New Specification

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ADVANCED SUBSIDIARY (AS)

General Certificate of Education

2019

Further Mathematics *StemUp Model Solution*

Assessment Unit AS 1

assessing

Pure Mathematics

SFM11

[SFM11]

WEDNESDAY 12 JUNE, MORNING

TIME

1 hour 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number on the Answer Booklet provided.
Answer **all seven** questions.
Show clearly the full development of your answers.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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Answer all seven questions.

Show clearly the full development of your answers.

Answers should be given to three significant figures unless otherwise stated.

1 **(a)** Given $\mathbf{A} = \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}$

find:

(i) $\mathbf{AB}$ [2]

(ii) $(\mathbf{AB})^{-1}$ [4]

(iii) Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$ [4]

(b) A transformation is defined by $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

This transformation maps any point on the line $y = mx$ onto another point on the line $y = mx$.

Find the possible values of m. [6]

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2 **Fig. 1** below shows a parallelepiped ABCDEFGH.

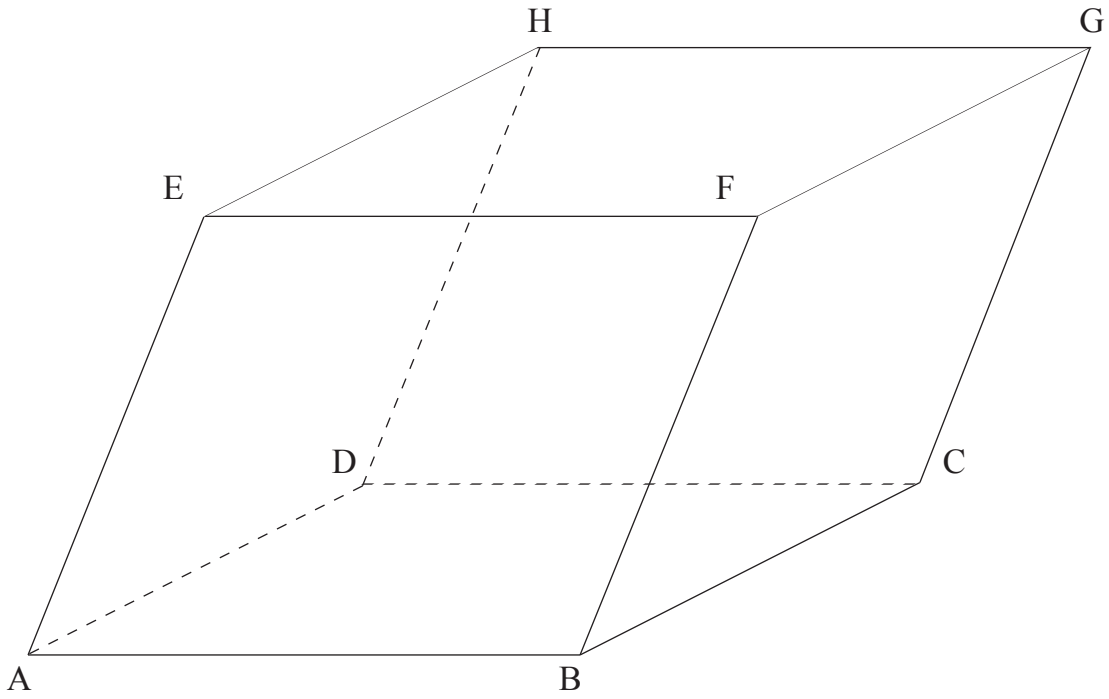


Fig. 1

The vertices A, B, D, E have coordinates (0, 0, 0), (5, −2, 3), (2, −3, 4) and (3, −2, −3) respectively.

Find the volume of the parallelepiped. [6]

3 **(a)** **(i)** Find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & -4 \\ 1 & 4 & -2 \\ -1 & -5 & 3 \end{pmatrix}$$ [4]

(ii) Find the matrix **B** which satisfies the equation

$$\mathbf{A}^2 + \mathbf{AB} = \mathbf{I}$$ [3]

(b) The roots of the equation $2x^2 + 5x - 6 = 0$ are α and β.

Find an equation with integer coefficients whose roots are α^2 and β^2 [6]

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4 **(a)** Express the following in the form $a + b\mathrm{i}$ where a and b are real numbers.

(i) $(3 - 2\mathrm{i})(2 + \mathrm{i})$ [2]

(ii) $\dfrac{3 - 2\mathrm{i}}{2 - \mathrm{i}}$ [3]

(iii) $\sqrt{16\mathrm{i}}$ [7]

(b) $z = 1 - \mathrm{i}$ is a root of the equation

$$z^3 - 5z^2 + az + b = 0 \quad \text{where } a \text{ and } b \text{ are integers.}$$

Find the other two roots and the values of a and b. [7]

5 **(a)** A complex number is given by $z_1 = -\sqrt{3} + 3\mathrm{i}$

Find:

(i) the modulus of z_1 [2]

(ii) the argument of z_1 in radians. [2]

The complex number z_2 has modulus $\sqrt{3}$ and argument $\dfrac{\pi}{4}$

Find in modulus argument form:

(iii) $z_1 z_2$ [4]

(iv) $\dfrac{z_1}{z_2}$ [4]

(b) The complex number z satisfies both $|z - 3\mathrm{i}| \leqslant 3$ and $\dfrac{\pi}{2} \leqslant \arg z \leqslant \dfrac{2\pi}{3}$

Sketch on an Argand diagram the region satisfied by z. [6]

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6 The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} a & 4 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

(i) Find the determinant of **M** [4]

(ii) Find the value of a for which the matrix equation

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}$$

does not have a unique solution. [2]

(iii) Find the general solution of the system of equations

$$\begin{aligned} 3x + 4y + z &= 6 \\ x + 2y - z &= 0 \\ x + y + z &= 3 \end{aligned}$$ [4]

7 **(a)** The lines l_1 and l_2 have equations

$$\begin{aligned} l_1: &\quad \mathbf{r} = 3\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ l_2: &\quad \mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k}) \end{aligned}$$

(i) Show that the lines l_1 and l_2 intersect. [6]

(ii) Find the coordinates of the point of intersection. [1]

(b) Two planes are given by

$$\begin{aligned} \mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) &= -5 \\ \mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) &= 15 \end{aligned}$$

(i) Find, in vector form, an equation of the line of intersection of the planes. [6]

(ii) Find, in degrees, the acute angle between the planes. [5]

THIS IS THE END OF THE QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)(i) $AB = \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 12 - 21 & -3 + 7 \\ 4 - 6 & -1 + 2 \end{pmatrix}$

$AB = \begin{pmatrix} -9 & 4 \\ -2 & 1 \end{pmatrix}$

(a)(ii) $\det(AB) = (-9)(1) - (4)(-2) = -9 + 8 = -1$

$(AB)^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -4 \\ 2 & -9 \end{pmatrix}$

$(AB)^{-1} = \begin{pmatrix} -1 & 4 \\ -2 & 9 \end{pmatrix}$

(a)(iii) $\det A = 6 - 7 = -1, \quad A^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & -7 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ 1 & -3 \end{pmatrix}$

$\det B = 4 - 3 = 1, \quad B^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$

$B^{-1}A^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 7 \\ 1 & -3 \end{pmatrix}$

$= \begin{pmatrix} -2 + 1 & 7 - 3 \\ -6 + 4 & 21 - 12 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -2 & 9 \end{pmatrix}$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$ as required

(b) A point on $y = mx$ is (x, mx):

Question number

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x + 4mx \\ 2x + 3mx \end{pmatrix}$$

The image lies on $Y = mX$, so

$2x + 3mx = m(x + 4mx)$

$2 + 3m = m + 4m^2$ (dividing by $x \neq 0$)

$4m^2 - 2m - 2 = 0$

$2m^2 - m - 1 = 0$

$(2m + 1)(m - 1) = 0$

$m = -\frac{1}{2}$ or $m = 1$

2

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \overrightarrow{AE} = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$$

Volume $= |\overrightarrow{AE} \cdot (\overrightarrow{AB} \times \overrightarrow{AD})|$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} (-2)(4) - (3)(-3) \\ (3)(2) - (5)(4) \\ (5)(-3) - (-2)(2) \end{pmatrix} = \begin{pmatrix} 1 \\ -14 \\ -11 \end{pmatrix}$$

$$\overrightarrow{AE} \cdot (\overrightarrow{AB} \times \overrightarrow{AD}) = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -14 \\ -11 \end{pmatrix}$$

$= 3 + 28 + 33 = 64$

Volume of parallelepiped = 64 cubic units

3 (a)(i) $\det A = 1(12 - 10) - 7(3 - 2) - 4(-5 + 4)$

$= 2 - 7 + 4 = -1$

Question number

Matrix of minors $= \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$

Matrix of cofactors $= \begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & -2 \\ 2 & -2 & -3 \end{pmatrix}$

$\text{adj } A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & -1 & -2 \\ -1 & -2 & -3 \end{pmatrix}$

$A^{-1} = \frac{1}{-1}\text{adj } A = \begin{pmatrix} -2 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

(a)(ii) $A^2 + AB = I$

$A(A + B) = I$

$A + B = A^{-1}$

$B = A^{-1} - A = \begin{pmatrix} -2 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 7 & -4 \\ 1 & 4 & -2 \\ -1 & -5 & 3 \end{pmatrix}$

$B = \begin{pmatrix} -3 & -6 & 2 \\ 0 & -3 & 4 \\ 2 & 7 & 0 \end{pmatrix}$

(b) $\alpha + \beta = -\frac{5}{2}, \quad \alpha\beta = -\frac{6}{2} = -3$

Sum of new roots: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \frac{25}{4} + 6 = \frac{49}{4}$

Question number

Product of new roots: $\alpha^2\beta^2 = (\alpha\beta)^2 = (-3)^2 = 9$

Equation: $x^2 - \frac{49}{4}x + 9 = 0$

$4x^2 - 49x + 36 = 0$

4 (a)(i) $(3 - 2i)(2 + i) = 6 + 3i - 4i - 2i^2$

$= 6 - i + 2 = 8 - i$

(a)(ii) $\frac{3 - 2i}{2 - i} = \frac{(3 - 2i)(2 + i)}{(2 - i)(2 + i)}$

$= \frac{8 - i}{4 + 1} = \frac{8 - i}{5}$

$= \frac{8}{5} - \frac{1}{5}i$

(a)(iii) Let $\sqrt{16i} = a + bi$, with a, b real

$(a + bi)^2 = 16i$

$a^2 - b^2 + 2abi = 16i$

Real parts: $a^2 - b^2 = 0$ Imaginary parts: $2ab = 16 \Rightarrow ab = 8$

$a^2 = b^2$ and $ab > 0$, so $a = b$

$a^2 = 8 \Rightarrow a = \pm 2\sqrt{2}$, $b = \pm 2\sqrt{2}$

$\sqrt{16i} = 2\sqrt{2} + 2\sqrt{2}\,i$ (or $-2\sqrt{2} - 2\sqrt{2}\,i$)

(b) Coefficients are real, so the conjugate $z = 1 + i$ is also a root

$(z - (1 - i))(z - (1 + i)) = (z - 1)^2 + 1 = z^2 - 2z + 2$

Let the third root be k (real):

$(z^2 - 2z + 2)(z - k) = z^3 - (2 + k)z^2 + (2 + 2k)z - 2k$

Compare with $z^3 - 5z^2 + az + b$:

Question number

z^2: $-(2 + k) = -5 \Rightarrow k = 3$

z: $a = 2 + 2k = 8$

constant: $b = -2k = -6$

Other two roots: $1 + i$ and 3; $a = 8$, $b = -6$

5 (a)(i) $|z_1| = \sqrt{(-\sqrt{3})^2 + 3^2} = \sqrt{3 + 9} = \sqrt{12}$

$|z_1| = 2\sqrt{3}$ $(= 3.46)$

(a)(ii) z_1 is in the second quadrant

$\arg z_1 = \pi - \tan^{-1}\left(\frac{3}{\sqrt{3}}\right) = \pi - \frac{\pi}{3}$

$\arg z_1 = \frac{2\pi}{3}$

(a)(iii) $|z_1 z_2| = |z_1||z_2| = 2\sqrt{3} \times \sqrt{3} = 6$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{12}$

$z_1 z_2 = 6\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

(a)(iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{2\sqrt{3}}{\sqrt{3}} = 2$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$

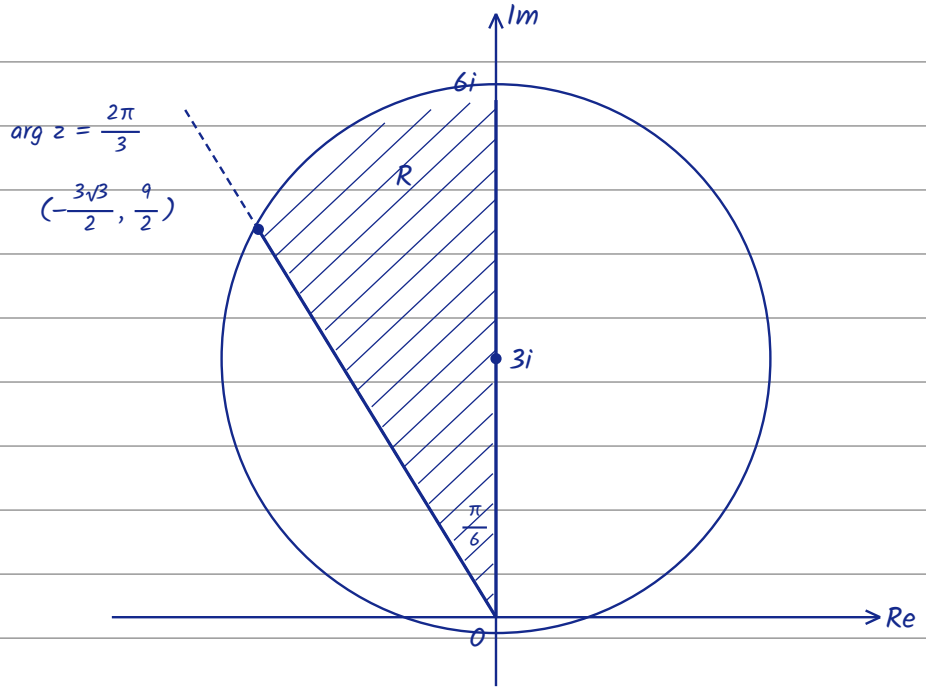
$\frac{z_1}{z_2} = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

(b) $|z - 3i| \leq 3$: on or inside the circle, centre $3i$, radius 3 (passes through 0)

Question number

$\frac{\pi}{2} \leq \arg z \leq \frac{2\pi}{3}$: between the half-lines from O at $\frac{\pi}{2}$ and $\frac{2\pi}{3}$

The half-line $\arg z = \frac{2\pi}{3}$ meets the circle at $(-\frac{3\sqrt{3}}{2}, \frac{9}{2})$



The required region R is shaded (including its boundary)

6 (i) Expanding along the first row:

$\det M = a[(2)(1) - (-1)(1)] - 4[(1)(1) - (-1)(1)] + 1[(1)(1) - (2)(1)]$

$= a(2 + 1) - 4(1 + 1) + 1(1 - 2)$

$= 3a - 8 - 1$

$\det M = 3a - 9$

(ii) No unique solution when $\det M = 0$

$3a - 9 = 0 \Rightarrow a = 3$

(iii) $3x + 4y + z = 6$ (1)

$x + 2y - z = 0$ (2)

Question number

$x + y + z = 3 \quad (3)$

$(2) - (3)$: $y - 2z = -3$

$(1) - 3 \times (3)$: $y - 2z = -3$ (same equation, so the system is consistent)

Let $z = \lambda$: $y = 2\lambda - 3$

From (3): $x = 3 - y - z = 3 - (2\lambda - 3) - \lambda = 6 - 3\lambda$

General solution: $x = 6 - 3\lambda$, $y = 2\lambda - 3$, $z = \lambda$

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

7 (a)(i) l_1: $(3 + \lambda, 1 + 2\lambda, -1 + 3\lambda)$ $\quad$ l_2: $(2 + \mu, 5 - \mu, \mu)$

Equate components:

i: $3 + \lambda = 2 + \mu \quad (1)$

j: $1 + 2\lambda = 5 - \mu \quad (2)$

k: $-1 + 3\lambda = \mu \quad (3)$

$(1) + (2)$: $4 + 3\lambda = 7 \Rightarrow \lambda = 1$

In (1): $\mu = 3 + 1 - 2 = 2$

Check (3): $-1 + 3(1) = 2 = \mu$ ✓

All three equations are satisfied, so l_1 and l_2 intersect

(a)(ii) $\lambda = 1$: $r = 3i + j - k + (i + 2j + 3k) = 4i + 3j + 2k$

Point of intersection (4, 3, 2)

(b)(i) Cartesian forms: $x + 2y + z = -5 \quad (1)$

$2x - 3y + z = 15 \quad (2)$

$(2) - (1)$: $x - 5y = 20 \Rightarrow x = 20 + 5y$

In (1): $20 + 5y + 2y + z = -5 \Rightarrow z = -25 - 7y$

Let $y = \lambda$: $x = 20 + 5\lambda$, $y = \lambda$, $z = -25 - 7\lambda$

Question number

Line of intersection: $r = 20i - 25k + \lambda(5i + j - 7k)$

(b)(ii) The angle between the planes is the angle between their normals

$n_1 = i + 2j + k, \quad n_2 = 2i - 3j + k$

$n_1 \cdot n_2 = 2 - 6 + 1 = -3$

$|n_1| = \sqrt{6}, \quad |n_2| = \sqrt{4 + 9 + 1} = \sqrt{14}$

$\cos\theta = \frac{-3}{\sqrt{6}\sqrt{14}} = \frac{-3}{\sqrt{84}}$

$\theta = 109.1°$

Acute angle between the planes $= 180° - 109.1° = 70.9°$

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