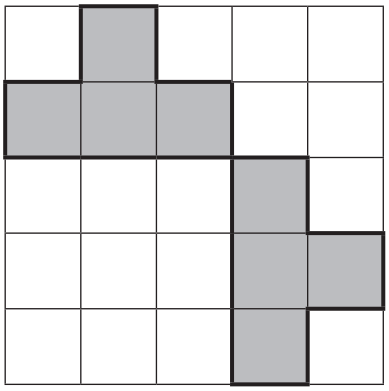


**Fig. 6**

**(ii)** Show that the rook polynomial for this compound board is

$$1 + 8x + 20x^2 + 16x^3 + 4x^4$$ [5]

**(iii)** The manager of a supermarket is assigning employees A, B, C, D and E to shifts 1, 2, 3, 4 and 5, subject to the following rules:

A cannot do shift 2

B cannot do shifts 1, 2, 3

C cannot do shift 4

D cannot do shifts 4 or 5

E cannot do shift 4

Using part **(ii)** and the Rook Inclusion-Exclusion Theorem, show that this can only be achieved in 20 different ways. [7]



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**5** **(a)** The cycle index for a polygon with $p$ vertices, where $p$ is prime, is

$$P_G = \frac{1}{2p}\left(x_1^p + (p-1)x_p^1 + px_1^1 x_2^{\frac{p-1}{2}}\right)$$

**(i)** Write down the cycle index for vertices of a regular seven-sided polygon. [2]

A jeweller designs a necklace with seven equally-spaced stones along its length. Each stone may be a beryl, a topaz or an amethyst.

**(ii)** How many different necklaces are possible? [3]

**(b)** The cycle index for the face symmetries of an octahedron is

$$P_G = \frac{1}{24}\left(x_1^8 + 6x_4^2 + 9x_2^4 + 8x_1^2 x_3^2\right)$$

**(i)** Find the pattern inventory for the colouring of the faces of an octahedron using two colours, red (R) and blue (B).
(You need not fully expand the expression.) [3]

**(ii)** How many of these possible colourings have five red faces and three blue faces? [5]



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**6** A linear programming problem is stated as:

Maximise $P = 2x + 3y$ subject to the constraints

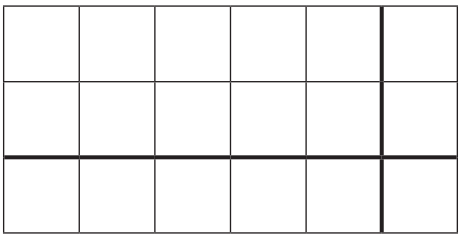
$$2x + y \leqslant 10$$

$$x + 3y \leqslant 12$$

where $x$ and $y$ are positive.

**(i)** Restate the problem in terms of equalities in preparation for the simplex method. [3]

**(ii)** Hence copy the grid below and fill out an initial simplex tableau for this problem.



[3]

**(iii)** Perform two iterations of the simplex method, identifying the pivot element and writing out the tableau in each case. [7]

**(iv)** Hence find the optimum value of $P$ and the corresponding values of $x$ and $y$. [3]

**THIS IS THE END OF THE QUESTION PAPER**



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| Surname | StemUp Model Solution |
|---|---|
| Answer booklet | Write the question number in the left-hand margin at the start of each answer. |

Question number

Section A Mechanics 1

A1 (i) Amplitude $a = \frac{1}{2} \times 4 = 2$ m

(ii) Maximum acceleration $= a\omega^2$

$6 = 2\omega^2 \Rightarrow \omega^2 = 3 \Rightarrow \omega = \sqrt{3}$

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3}} = 3.6276...$

$T = 3.63$ s (3 s.f.)

(iii) $v^2 = \omega^2(a^2 - x^2)$

$v^2 = 3(2^2 - 0.5^2) = 3 \times 3.75 = 11.25$

$v = \frac{3\sqrt{5}}{2} = 3.354...$

Speed at C = 3.35 m $s^{-1}$ (3 s.f.)

(iv) Starting at O: $x = a \sin \omega t$

$0.5 = 2 \sin(\sqrt{3}\, t)$

$\sin(\sqrt{3}\, t) = 0.25$

$\sqrt{3}\, t = \sin^{-1}(0.25) \Rightarrow t = \frac{\sin^{-1}(0.25)}{\sqrt{3}} = \frac{0.25268}{1.73205}$

Least time = 0.146 s (3 s.f.)

A2 (i) Take distances of centres of mass from DC.

Triangle: base is 4 + 15 = 19 cm below AB, so 21 cm above DC;

Question number

its centroid is ⅓ × 15 = 5 cm above its base: 26 cm from DC.

Circle: centre is 4 + 6 = 10 cm above DC.

| Shape | Mass | Distance | Mass × distance |
|---|---|---|---|
| Rectangle | 2400m | 20 | 48000m |
| Triangle | 90m | 26 | 2340m |
| Circle | 36πm | 10 | 360πm |
| Sign | $(2310 - 36\pi)m$ | $\bar{y}$ | $(2310 - 36\pi)m\bar{y}$ |

(rectangle 60 × 40 = 2400, triangle ½ × 12 × 15 = 90, circle $\pi \times 6^2 = 36\pi$)

Moments about DC:

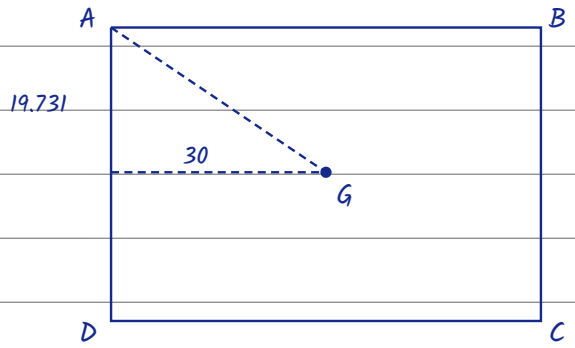
$(2400m - 90m - 36\pi m)\bar{y} = 48000m - 2340m - 360\pi m$

$$\bar{y} = \frac{45660 - 360\pi}{2310 - 36\pi} = \frac{44529.03}{2196.90} = 20.269...$$

Centre of mass is 20.3 cm from DC (3 s.f.)

(and 30 cm from AD, on the axis of symmetry).

(ii)




When suspended from A, AG is vertical.

Vertical distance of G below AB = 40 − 20.269 = 19.731 cm

Horizontal distance of G from AD = 30 cm

$$\tan\theta = \frac{30}{40 - 20.269} = \frac{30}{19.731}$$

$\theta = 56.7°$ (3 s.f.), the angle AD makes with the vertical.

Question number

**A3 (i)** $R_B = 0$, since on the point of toppling inwards (about A) wheel B is about to lose contact with the road.

**(ii)** The centre of the circle is towards the lower side (A).

Forces: $R_A$ at A perpendicular to road, Fr up the slope, weight mg at G. G is 0.4 m (along the road) from A and 1 m above the road.

Bike is not rotating, so take moments about G:

$R_A \times 0.4 = Fr \times 1$

$Fr = 0.4R_A$

Resolve horizontally (towards the centre):

$$R_A \sin 30° - Fr \cos 30° = \frac{mv^2}{20}$$

Resolve vertically:

$R_A \cos 30° + Fr \sin 30° = mg$

Divide the first equation by the second:

$$\frac{R_A \sin 30° - Fr \cos 30°}{R_A \cos 30° + Fr \sin 30°} = \frac{v^2}{20g}$$

Substitute $Fr = 0.4R_A$ and divide through by $R_A \cos 30°$:

$$\frac{\tan 30° - 0.4}{1 + 0.4 \tan 30°} = \frac{v^2}{20g}$$

$$v^2 = \frac{20 \times 9.8(\tan 30° - 0.4)}{1 + 0.4 \tan 30°} = \frac{196 \times 0.17735}{1.23094} = 28.24$$

$v = 5.31 \text{ m s}^{-1}$ (3 s.f.)

**A4 (i)** Take x (compression) positive into the buffer.

Spring thrust (Hooke): $T = \frac{\lambda x}{l} = \frac{\lambda x}{0.25} = 4\lambda x$

Question number

Dashpot resistance = $k\dot{x} = 3\dot{x}$

Both forces oppose x increasing. Newton II:

$-T - 3\dot{x} = 0.6\ddot{x}$

$-4\lambda x - 3\dot{x} = 0.6\ddot{x}$

$0.6\ddot{x} + 3\dot{x} + 4\lambda x = 0$

Divide by 0.6:

$\ddot{x} + 5\dot{x} + \frac{20}{3}\lambda x = 0$ as required

(ii) Auxiliary equation: $m^2 + 5m + \frac{20}{3}\lambda = 0$

Under-damped $\Leftrightarrow$ no real roots $\Leftrightarrow$ discriminant $< 0$

$25 - 4 \times \frac{20}{3}\lambda < 0$

$25 < \frac{80}{3}\lambda$

$\lambda > \frac{15}{16}$ $(\lambda > 0.9375)$

(iii) $\lambda = 0.9$: $\frac{20}{3} \times 0.9 = 6$

$\ddot{x} + 5\dot{x} + 6x = 0$

$m^2 + 5m + 6 = 0$

$(m + 2)(m + 3) = 0 \Rightarrow m = -2, -3$

$x = Ae^{-2t} + Be^{-3t}$

AS (i) In triangle BFC, $\angle FBC = \angle FCB = 60°$, so $\angle BFC = 60°$:

BFC is equilateral, so $BF = BC = 2$ m.

Question number

$\angle ABF = 180° - 60° = 120°$, so in triangle ABF

$\angle AFB = 180° - 30° - 120° = 30° = \angle FAB$

Triangle ABF is isosceles, so $AB = BF = 2$ m as required

(ii)

By symmetry $CD = AB = 2$ m, so $AD = 6$ m.

Moments about A: $90 \times 2 = R_D \times 6$

$R_D = 30$ N

Resolve vertically: $R_A + R_D = 90$

$R_A = 60$ N

(iii)

Take all rod forces as tensions.

At E: rods ED and EF lie along the line DF, and EC is perpendicular to DF ($\angle CED = 90°$). No external load at E.

Resolve perpendicular to DF at E: $T_{CE} = 0$ N

At C: rods CB and CD are horizontal, CF and CE make 60° with the horizontal. No external load at C.

Resolve vertically at C: $T_{FC} \sin 60° + T_{CE} \sin 60° = 0$

Since $T_{CE} = 0$, $T_{FC} = 0$ N as required

(iv)

At D: $R_D = 30$ N up, DE at 30° above horizontal (towards E), DC horizontal (towards C).

Resolve vertically at D: $T_{DE} \sin 30° + 30 = 0$

$T_{DE} = -60$ N

Force in DE is 60 N, a thrust.

Resolve horizontally at D: $T_{CD} + T_{DE} \cos 30° = 0$

$T_{CD} = 60 \cos 30° = 30\sqrt{3} = 52.0$ N (3 s.f.)

Force in CD is $30\sqrt{3}$ N (52.0 N), a tension.

Resolve horizontally at C ($T_{CE} = T_{FC} = 0$): $T_{BC} = T_{CD}$

Question number

Force in BC is $30\sqrt{3}$ N (52.0 N), a tension.

Section B Mechanics 2

B1 (i) $\angle CAB = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

Resolve parallel to AB (→):

$X = 8 - 6 + 12\cos 60° = 8$ N

Resolve parallel to BC (↑):

$Y = 6 + 12\sin 60° = 6 + 6\sqrt{3} = 16.39$ N

$R = \sqrt{(8^2 + (6 + 6\sqrt{3})^2)} = \sqrt{332.71}$

$R = 18.2$ N (3 s.f.)

$\theta = \tan^{-1}\left(\frac{6 + 6\sqrt{3}}{8}\right) = 64.0°$

R = 18.2 N acting at 64.0° to AB (measured from AB towards AD).

(ii) Let R cut line AB at distance x from A.

Moments about A (anticlockwise): 8 N and 12 N pass through A.

$6 \times 1 + 6 \times \sqrt{3} = (6 + 6\sqrt{3})x$

(the 8 N component of R acts along AB, so has no moment)

$x = 1$, and AB = 1 m

So R acts through B as required

B2 (i) $a = \frac{dv}{dt} = (-6t)i + 4j + (3t^2)k \ \text{m s}^{-2}$

(ii) $r = \int v\,dt = (-t^3 + 12t)i + (2t^2 - t)j + (\tfrac{1}{4}t^4 - 8t)k + c$

$t = 0, r = i + 2j \Rightarrow c = i + 2j$

Question number

$r = (-t^3 + 12t + 1)i + (2t^2 - t + 2)j + (\frac{1}{4}t^4 - 8t)k$

$t = 4$: $r = (-64 + 48 + 1)i + (32 - 4 + 2)j + (64 - 32)k$

$r = -15i + 30j + 32k$

Distance $= \sqrt{(15^2 + 30^2 + 32^2)} = \sqrt{2149}$

Distance = 46.4 m (3 s.f.)

(iii) Parallel to j: the i and k components of v are zero

(and the j component is non-zero).

i: $-3t^2 + 12 = 0 \Rightarrow t = \pm 2$

k: $t^3 - 8 = 0 \Rightarrow t = 2$

Both hold at $t = 2$, where the j component $= 4(2) - 1 = 7 \neq 0$.

The particle moves parallel to j when $t = 2$ s.

B3 (i) Let $v_A$, $v_B$ be the velocities after impact (→ positive).

Momentum: $6mu = 6mv_A + 4mv_B \Rightarrow 3u = 3v_A + 2v_B$ ...(1)

Restitution: $v_B - v_A = 0.25u \Rightarrow v_A - v_B = -0.25u$ ...(2)

(1) + 2 × (2): $2.5u = 5v_A$

$v_A = 0.5u$

$v_B = 0.5u + 0.25u = 0.75u$

Both move in the original direction of A.

(ii) B (4m, speed $0.75u$) hits C (m, at rest).

Momentum: $4m(0.75u) = 4mv_B + mv_C \Rightarrow 3u = 4v_B + v_C$ ...(3)

Restitution: $v_C - v_B = 0.75eu$ ...(4)

(3) − (4): $3u - 0.75eu = 5v_B$

$v_B = 0.15u(4 - e)$

$v_C = 0.15u(4 - e) + 0.75eu = 0.6u + 0.6eu$

$v_C = 0.6u(1 + e)$

Question number

(iii) A moves at $0.5u$ and B at $0.15u(4 - e)$, both in the same direction. A and B collide again if A is faster than B:

$0.5u > 0.15u(4 - e)$

$0.5 > 0.6 - 0.15e$

$0.15e > 0.1 \Rightarrow e > \frac{2}{3}$

Since $e \leq 1$: $\frac{2}{3} < e \leq 1$

B4 Newton II along the road:

$10e^{-0.02t} - 3v = 100\frac{dv}{dt}$

$\frac{dv}{dt} + 0.03v = 0.1e^{-0.02t}$

Integrating factor $= e^{\int 0.03\,dt} = e^{0.03t}$

$e^{0.03t}\frac{dv}{dt} + 0.03ve^{0.03t} = 0.1e^{0.01t}$

$\frac{d}{dt}(ve^{0.03t}) = 0.1e^{0.01t}$

Integrate: $ve^{0.03t} = 10e^{0.01t} + c$

$v = 10e^{-0.02t} + ce^{-0.03t}$

$t = 0, v = 9$: $9 = 10 + c \Rightarrow c = -1$

$v = 10e^{-0.02t} - e^{-0.03t}$

$t = 30$: $v = 10e^{-0.6} - e^{-0.9} = 5.4881 - 0.4066$

$v = 5.08\ m\ s^{-1}$ (3 s.f.)

B5 (i) Place the base on the y-axis with the axis of symmetry

Question number

along the x-axis: the curved surface is $x^2 + y^2 = r^2$, $0 \leq x \leq r$.

Moments about the base: $(\frac{2}{3}\pi r^3\rho)d = \rho\int_0^r \pi xy^2\,dx$

$\rho\int_0^r \pi xy^2\,dx = \rho\pi\int_0^r x(r^2 - x^2)\,dx = \rho\pi\int_0^r (r^2x - x^3)\,dx$

$= \rho\pi[\frac{1}{2}r^2x^2 - \frac{1}{4}x^4]$ from 0 to r

$= \rho\pi(\frac{1}{2}r^4 - \frac{1}{4}r^4) = \frac{1}{4}\rho\pi r^4$

$\frac{2}{3}\pi\rho r^3 d = \frac{1}{4}\rho\pi r^4$

$d = \frac{3\rho\pi r^4}{2 \times 4\pi\rho r^3} = \frac{3}{8}r$ as required

(ii) Distances measured from O along the axis, towards the curved surface of the hemisphere positive.

| Part | Mass | Distance | Mass × distance |
|---|---|---|---|
| Hemisphere | $\frac{2}{3}\pi\rho(3a)^3 = 18\pi\rho a^3$ | $\frac{3}{8}(3a) = \frac{9}{8}a$ | $\frac{81}{4}\pi\rho a^4$ |
| Cylinder | $\pi\rho a^2(3a) = 3\pi\rho a^3$ | $-\frac{3}{2}a$ | $-\frac{9}{2}\pi\rho a^4$ |
| Toy | $21\pi\rho a^3$ | $\bar{y}$ | $21\pi\rho a^3\bar{y}$ |

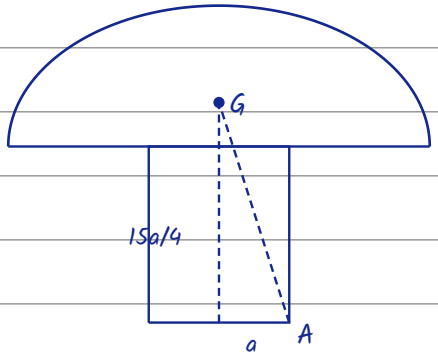
$21\pi\rho a^3\bar{y} = \frac{81}{4}\pi\rho a^4 - \frac{9}{2}\pi\rho a^4 = \frac{63}{4}\pi\rho a^4$

$\bar{y} = \frac{63}{4 \times 21}a = \frac{3}{4}a$

Centre of mass is $\frac{3}{4}a$ from O (on the hemisphere side).

Question number

(iii)




The toy stands on the plane face of the cylinder (radius a).

Height of G above that face = 3a + ¾a = 15a/4

On the point of toppling, G lies vertically above the lowest point A of the base, so the angle at G between the axis and GA equals θ:

$$\tan\theta = \frac{a}{3a + \frac{3}{4}a} = \frac{4}{15}$$

θ = 14.9° (3 s.f.)

Maximum θ = 14.9°

Section C Statistics

C1 (i) $E(X) = E(2W - 0.5Y) = 2E(W) - 0.5E(Y)$

$= 2(5) - 0.5(4) = 8$ as required

W and Y are independent, so

$Var(X) = 2^2Var(W) + 0.5^2Var(Y)$

$= 4(0.5) + 0.25(4) = 3$ as required

(ii) For random samples of size n from any population with finite mean μ and variance $\sigma^2$, the sample mean $\bar{X}$ is approximately Normally distributed when n is large:

Question number

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately.

(iii) $n = 40$ is large, so $\bar{X} \sim N(8, \frac{3}{40})$ approximately.

$P(\bar{X} > 8.5) = P(Z > \frac{8.5 - 8}{\sqrt{3/40}}) = P(Z > 1.826)$

$= 1 - \Phi(1.826)$

$= 0.0339$ (3 s.f.), i.e. about 0.034

C2 $H_0$: school age group and level of attendance are independent

$H_1$: school age group and level of attendance are not independent

| Observed | Good | Satisfactory | Poor | Total |
|---|---|---|---|---|
| Junior | 46 | 22 | 12 | 80 |
| Senior | 24 | 20 | 16 | 60 |
| Total | 70 | 42 | 28 | 140 |

Expected (row total × column total ÷ 140), e.g. $\frac{80 \times 70}{140} = 40$, $\frac{80 \times 42}{140} = 24$

| Expected | Good | Satisfactory | Poor | Total |
|---|---|---|---|---|
| Junior | 40 | 24 | 16 | 80 |
| Senior | 30 | 18 | 12 | 60 |

All expected frequencies ≥ 5. $\chi^2 = \Sigma\frac{(O - E)^2}{E}$

| O | E | $(O - E)^2$ | $(O - E)^2/E$ |
|---|---|---|---|
| 46 | 40 | 36 | 0.9 |
| 22 | 24 | 4 | 0.1667 |
| 12 | 16 | 16 | 1 |
| 24 | 30 | 36 | 1.2 |

Question number

| | | | |
|---|---|---|---|
| 20 | 18 | 4 | 0.2222 |
| 16 | 12 | 16 | 1.3333 |

$\chi^2 = \frac{217}{45} = 4.82$ (3 s.f.)

Degrees of freedom $\nu = (2 - 1)(3 - 1) = 2$

Critical value at 5%: $\chi^2_{5\%}(2) = 5.991$

$4.82 < 5.991$, so do not reject $H_0$.

There is insufficient evidence at the 5% level to suggest that school age group and level of attendance are dependent.

C3 (i)

$X \sim N(55, 4^2)$

$E(D) = 3.5E(X) - 3E(X) = 0.5 \times 55 = 27.5$ g

$Var(D) = 3.5^2 Var(X) + Var(X_1) + Var(X_2) + Var(X_3)$

$= 12.25 \times 16 + 3 \times 16 = 15.25 \times 16 = 244$ g$^2$

$D \sim N(27.5, 244)$

King size heavier $\Leftrightarrow D > 0$

$P(D > 0) = P\left(Z > \frac{0 - 27.5}{\sqrt{244}}\right) = P(Z > -1.761)$

$= 1 - \Phi(-1.761) = \Phi(1.761)$

$= 0.961$ (3 s.f.)

(ii)

It is assumed that the variance of the king size bar is $3.5^2 = 12.25$ times the variance of a standard bar, i.e. $3.5^2 \times 16 = 196$ g$^2$.

In practice the king size bar is made as one bar, so its variance is likely to be much smaller than this.

Question number

C4 (i)

$\bar{x}_A = \frac{520.1}{7} = 74.3$

$s_A^2 = \frac{1}{6}(38740.85 - 7 \times 74.3^2) = \frac{97.42}{6} = 16.2367$

$\bar{x}_B = \frac{703.8}{9} = 78.2$

$s_B^2 = \frac{1}{8}(55161.96 - 9 \times 78.2^2) = \frac{124.8}{8} = 15.6$

$s_p^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2}$

$s_p^2 = \frac{6(16.2367) + 8(15.6)}{14} = \frac{222.22}{14}$

$s_p^2 = 15.873$ (3 d.p.) as required

(ii)

$H_0$: $\mu_A = \mu_B$

$H_1$: $\mu_A < \mu_B$ (new format faster, i.e. shorter mean time)

Small samples, Normal populations, common variance:

two-sample t-test, lower one-tailed, $\nu = 7 + 9 - 2 = 14$

Critical value at 5%: $t = -1.761$

Test statistic: $t = \frac{\bar{x}_A - \bar{x}_B}{s_p\sqrt{\frac{1}{7} + \frac{1}{9}}}$

$t = \frac{74.3 - 78.2}{\sqrt{15.873} \times \sqrt{\frac{1}{7} + \frac{1}{9}}} = \frac{-3.9}{2.0078}$

$t = -1.942$

$-1.942 < -1.761$, so reject $H_0$.

Question number

There is sufficient evidence at the 5% level to support the claim that the new keyboard format enables faster texting.

C5 (i) $\bar{x} = \frac{900}{40} = 22.5$

$S^2 = \frac{1}{39}(20330 - 40 \times 22.5^2) = \frac{80}{39} = 2.0513$

95% CI: $\bar{x} \pm 1.96\frac{S}{\sqrt{n}}$

$= 22.5 \pm 1.96 \times \frac{\sqrt{80/39}}{\sqrt{40}} = 22.5 \pm 0.4439$

95% CI = (22.1, 22.9) seconds (3 s.f.)

(ii) Width $= 2z\frac{S}{\sqrt{n}}$

$0.75 = 2z \times \frac{\sqrt{80/39}}{\sqrt{40}} = 2z \times 0.22646$

$z = 1.656$

$P(Z < 1.656) = 0.9511$

$1 - 0.9511 = 0.0489$ in each tail

Confidence = 100% − 2 × 4.89% = 90.2%

Confidence level = 90% (nearest whole number)

Section D Discrete and Decision Mathematics

D1 (a) In $K_6$ every vertex is adjacent to every other vertex, so all six vertices must have different colours.

Minimum number of colours = 6

Question number

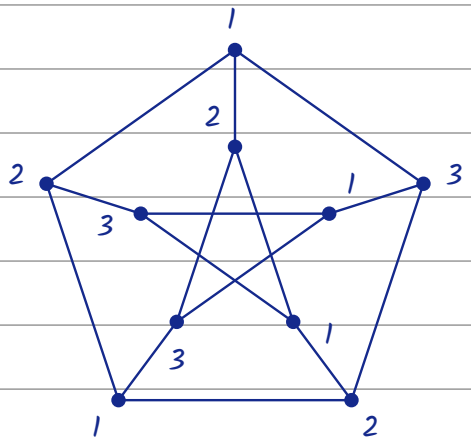
(b)(i) In $K_{10}$ each vertex is joined to the other 9 vertices.

Edges in $K_{10} = \frac{10 \times 9}{2} = 45$

The Petersen graph has 15 edges.

Edges to add = 45 − 15 = 30

(ii)




Colours 1, 2, 3 marked on the vertices above (no two adjacent vertices share a colour).

The outer pentagon is an odd cycle, so 2 colours are not enough: the minimum number of colours is 3.

(c) Axel – Hairdresser (A – H)

Brian – Goatherd (B – G)

Chloé – Ichthyologist (C – I)

Darwin – Joiner (D – J)

Edith – Lawyer (E – L)

Fergus – Karate coach (F – K)

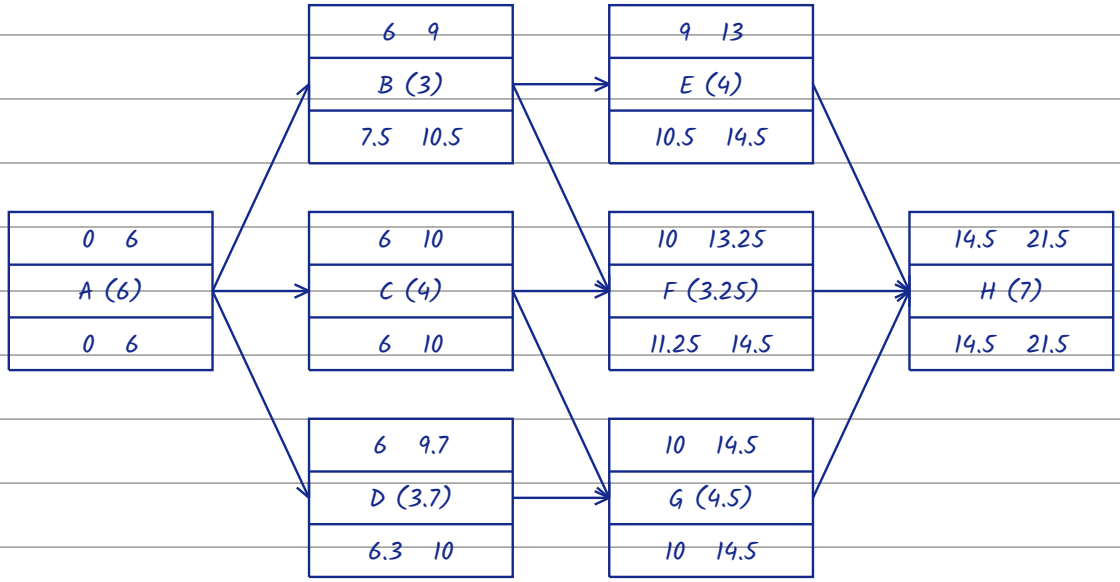
D2 (i) Expected time $= \frac{a + 4m + b}{6}$

Question number

E: $\frac{1 + 16 + 7}{6} = 4$ F: $\frac{2 + 12 + 5.5}{6} = 3.25$

G: $\frac{2 + 18 + 7}{6} = 4.5$ H: $\frac{5.5 + 28 + 8.5}{6} = 7$

(ii)




Key: top row ES EF; middle: activity (expected time);

bottom row LS LF.

| Activity | ES | EF | LS | LF | Float |
|---|---|---|---|---|---|
| A | 0 | 6 | 0 | 6 | 0 |
| B | 6 | 9 | 7.5 | 10.5 | 1.5 |
| C | 6 | 10 | 6 | 10 | 0 |
| D | 6 | 9.7 | 6.3 | 10 | 0.3 |
| E | 9 | 13 | 10.5 | 14.5 | 1.5 |
| F | 10 | 13.25 | 11.25 | 14.5 | 1.25 |
| G | 10 | 14.5 | 10 | 14.5 | 0 |
| H | 14.5 | 21.5 | 14.5 | 21.5 | 0 |

Minimum project completion time = 21.5 weeks

Question number

(iii) Critical path: A C G H (all have zero float)

(iv) Expected time = 6 + 4 + 4.5 + 7 = 21.5 weeks

Variance of each activity = $(\frac{b-a}{6})^2$

A: $(\frac{4}{6})^2 = \frac{16}{36}$ C: $(\frac{3}{6})^2 = \frac{9}{36}$ G: $(\frac{5}{6})^2 = \frac{25}{36}$ H: $(\frac{3}{6})^2 = \frac{9}{36}$

Project variance = $\frac{16+9+25+9}{36} = \frac{59}{36} = 1.639$

Project time $T \sim N(21.5, 1.639)$

$P(T > 23) = P(Z > \frac{23-21.5}{\sqrt{1.639}}) = P(Z > 1.1717)$

$= 1 - \Phi(1.1717)$

$= 0.121$ (3 s.f.)

D3 (i) $\frac{d}{dt}[f(t)] = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \ldots$

Sequence: $a_1, 2a_2, 3a_3, 4a_4, \ldots$

(ii) $g(t) = \frac{1}{(1-t)^2} = \frac{d}{dt}(\frac{1}{1-t})$

$= \frac{d}{dt}(1 + t + t^2 + t^3 + t^4 + \ldots)$

$g(t) = 1 + 2t + 3t^2 + 4t^3 + \ldots$

(iii) Sequence: 1, 2, 3, 4, ...

Question number

(iv) $1 + \frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \ldots = g(\frac{1}{5})$

$= \frac{1}{(1-\frac{1}{5})^2} = \frac{1}{(\frac{4}{5})^2} = \frac{25}{16}$

(v) $g'(t) = \frac{2}{(1-t)^3} = 2 + 6t + 12t^2 + 20t^3 + \ldots$

$tg'(t) = \frac{2t}{(1-t)^3} = 2t + 6t^2 + 12t^3 + 20t^4 + \ldots$

$\frac{2}{3} + \frac{6}{9} + \frac{12}{27} + \frac{20}{81} + \ldots = tg'(t)$ with $t = \frac{1}{3}$

$= \frac{2 \times \frac{1}{3}}{(1-\frac{1}{3})^3} = \frac{2}{3} \div \frac{8}{27} = \frac{9}{4}$

D4 (i) 0 rooks: 1 way

1 rook: 4 ways (4 squares)

2 rooks (non-attacking): 2 ways

3 rooks: impossible

Rook polynomial = $1 + 4x + 2x^2$ as required

(ii) The compound board is made of two copies of the board in (i) with no rows or columns in common, so its rook polynomial is the product of their rook polynomials:

$(1 + 4x + 2x^2)(1 + 4x + 2x^2)$

$= 1 + 4x + 2x^2 + 4x + 16x^2 + 8x^3 + 2x^2 + 8x^3 + 4x^4$

$= 1 + 8x + 20x^2 + 16x^3 + 4x^4$ as required

Question number

(iii) Forbidden positions (X):

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | | X | | | |
| B | X | X | X | | |
| C | | | | X | |
| D | | | | X | X |
| E | | | | X | |

Rows A, B with columns 1, 2, 3 give the board in (i);

rows C, D, E with columns 4, 5 give another copy of it.

They share no rows or columns, so the forbidden board is the

compound board of (ii), with rook polynomial

$1 + 8x + 20x^2 + 16x^3 + 4x^4$, so $r_1 = 8$, $r_2 = 20$, $r_3 = 16$, $r_4 = 4$, $r_5 = 0$.

Rook Inclusion-Exclusion Theorem:

Number of ways $= 5! - r_1 4! + r_2 3! - r_3 2! + r_4 1! - r_5 0!$

$= 120 - 8 \times 24 + 20 \times 6 - 16 \times 2 + 4 \times 1 - 0$

$= 120 - 192 + 120 - 32 + 4$

$= 20$ ways as required

D5 (a)(i) $p = 7$: $P_G = \frac{1}{14}(x_1^7 + 6x_7 + 7x_1x_2^3)$

(ii) Three colours: replace every $x_i$ by 3.

Number $= \frac{1}{14}(3^7 + 6 \times 3 + 7 \times 3 \times 3^3)$

$= \frac{1}{14}(2187 + 18 + 567) = \frac{2772}{14}$

$= 198$ different necklaces

Question number

(b)(i) Replace $x_i$ by $R^i + B^i$:

$$\frac{1}{24}[(R+B)^8 + 6(R^4+B^4)^2 + 9(R^2+B^2)^4 + 8(R+B)^2(R^3+B^3)^2]$$

(ii) Coefficient of $R^5B^3$ in each term:

$(R+B)^8$: ${}^8C_5 = 56$

$6(R^4+B^4)^2$ and $9(R^2+B^2)^4$: no $R^5B^3$ term (wrong powers)

$8(R+B)^2(R^3+B^3)^2 = 8(R^2+2RB+B^2)(R^6+2R^3B^3+B^6)$:

$R^5B^3$ comes only from $R^2 \times 2R^3B^3$, giving $8 \times 2 = 16$

Total $= \frac{1}{24}(56+16) = \frac{72}{24} = 3$

3 colourings have five red faces and three blue faces.

D6 (i) Introduce slack variables $u, v \geq 0$:

Maximise $P$ where $P - 2x - 3y = 0$

subject to $2x + y + u = 10$

$x + 3y + v = 12$

with $x, y, u, v \geq 0$

(ii)

| | $x$ | $y$ | $u$ | $v$ | $P$ | Value |
|---|---|---|---|---|---|---|
| $u$ | 2 | 1 | 1 | 0 | 0 | 10 |
| $v$ | 1 | 3 | 0 | 1 | 0 | 12 |
| $P$ | −2 | −3 | 0 | 0 | 1 | 0 |

(iii) Iteration 1: pivot column $y$ (most negative entry, $-3$).

Ratios: $10 \div 1 = 10$, $12 \div 3 = 4$ (smallest)

Question number

Pivot element: 3 in the v row, y column.

Divide pivot row by 3, then R1 – new R2, R3 + 3 × new R2:

| | x | y | u | v | P | Value |
|---|---|---|---|---|---|---|
| u | 5/3 | 0 | 1 | −1/3 | 0 | 6 |
| y | 1/3 | 1 | 0 | 1/3 | 0 | 4 |
| P | −1 | 0 | 0 | 1 | 1 | 12 |

Iteration 2: pivot column x (only negative entry, −1).

Ratios: $6 \div \frac{5}{3} = 3.6$, $4 \div \frac{1}{3} = 12$; 3.6 is smallest

Pivot element: 5/3 in the u row, x column.

Divide pivot row by 5/3, then R2 – ⅓ new R1, R3 + new R1:

| | x | y | u | v | P | Value |
|---|---|---|---|---|---|---|
| x | 1 | 0 | 3/5 | −1/5 | 0 | 18/5 |
| y | 0 | 1 | −1/5 | 2/5 | 0 | 14/5 |
| P | 0 | 0 | 3/5 | 4/5 | 1 | 78/5 |

(iv) No negative entries in the P row, so this is optimal.

Maximum P = 78/5 = 15.6

when x = 18/5 = 3.6 and y = 14/5 = 2.8

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