AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

AS
FURTHER MATHEMATICS

Paper 1

Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

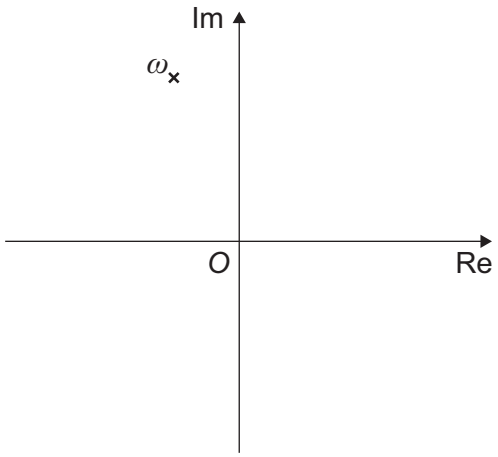
Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
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TOTAL	

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Answer **all** questions in the spaces provided.

1 The complex number ω is shown below on the Argand diagram.



Which of the following complex numbers could be ω?

Tick (✓) **one** box.

[1 mark]

$\cos(-2) + i\sin(-2)$ ☐

$\cos(-1) + i\sin(-1)$ ☐

$\cos(1) + i\sin(1)$ ☐

$\cos(2) + i\sin(2)$ ☑

2 Given that $f(x) = 3x - 1$ find the mean value of $f(x)$ over the interval $4 \leq x \leq 8$

Circle your answer.

[1 mark]

6 11 (17) 23

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3 The matrix **M** represents a rotation about the x-axis.

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & \frac{\sqrt{3}}{2} \\ 0 & b & -\frac{1}{2} \end{bmatrix}$$

Which of the following pairs of values is correct?

Tick (✓) **one** box.

[1 mark]

$a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ ☐

$a = \frac{1}{2}$ and $b = -\frac{\sqrt{3}}{2}$ ☐

$a = -\frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ ☐

$a = -\frac{1}{2}$ and $b = -\frac{\sqrt{3}}{2}$ ☑

4 The point $(2, -1)$ is invariant under the transformation represented by the matrix **N**

Which of the following matrices could be **N**?

Circle your answer.

[1 mark]

$\begin{bmatrix} 4 & 6 \\ 2 & 5 \end{bmatrix}$ $\begin{bmatrix} 6 & 5 \\ 4 & 2 \end{bmatrix}$ $\begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix}$ $\begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$

Turn over for the next question

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5 Show that the vectors $\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$ are perpendicular.

[2 marks]

$\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} = (1)(7) + (-3)(4) + (5)(1)$

$= 7 - 12 + 5 = 0$

The scalar product is 0, so the vectors are perpendicular.

6 Prove the identity

$$\cosh^2 x - \sinh^2 x = 1$$

[2 marks]

$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$

$= \frac{4}{4}$

$= 1$ as required

Do not write outside the box

7 Show that the Maclaurin series for $\ln(e + 2ex)$ is

$$1 + 2x - 2x^2 + ax^3 - \ldots$$

where a is to be determined.

[3 marks]

$\ln(e + 2ex) = \ln[\, e(1 + 2x)\,]$

$= \ln e + \ln(1 + 2x)$

$= 1 + \ln(1 + 2x)$

Using $\ln(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots$ with $y = 2x$:

$\ln(e + 2ex) = 1 + 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \ldots$

$= 1 + 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \ldots$

$= 1 + 2x - 2x^2 + \frac{8}{3}x^3 - \ldots$

$\therefore\ a = \frac{8}{3}$

Turn over for the next question

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8 Stephen is correctly told that $(1 + \mathrm{i})$ and -1 are two roots of the polynomial equation

$$z^3 - 2\mathrm{i}z^2 + pz + q = 0$$

where p and q are complex numbers.

8 (a) Stephen states that $(1 - \mathrm{i})$ **must** also be a root of the equation because roots of polynomial equations occur in conjugate pairs.

Explain why Stephen's reasoning is wrong.

[1 mark]

Roots only have to occur in conjugate pairs when all the coefficients of the polynomial are real. Here the coefficient of z^2 is $-2i$ (and p, q may be complex), so $(1 - i)$ need not be a root.

8 (b) Find p and q

[5 marks]

Let the third root be α.

Sum of roots: $(1 + i) + (-1) + \alpha = 2i$

$i + \alpha = 2i \Rightarrow \alpha = i$

Sum of products of pairs of roots = p:

$p = (1 + i)(-1) + (-1)(i) + i(1 + i)$

$= -1 - i - i + i + i^2$

$= -1 - i - 1$

$= -2 - i$

Product of roots = $-q$:

$(1 + i)(-1)(i) = -i - i^2 = 1 - i$

$-q = 1 - i \Rightarrow q = -1 + i$

$p = -2 - i, \quad q = -1 + i$

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Turn over for the next question

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9 (a) Use the standard formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r(r+3) = an(n+1)(n+b)$$

where a and b are constants to be determined.

[4 marks]

$\sum_{r=1}^{n} r(r+3) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$

$= \frac{1}{6} n(n+1)(2n+1) + 3 \times \frac{1}{2} n(n+1)$

$= \frac{1}{6} n(n+1)[(2n+1) + 9]$

$= \frac{1}{6} n(n+1)(2n+10)$

$= \frac{1}{3} n(n+1)(n+5)$

$\therefore a = \frac{1}{3}, \quad b = 5$

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9 (b) Hence, or otherwise, find a fully factorised expression for

$$\sum_{r=n+1}^{5n} r(r+3)$$

[3 marks]

$\sum_{r=n+1}^{5n} r(r+3) = \sum_{r=1}^{5n} r(r+3) - \sum_{r=1}^{n} r(r+3)$

$= \frac{1}{3}(5n)(5n+1)(5n+5) - \frac{1}{3}n(n+1)(n+5)$

$= \frac{25}{3}n(n+1)(5n+1) - \frac{1}{3}n(n+1)(n+5)$

$= \frac{1}{3}n(n+1)[\,25(5n+1) - (n+5)\,]$

$= \frac{1}{3}n(n+1)(125n + 25 - n - 5)$

$= \frac{1}{3}n(n+1)(124n+20)$

$= \frac{4}{3}n(n+1)(31n+5)$

Turn over ▶

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10 Matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 3 & \mathrm{i} - 1 \\ \mathrm{i} & 2 \end{bmatrix}$$

10 (a) Show that $\det \mathbf{A} = a + \mathrm{i}$ where a is an integer to be determined.

[2 marks]

$\det A = (3)(2) - (i - 1)(i)$

$= 6 - (i^2 - i)$

$= 6 - (-1) + i$

$= 7 + i$

$\therefore \ a = 7$

10 (b) Matrix **B** is given by

$$\mathbf{B} = \begin{bmatrix} 14 - 2\mathrm{i} & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{AB} = p\mathbf{I}$$

where $b, c, d \in \mathbb{C}$ and $p \in \mathbb{N}$

Find b, c, d and p

[6 marks]

$AB = pI \Rightarrow B = pA^{-1}$

$A^{-1} = \frac{1}{7 + i}\begin{bmatrix} 2 & 1-i \\ -i & 3 \end{bmatrix}$

$\begin{bmatrix} 14-2i & b \\ c & d \end{bmatrix} = \frac{p}{7 + i}\begin{bmatrix} 2 & 1-i \\ -i & 3 \end{bmatrix}$

Top left: $14 - 2i = \frac{2p}{7 + i}$

$p = (7 - i)(7 + i) = 49 - i^2 = 50$

$b = \frac{50(1 - i)}{7 + i} = \frac{50(1 - i)(7 - i)}{50} = 7 - 8i + i^2 = 6 - 8i$

$c = \frac{-50i}{7 + i} = -i(7 - i) = -7i + i^2 = -1 - 7i$

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$d = \frac{3 \times 50}{7 + i} = 3(7 - i) = 21 - 3i$

$p = 50, \quad b = 6 - 8i, \quad c = -1 - 7i, \quad d = 21 - 3i$

Turn over for the next question

Do not write outside the box

11 (a) Show that, for all positive integers r,

$$\frac{1}{(r-1)!} - \frac{1}{r!} = \frac{r-1}{r!}$$

[1 mark]

$\frac{1}{(r-1)!} - \frac{1}{r!} = \frac{r}{r(r-1)!} - \frac{1}{r!}$

$= \frac{r}{r!} - \frac{1}{r!} = \frac{r-1}{r!}$ as required

11 (b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \frac{r-1}{r!} = a + \frac{b}{n!}$$

where a and b are integers to be determined.

[3 marks]

$\sum_{r=1}^{n} \frac{r-1}{r!} = \sum_{r=1}^{n} \left(\frac{1}{(r-1)!} - \frac{1}{r!} \right)$

$r = 1$: $\frac{1}{0!} - \frac{1}{1!}$

$r = 2$: $\frac{1}{1!} - \frac{1}{2!}$

$r = 3$: $\frac{1}{2!} - \frac{1}{3!}$

$\vdots$

$r = n-1$: $\frac{1}{(n-2)!} - \frac{1}{(n-1)!}$

$r = n$: $\frac{1}{(n-1)!} - \frac{1}{n!}$

Adding, all the terms cancel except the first and the last:

$\sum_{r=1}^{n} \frac{r-1}{r!} = \frac{1}{0!} - \frac{1}{n!} = 1 - \frac{1}{n!}$

$\therefore \; a = 1, \quad b = -1$

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Turn over for the next question

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12 The equation $x^3 - 2x^2 - x + 2 = 0$ has three roots. One of the roots is 2

12 (a) Find the other two roots of the equation.

[1 mark]

$x^3 - 2x^2 - x + 2 = (x - 2)(x^2 - 1)$

$= (x - 2)(x - 1)(x + 1)$

Other two roots: $x = 1$ and $x = -1$

12 (b) Hence, or otherwise, solve

$$\cosh^3\theta - 2\cosh^2\theta - \cosh\theta + 2 = 0$$

giving your answers in an exact form.

[4 marks]

Let $x = \cosh\theta$: $x^3 - 2x^2 - x + 2 = 0$

so $\cosh\theta = 2$, $\cosh\theta = 1$ or $\cosh\theta = -1$

But $\cosh\theta \geq 1$ for all real θ, so $\cosh\theta = -1$ is rejected.

$\cosh\theta = 1 \Rightarrow \theta = 0$

$\cosh\theta = 2 \Rightarrow \theta = \pm\cosh^{-1} 2 = \pm\ln(2 + \sqrt{(2^2 - 1)})$

$= \pm\ln(2 + \sqrt{3})$

$\theta = 0$, $\theta = \ln(2 + \sqrt{3})$, $\theta = -\ln(2 + \sqrt{3})$

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13 Prove by induction that, for all integers $n \geq 1$

$$\sum_{r=1}^{n} 2^{-r} = 1 - 2^{-n}$$

[4 marks]

$n = 1$: $LHS = \sum_{r=1}^{1} 2^{-r} = 2^{-1} = \frac{1}{2}$, $RHS = 1 - 2^{-1} = \frac{1}{2}$

LHS = RHS, so the result is true for $n = 1$

Assume the result is true for $n = k$:

$\sum_{r=1}^{k} 2^{-r} = 1 - 2^{-k}$

Then $\sum_{r=1}^{k+1} 2^{-r} = \sum_{r=1}^{k} 2^{-r} + 2^{-(k+1)}$

$= 1 - 2^{-k} + 2^{-(k+1)}$

$= 1 - 2 \times 2^{-(k+1)} + 2^{-(k+1)}$

$= 1 - 2^{-(k+1)}$

which is the formula with $n = k + 1$, so it is true for $n = k + 1$.

The result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, so by induction

$\sum_{r=1}^{n} 2^{-r} = 1 - 2^{-n}$ is true for all integers $n \geq 1$

Turn over ▶

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14 Curve C_1 has equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

14 (a) Curve C_2 is a reflection of C_1 in the line $y=x$

Write down an equation of C_2

[1 mark]

$\frac{y^2}{16} + \frac{x^2}{4} = 1$

14 (b) Curve C_3 is a circle of radius 4, centred at the origin.

Describe a single transformation which maps C_1 onto C_3

[2 marks]

Stretch parallel to the y-axis, scale factor 2

$(y \to \frac{y}{2}: \quad \frac{x^2}{16} + \frac{y^2}{16} = 1, \text{ i.e. } x^2 + y^2 = 16)$

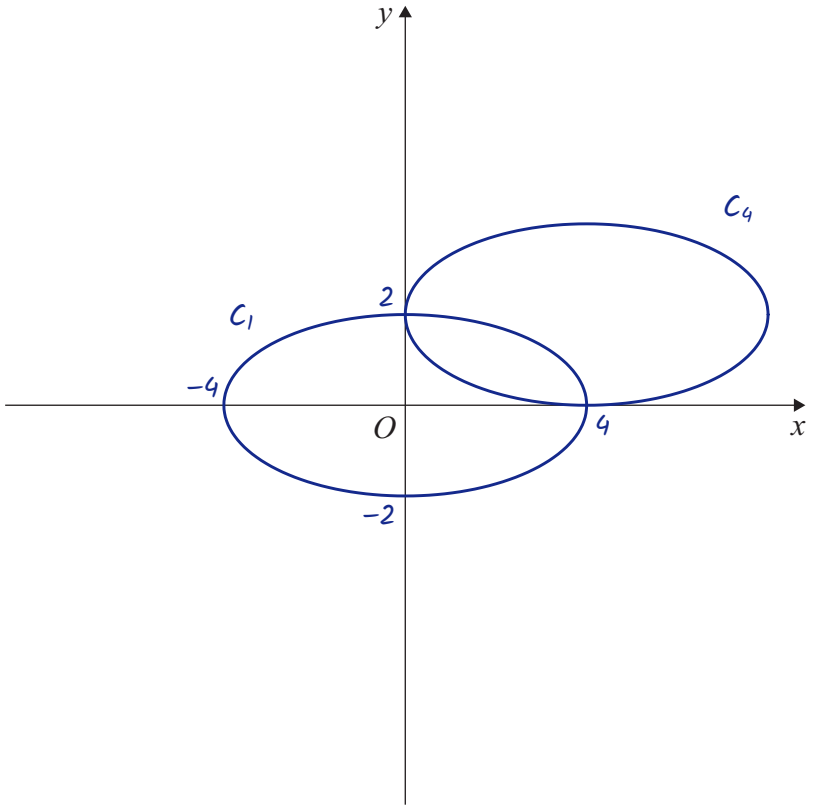
14 (c) Curve C_4 is a translation of C_1
The positive x-axis and the positive y-axis are tangents to C_4

14 (c) (i) Sketch the graphs of C_1 and C_4 on the axes opposite. Indicate the coordinates of the x and y intercepts on your graphs.

[2 marks]

C_4 has centre (4, 2) and touches the axes at (4, 0) and (0, 2)

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14 (c) (ii) Determine the translation vector.

[2 marks]

C_4 touches the positive y-axis, so its centre is 4 units right of O,

and touches the positive x-axis, so its centre is 2 units above O.

Translation vector $= \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

14 (c) (iii) The line $y = mx + c$ is a tangent to both C_1 and C_4
Find the value of m

[2 marks]

A line touching both C_1 and C_4 must be parallel to the

translation vector $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$, so $m = \frac{2}{4} = \frac{1}{2}$

Turn over ▶

Do not write outside the box

15 Two submarines are travelling on different straight lines.
The two lines are described by the equations

$$\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \frac{x-5}{4} = \frac{y}{2} = 4 - z$$

15 (a) (i) Show that the two lines intersect.

[3 marks]

First line: $x = 2 + 5\lambda$, $y = -1 + 3\lambda$, $z = 4 - 2\lambda$

Substitute into $\frac{x-5}{4} = \frac{y}{2}$:

$\frac{2 + 5\lambda - 5}{4} = \frac{-1 + 3\lambda}{2}$

$2(5\lambda - 3) = 4(3\lambda - 1)$

$10\lambda - 6 = 12\lambda - 4$

$-2\lambda = 2 \Rightarrow \lambda = -1$

Check in $\frac{x-5}{4} = 4 - z$ with $\lambda = -1$: $x = -3$, $y = -4$, $z = 6$

$\frac{-3-5}{4} = -2$, $4 - 6 = -2$, $\frac{-4}{2} = -2$

$\lambda = -1$ satisfies all three equations, so the point $(-3, -4, 6)$ lies on both lines, so the two lines intersect.

15 (a) (ii) Find the position vector of the point of intersection.

[1 mark]

$$\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - 1 \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 6 \end{bmatrix}$$

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15 (b) Tracey says that the submarines will collide because there is a common point on the two lines.

Explain why Tracey is not necessarily correct.

[1 mark]

The submarines might not be at the point of intersection at the same time, so they will not necessarily collide.

15 (c) Calculate the acute angle between the lines

$$\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \frac{x-5}{4} = \frac{y}{2} = 4 - z$$

Give your angle to the nearest 0.1°

[3 marks]

Direction of the second line:

$\frac{x-5}{4} = \frac{y}{2} = \frac{z-4}{-1}$ so $d_2 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = 5 \times 4 + 3 \times 2 + (-2)(-1) = 28$

$|d_1| = \sqrt{(5^2 + 3^2 + 2^2)} = \sqrt{38}, \quad |d_2| = \sqrt{(4^2 + 2^2 + 1^2)} = \sqrt{21}$

$\cos\theta = \frac{28}{\sqrt{38} \times \sqrt{21}} = \frac{28}{\sqrt{798}}$

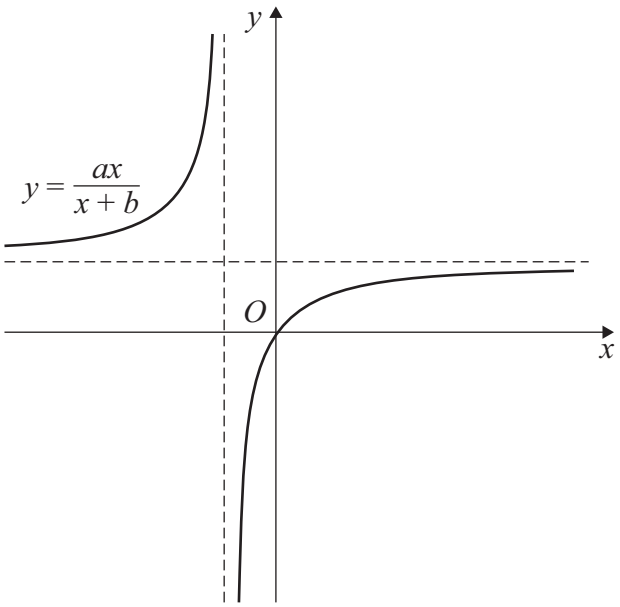
$\theta = 7.6128...° = 7.6°$ (nearest 0.1°)

Turn over ▶

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16 Curve C has equation $y = \dfrac{ax}{x+b}$ where a and b are constants.

The equations of the asymptotes to C are $x = -2$ and $y = 3$



16 (a) Write down the value of a and the value of b

[2 marks]

As $x \to \pm\infty$, $y \to a$, so $a = 3$

Denominator $x + b = 0$ when $x = -2$, so $b = 2$

16 (b) The gradient of C at the origin is $\dfrac{3}{2}$

With reference to the graph, explain why there is exactly one root of the equation

$$\frac{ax}{x+b} = \frac{3x}{2}$$

[2 marks]

The line $y = \frac{3x}{2}$ passes through O with gradient $\frac{3}{2}$, so it is the tangent to C at the origin.

C is a hyperbola, and a tangent to a hyperbola meets it only once, at the point of contact (the origin). So the equation has exactly one root ($x = 0$).

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16 (c) Using the values found in part **(a)**, solve the inequality

$$\frac{ax}{x+b} \leq 1 - x$$

[4 marks]

$\frac{3x}{x+2} \leq 1 - x$

Multiply both sides by $(x + 2)^2$, which is positive:

$3x(x + 2) \leq (1 - x)(x + 2)^2$

$0 \leq (x + 2)[\,(1 - x)(x + 2) - 3x\,]$

$0 \leq (x + 2)(-x^2 - 4x + 2)$

$(x + 2)(x^2 + 4x - 2) \leq 0$

Critical values: $x = -2$ (but $x \neq -2$) and

$x^2 + 4x - 2 = 0 \Rightarrow x = \frac{-4 \pm \sqrt{(16 + 8)}}{2} = -2 \pm \sqrt{6}$

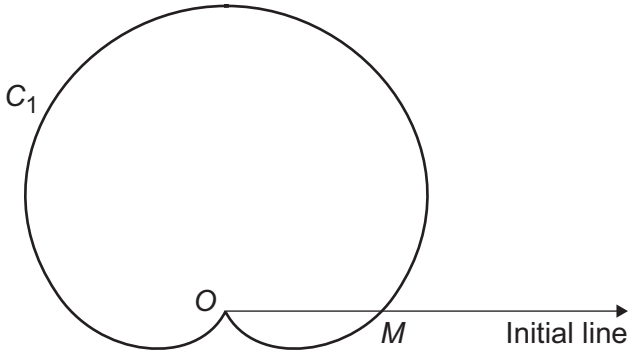
The cubic $(x + 2)(x^2 + 4x - 2)$ has a positive x^3 coefficient and

roots $-2 - \sqrt{6} < -2 < -2 + \sqrt{6}$, so it is ≤ 0 when

$x \leq -2 - \sqrt{6}$ or $-2 < x \leq -2 + \sqrt{6}$

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17 The curve C_1 has polar equation $r = 2a(1 + \sin\theta)$ for $-\pi < \theta \leq \pi$ where a is a positive constant.



The point M lies on C_1 and the initial line.

17 (a) Write down, in terms of a, the polar coordinates of M

[1 mark]

$\theta = 0$: $r = 2a(1 + \sin 0) = 2a$

$M = (2a, 0)$

17 (b) N is the point on C_1 that is furthest from the pole O

Find, in terms of a, the polar coordinates of N

[2 marks]

r is greatest when $\sin\theta = 1$, i.e. $\theta = \frac{\pi}{2}$

$r = 2a(1 + 1) = 4a$

$N = \left(4a, \frac{\pi}{2}\right)$

Do not write outside the box

17 (c) The curve C_2 has polar equation $r = 3a$ for $-\pi < \theta \leq \pi$
C_2 intersects C_1 at points P and Q

Show that the area of triangle NPQ can be written in the form

$$m\sqrt{3}a^2$$

where m is a rational number to be determined.

[5 marks]

At P and Q: $2a(1 + \sin\theta) = 3a$

$1 + \sin\theta = \frac{3}{2} \quad \Rightarrow \quad \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$ (P and Q symmetric about $\theta = \frac{\pi}{2}$)

$x = \pm 3a\cos\frac{\pi}{6} = \pm\frac{3\sqrt{3}a}{2}, \qquad y = 3a\sin\frac{\pi}{6} = \frac{3a}{2}$

$PQ = 2 \times \frac{3\sqrt{3}a}{2} = 3\sqrt{3}a$

N is the point $(0, 4a)$, so the height of $NPQ = 4a - \frac{3a}{2} = \frac{5a}{2}$

Area of $NPQ = \frac{1}{2} \times 3\sqrt{3}a \times \frac{5a}{2}$

$= \frac{15}{4}\sqrt{3}a^2$

$\therefore \quad m = \frac{15}{4}$

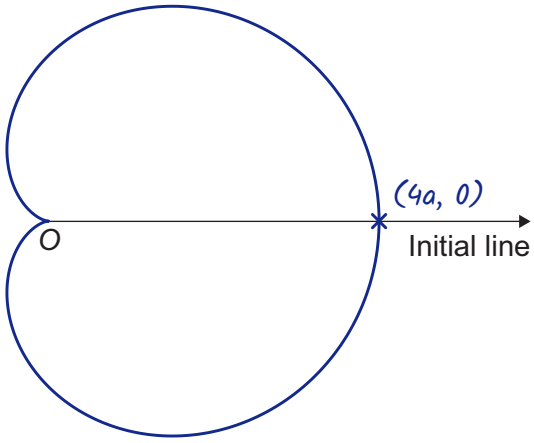
Question 17 continues on the next page

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17 (d) On the initial line below, sketch the graph of $r = 2a(1 + \cos\theta)$ for $-\pi < \theta \leq \pi$

Include the polar coordinates, in terms of a, of any intersection points with the initial line.

[2 marks]



END OF QUESTIONS

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