New Specification

Rewarding Learning

ADVANCED

General Certificate of Education

Further Mathematics *StemUp Model Solution*

AFM11

Assessment Unit A2 1

assessing

Pure Mathematics

[AFM11]

Assessment

Assessment Level of Control:

Tick the relevant box (✓)

Controlled Conditions	
Other	

TIME

2 hours 15 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided.
Answer **all twelve** questions.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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Answer all twelve questions.

Show clearly the full development of your answers.

Answers should be given to three significant figures unless otherwise stated.

1 Show that

$$\sum_{r=1}^{2n} (2r-1)^2 \equiv \frac{2n}{3}(16n^2-1) \qquad n \geqslant 1 \qquad [6]$$

2 **(a)** Using the substitution $u = 2x + 1$ show that

$$\int \frac{\mathrm{d}x}{\sqrt{4x^2+4x+3}} = \frac{1}{2}\ln\left(2x+1+\sqrt{4x^2+4x+3}\right) + c \qquad [6]$$

(b) Evaluate the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \qquad [4]$$

3 **(i)** Express

$$\mathrm{f}(x) = \frac{2x^2+x+7}{(x-1)(x^2+4)}$$

in partial fractions. [7]

(ii) Hence show that

$$\int \mathrm{f}(x)\,\mathrm{d}x = 2\ln(x-1) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c \qquad [3]$$

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4 **(i)** Given that

$$I_n = \int (\ln x)^n \, dx$$

show that for $n \geqslant 1$

$$I_n = x\,(\ln x)^n - n\,I_{n-1}$$ [6]

(ii) Evaluate

$$\int_1^2 (\ln x)^2 \, dx$$ [4]

5 Using mathematical induction, prove that

$$\sum_{r=1}^{n} 3^{r-1} = \frac{3^n - 1}{2} \qquad n \geqslant 1$$ [6]

6 The rate at which a bank's assets grow is modelled by

$$t\,\frac{dA}{dt} - A = t$$

where A represents the assets of the bank in £ millions and t is the number of years since the setting up of the bank.

(i) Given that the assets of the bank are £300 million after 20 years, find a formula expressing A in terms of t. [8]

(ii) Estimate the assets of the bank 50 years after set up. [2]

(iii) State possible limitations of the model. [2]

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7 **(a)** **(i)** Given that

$$f(x) = \ln\left(2x + \sqrt{1 + 4x^2}\right)$$

show that

$$f'(x) = \frac{2}{\sqrt{1 + 4x^2}}$$ [5]

(ii) Hence find the Maclaurin expansion of $f(x)$ up to and including the term in x^3 [4]

(b) Using small angle approximations, show that

$$\lim_{x \to 0} \left(\frac{1 - \cos 6x - 2x \sin 3x}{x^2}\right) = 12$$ [4]

8 **(a)** Using the exponential definitions of hyperbolic functions, prove that

$$\cosh (x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$$ [6]

(b) Solve the equation

$$4 \cosh x + 5 \sinh x = 2$$

leaving the answer in logarithmic form. [7]

9 **(i)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 20x - 6$$ [10]

(ii) Given that $\frac{dy}{dx} = 6$ and $y = 0$, when $x = 0$, find y in terms of x. [5]

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10 (a) The polar equation of a curve C is

$$r = 3 \sin 2\theta$$

Find a cartesian equation for C. [6]

(b) **Fig. 1** below shows the graph of the finite region bounded by the half-lines $\theta = 0$ and $\theta = \frac{\pi}{4}$ and the curve with polar equation $r = a(1 + \tan\theta)$ where a is a positive constant.

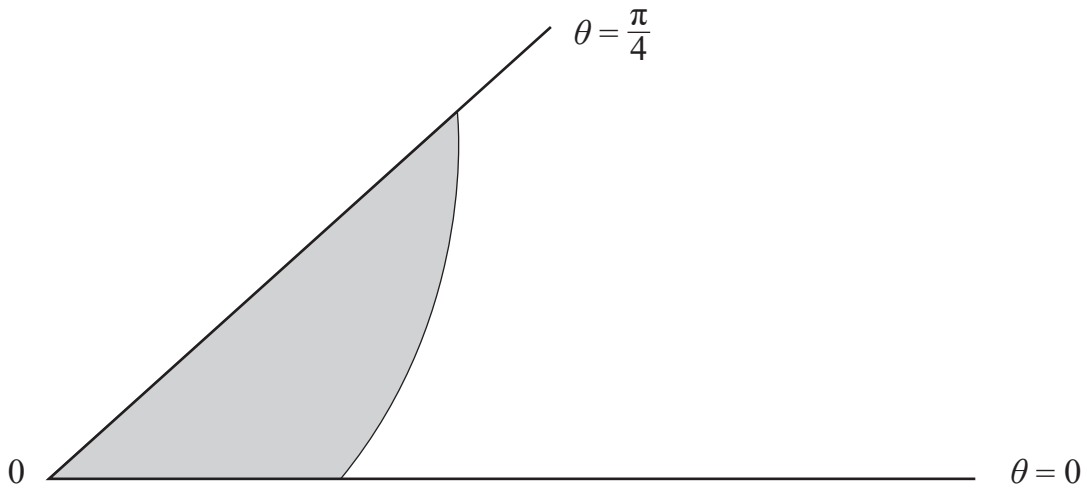


Fig. 1

Find, in terms of a, the area of the shaded region. [8]

11 (a) Given that n is a positive integer, show that

$$(1 + \mathrm{i})^{4n} - (1 - \mathrm{i})^{4n} = 0$$ [8]

(b) (i) Using De Moivre's theorem, express $\cos 4\theta$ in terms of $\sin\theta$ and $\cos\theta$ [4]

(ii) Using **(i)** show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$ [2]

(iii) Hence using $\theta = \frac{\pi}{12}$, show that

$$\cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2 + \sqrt{3}}$$ [7]

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12 **(i)** By using exponential definitions, show that

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$ [6]

(ii) Using **(i)**, or otherwise, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1}(\sin x)\right) = \sec x$$

where $x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$ [6]

(iii) Hence, or otherwise, find

$$\int_0^{\frac{\pi}{6}} \sec^2 x \ \tanh^{-1}(\sin x)\,\mathrm{d}x$$ [8]

THIS IS THE END OF THE QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 $(2r-1)^2 = 4r^2 - 4r + 1$

$$\sum_{r=1}^{2n}(2r-1)^2 = 4\sum_{r=1}^{2n} r^2 - 4\sum_{r=1}^{2n} r + \sum_{r=1}^{2n} 1$$

$$= 4 \times \frac{2n(2n+1)(4n+1)}{6} - 4 \times \frac{2n(2n+1)}{2} + 2n$$

$$= \frac{4n(2n+1)(4n+1)}{3} - 4n(2n+1) + 2n$$

$$= \frac{2n}{3}[2(2n+1)(4n+1) - 6(2n+1) + 3]$$

$$= \frac{2n}{3}[16n^2 + 12n + 2 - 12n - 6 + 3]$$

$$= \frac{2n}{3}(16n^2 - 1) \quad \text{as required}$$

2 (a) $u = 2x + 1 \Rightarrow 4x^2 + 4x + 3 = (2x+1)^2 + 2 = u^2 + 2$

$$\frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2}\,du$$

$$\int \frac{dx}{\sqrt{4x^2 + 4x + 3}} = \int \frac{1}{2\sqrt{u^2 + 2}}\,du$$

$$= \frac{1}{2}\ln\left(u + \sqrt{u^2 + 2}\right) + c$$

$$= \frac{1}{2}\ln\left(2x + 1 + \sqrt{4x^2 + 4x + 3}\right) + c \quad \text{as required}$$

(b) The integrand is undefined at $x = 1$, so

Question number

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{t\to 1}\left[\sin^{-1}x\right]_0^t$$

$$= \lim_{t\to 1}\left(\sin^{-1}t - \sin^{-1}0\right)$$

$$= \sin^{-1}1 = \frac{\pi}{2}$$

3 (i)

$$\frac{2x^2+x+7}{(x-1)(x^2+4)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$2x^2+x+7 \equiv A(x^2+4) + (Bx+C)(x-1)$$

$x = 1$: $10 = 5A \Rightarrow A = 2$

Coefficient of x^2: $2 = A + B \Rightarrow B = 0$

Constant term: $7 = 4A - C \Rightarrow C = 1$

(Check x: $1 = -B + C = 1$ ✓)

$$f(x) = \frac{2}{x-1} + \frac{1}{x^2+4}$$

(ii)

$$\int f(x)\,dx = \int \frac{2}{x-1}\,dx + \int \frac{1}{x^2+2^2}\,dx$$

$$= 2\ln(x-1) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c \quad \text{as required}$$

4 (i)

$I_n = \int (\ln x)^n \times 1\,dx$. By parts:

$$u = (\ln x)^n \quad \Rightarrow \quad \frac{du}{dx} = n(\ln x)^{n-1} \times \frac{1}{x}$$

$$\frac{dv}{dx} = 1 \quad \Rightarrow \quad v = x$$

Question number

$I_n = x(\ln x)^n - \int x \times \frac{n(\ln x)^{n-1}}{x}\, dx$

$= x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$

$I_n = x(\ln x)^n - nI_{n-1}$ as required

(ii) $\int_1^2 (\ln x)^2\, dx = [x(\ln x)^2]_1^2 - 2I_1$

$I_1 = x \ln x - I_0 = x \ln x - \int 1\, dx = x \ln x - x$

$\int_1^2 (\ln x)^2\, dx = [x(\ln x)^2 - 2x \ln x + 2x]_1^2$

$= (2(\ln 2)^2 - 4 \ln 2 + 4) - (0 - 0 + 2)$

$= 2(\ln 2)^2 - 4 \ln 2 + 2$

$= 0.188$ (3 s.f.)

5 $n = 1$: LHS $= 3^0 = 1$, RHS $= \frac{3^1 - 1}{2} = 1$

LHS = RHS, so the result is true for $n = 1$.

Assume true for $n = k$: $\sum_{r=1}^{k} 3^{r-1} = \frac{3^k - 1}{2}$

Then for $n = k + 1$: $\sum_{r=1}^{k+1} 3^{r-1} = \sum_{r=1}^{k} 3^{r-1} + 3^k$

$= \frac{3^k - 1}{2} + 3^k = \frac{3^k - 1 + 2 \times 3^k}{2}$

$= \frac{3 \times 3^k - 1}{2} = \frac{3^{k+1} - 1}{2}$

which is the result with $n = k + 1$.

So if the result is true for $n = k$, it is true for $n = k + 1$.

Since it is true for $n = 1$, by mathematical induction

Question number

it is true for all integers $n \geqslant 1$.

6 (i) $\frac{dA}{dt} - \frac{1}{t}A = 1$

Integrating factor $= e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = \frac{1}{t}$

$\frac{1}{t}\frac{dA}{dt} - \frac{1}{t^2}A = \frac{1}{t}$

$\frac{d}{dt}\left(\frac{A}{t}\right) = \frac{1}{t}$

$\frac{A}{t} = \ln t + c \quad \Rightarrow \quad A = t(\ln t + c)$

$t = 20, A = 300: \quad 300 = 20(\ln 20 + c)$

$15 = \ln 20 + c \quad \Rightarrow \quad c = 15 - \ln 20$

$A = t(\ln t + 15 - \ln 20)$

(ii) $t = 50: \quad A = 50(\ln 50 + 15 - \ln 20)$

$= 50(\ln 2.5 + 15) = 795.8\ldots$

Assets are about £796 million (3 s.f.)

(iii) The model assumes steady growth forever; 50 years is a long time to extrapolate. Banking is volatile (crashes, recessions, interest rate changes), which the model does not include.

7 (a)(i) $f(x) = \ln(2x + (1 + 4x^2)^{\frac{1}{2}})$

$$f'(x) = \frac{2 + \frac{1}{2}(1 + 4x^2)^{-\frac{1}{2}} \times 8x}{2x + \sqrt{1 + 4x^2}}$$

Question number

$$= \frac{2 + \frac{4x}{\sqrt{1+4x^2}}}{2x + \sqrt{1+4x^2}}$$

$$= \frac{2(\sqrt{1+4x^2} + 2x)}{\sqrt{1+4x^2}} \times \frac{1}{2x + \sqrt{1+4x^2}}$$

$f'(x) = \frac{2}{\sqrt{1+4x^2}}$ as required

(ii) $f'(x) = 2(1+4x^2)^{-\frac{1}{2}}$

$f''(x) = -(1+4x^2)^{-3/2} \times 8x = -8x(1+4x^2)^{-3/2}$

$f'''(x) = -8(1+4x^2)^{-3/2} + (-8x)(-3/2)(1+4x^2)^{-5/2} \times 8x$

$= -8(1+4x^2)^{-3/2} + 96x^2(1+4x^2)^{-5/2}$

$f(0) = \ln 1 = 0, \quad f'(0) = 2, \quad f''(0) = 0, \quad f'''(0) = -8$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$f(x) \approx 2x - \frac{8}{6}x^3 = 2x - \frac{4}{3}x^3$

(b) For small x: $\cos 6x \approx 1 - \frac{(6x)^2}{2} = 1 - 18x^2, \quad \sin 3x \approx 3x$

$1 - \cos 6x - 2x \sin 3x \approx 1 - (1 - 18x^2) - 2x(3x)$

$= 18x^2 - 6x^2 = 12x^2$

$\lim_{x\to 0}\left(\frac{1 - \cos 6x - 2x \sin 3x}{x^2}\right) = \lim_{x\to 0} \frac{12x^2}{x^2} = 12$ as required

8 (a) RHS = $\cosh x \cosh y + \sinh x \sinh y$

$= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$

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$= \frac{1}{4}(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y})$

$+ \frac{1}{4}(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})$

$= \frac{1}{4}(2e^{x+y} + 2e^{-(x+y)})$

$= \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x + y) = LHS$ as required

(b) $4\left(\frac{e^x + e^{-x}}{2}\right) + 5\left(\frac{e^x - e^{-x}}{2}\right) = 2$

$4e^x + 4e^{-x} + 5e^x - 5e^{-x} = 4$

$9e^x - e^{-x} - 4 = 0$

Multiply by e^x: $9e^{2x} - 4e^x - 1 = 0$

$e^x = \frac{4 \pm \sqrt{16 + 36}}{18} = \frac{4 \pm 2\sqrt{13}}{18} = \frac{2 \pm \sqrt{13}}{9}$

$e^x > 0$, so $e^x = \frac{2 + \sqrt{13}}{9}$ (reject $\frac{2 - \sqrt{13}}{9} < 0$)

$x = \ln\left(\frac{2 + \sqrt{13}}{9}\right)$

9 (i) Auxiliary equation: $m^2 + 2m + 10 = 0$

$m = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$

Complementary function: $y = e^{-x}(A \cos 3x + B \sin 3x)$

Particular integral: try $y = Cx + D$, $\frac{dy}{dx} = C$, $\frac{d^2y}{dx^2} = 0$

$0 + 2C + 10(Cx + D) = 20x - 6$

Question number

x terms: $10C = 20 \Rightarrow C = 2$

Constants: $2C + 10D = -6 \Rightarrow 10D = -10 \Rightarrow D = -1$

General solution:

$y = e^{-x}(A \cos 3x + B \sin 3x) + 2x - 1$

(ii) $x = 0, y = 0$: $0 = A - 1 \Rightarrow A = 1$

$\frac{dy}{dx} = -e^{-x}(A \cos 3x + B \sin 3x)$

$+ e^{-x}(-3A \sin 3x + 3B \cos 3x) + 2$

$x = 0, \frac{dy}{dx} = 6$: $6 = -A + 3B + 2$

$6 = -1 + 3B + 2 \Rightarrow 3B = 5 \Rightarrow B = \frac{5}{3}$

$y = e^{-x}(\cos 3x + \frac{5}{3} \sin 3x) + 2x - 1$

10 (a) $x = r \cos \theta$, $y = r \sin \theta$, $x^2 + y^2 = r^2$

$r = 3 \sin 2\theta = 6 \sin \theta \cos \theta$

Multiply by r^2: $r^3 = 6(r \sin \theta)(r \cos \theta) = 6xy$

$r^3 = (r^2)^{3/2} = (x^2 + y^2)^{3/2}$

$(x^2 + y^2)^{3/2} = 6xy$

or $(x^2 + y^2)^3 = 36x^2y^2$

(b) Area $= \frac{1}{2}\int_0^{\pi/4} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/4} a^2(1 + \tan \theta)^2 \, d\theta$

$= \frac{a^2}{2}\int_0^{\pi/4} (1 + 2 \tan \theta + \tan^2\theta) \, d\theta$

Using $1 + \tan^2\theta = \sec^2\theta$:

Question number

$= \frac{a^2}{2}\int_0^{\pi/4} (\sec^2\theta + 2\tan\theta)\, d\theta$

$= \frac{a^2}{2}[\tan\theta + 2\ln\sec\theta]_0^{\pi/4}$

$= \frac{a^2}{2}[(1 + 2\ln\sqrt{2}) - (0 + 2\ln 1)]$

$= \frac{a^2}{2}(1 + \ln 2)$

11 (a) $1 + i$: $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\theta = \tan^{-1} 1 = \frac{\pi}{4}$

$1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, $1 - i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$

By De Moivre's theorem:

$(1 + i)^{4n} = (\sqrt{2})^{4n}(\cos\frac{4n\pi}{4} + i\sin\frac{4n\pi}{4}) = 4^n(\cos n\pi + i\sin n\pi)$

$(1 - i)^{4n} = (\sqrt{2})^{4n}(\cos(-n\pi) + i\sin(-n\pi)) = 4^n(\cos n\pi - i\sin n\pi)$

$(1 + i)^{4n} - (1 - i)^{4n} = 4^n(2i\sin n\pi) = 2i \times 4^n \sin n\pi$

n is a positive integer, so $\sin n\pi = 0$

$\therefore (1 + i)^{4n} - (1 - i)^{4n} = 0$ as required

(b)(i) Let $c = \cos\theta$, $s = \sin\theta$.

$\cos 4\theta + i\sin 4\theta = (c + is)^4$

$= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

$= c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$

$= (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)$

Equating real parts:

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

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(ii) Using $\sin^2\theta = 1 - \cos^2\theta$:

$\cos 4\theta = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2$

$= c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4$

$= 8\cos^4\theta - 8\cos^2\theta + 1$ as required

(iii) $\theta = \frac{\pi}{12}$: $\cos 4\theta = \cos\frac{\pi}{3} = \frac{1}{2}$

Let $u = \cos^2\frac{\pi}{12}$: $8u^2 - 8u + 1 = \frac{1}{2}$

$16u^2 - 16u + 1 = 0$

$u = \frac{16 \pm \sqrt{256 - 64}}{32} = \frac{16 \pm 8\sqrt{3}}{32} = \frac{2 \pm \sqrt{3}}{4}$

$\cos\frac{\pi}{12} = \pm\frac{1}{2}\sqrt{2 + \sqrt{3}}$ or $\pm\frac{1}{2}\sqrt{2 - \sqrt{3}}$

$0 < \frac{\pi}{12} < \frac{\pi}{2}$, so $\cos\frac{\pi}{12} > 0$: reject the negative values.

Also $\frac{\pi}{12} < \frac{\pi}{3}$, so $\cos\frac{\pi}{12} > \cos\frac{\pi}{3} = \frac{1}{2}$

But $\frac{1}{2}\sqrt{2 - \sqrt{3}} = \frac{1}{2}\sqrt{0.268\ldots} = 0.259 < \frac{1}{2}$, so reject it.

$\therefore \cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2 + \sqrt{3}}$ as required

12 (i) Let $y = \tanh^{-1} x$, so $x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

Multiply top and bottom by e^y: $x = \frac{e^{2y} - 1}{e^{2y} + 1}$

Question number

$x\,e^{2y} + x = e^{2y} - 1$

$e^{2y}(1 - x) = 1 + x$

$e^{2y} = \frac{1 + x}{1 - x}$

$2y = \ln\left(\frac{1 + x}{1 - x}\right)$

$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$ as required

(ii) $\tanh^{-1}(\sin x) = \frac{1}{2} \ln\left(\frac{1 + \sin x}{1 - \sin x}\right)$

$= \frac{1}{2} \ln(1 + \sin x) - \frac{1}{2} \ln(1 - \sin x)$

$\frac{d}{dx}(\tanh^{-1}(\sin x)) = \frac{1}{2}\left(\frac{\cos x}{1 + \sin x} + \frac{\cos x}{1 - \sin x}\right)$

$= \frac{\cos x}{2} \times \frac{(1 - \sin x) + (1 + \sin x)}{1 - \sin^2 x}$

$= \frac{\cos x}{2} \times \frac{2}{\cos^2 x}$

$= \frac{1}{\cos x} = \sec x$ as required

(iii) By parts: $u = \tanh^{-1}(\sin x) \Rightarrow \frac{du}{dx} = \sec x$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$\int_0^{\pi/6} \sec^2 x \tanh^{-1}(\sin x)\, dx = [\tan x \tanh^{-1}(\sin x)]_0^{\pi/6} - \int_0^{\pi/6} \tan x \sec x\, dx$

Question number

$= [\tan x \tanh^{-1}(\sin x)]_0^{\pi/6} - [\sec x]_0^{\pi/6}$

At $\frac{\pi}{6}$: $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $\tanh^{-1}(\frac{1}{2}) = \frac{1}{2}\ln(\frac{3/2}{1/2}) = \frac{1}{2}\ln 3$, $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

$= (\frac{1}{\sqrt{3}} \times \frac{1}{2}\ln 3 - 0) - (\frac{2}{\sqrt{3}} - 1)$

$= \frac{\ln 3}{2\sqrt{3}} + 1 - \frac{2}{\sqrt{3}}$

$= 0.162$ (3 s.f.)

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