AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

AS
FURTHER MATHEMATICS

Paper 1

Monday 11 May 2020 Afternoon Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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TOTAL	

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Answer **all** questions in the spaces provided.

1 Express the complex number $1 - i\sqrt{3}$ in modulus-argument form.

Tick (✓) **one** box.

[1 mark]

$2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ ☐

$2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ ☐

$2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ ✓

$2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$ ☐

2 Given that $1 - i$ is a root of the equation $z^3 - 3z^2 + 4z - 2 = 0$, find the other two roots.

Tick (✓) **one** box.

[1 mark]

$-1 + i$ and -1 ☐

$1 + i$ and 1 ✓

$-1 + i$ and 1 ☐

$1 + i$ and -1 ☐

Do not write outside the box

3 Given $(x-1)(x-2)(x-a) < 0$ and $a > 2$

Find the set of possible values of x.

Tick (✓) **one** box.

[1 mark]

$\{x : x < 1\} \cup \{x : 2 < x < a\}$ ☑

$\{x : 1 < x < 2\} \cup \{x : x > a\}$ ☐

$\{x : x < -a\} \cup \{x : -2 < x < -1\}$ ☐

$\{x : -a < x < -2\} \cup \{x : x > -1\}$ ☐

Turn over for the next question

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4 The matrices **A** and **B** are such that

$$\mathbf{A} = \begin{bmatrix} 2 & a & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -3 \\ -2 & 4a \\ 0 & 5 \end{bmatrix}$$

4 (a) Find the product **AB** in terms of a.

[2 marks]

Row 1 × column 1: $2(1) + a(-2) + 3(0) = 2 - 2a$

Row 1 × column 2: $2(-3) + a(4a) + 3(5) = 4a^2 + 9$

Row 2 × column 1: $0(1) + (-2)(-2) + 1(0) = 4$

Row 2 × column 2: $0(-3) + (-2)(4a) + 1(5) = 5 - 8a$

$$\mathbf{AB} = \begin{bmatrix} 2 - 2a & 4a^2 + 9 \\ 4 & 5 - 8a \end{bmatrix}$$

4 (b) Find the determinant of **AB** in terms of a.

[1 mark]

$\det(\mathbf{AB}) = (2 - 2a)(5 - 8a) - 4(4a^2 + 9)$

$= 10 - 16a - 10a + 16a^2 - 16a^2 - 36$

$= -26 - 26a$

Do not write outside the box

4 (c) Show that **AB** is singular when $a = -1$

[2 marks]

AB is singular ⇔ det(AB) = 0

When a = −1: det(AB) = −26 − 26(−1)

= −26 + 26

= 0

det(AB) = 0, so AB is singular when a = −1

Turn over for the next question

Do not write outside the box

5 (a) Show that

$$r^2(r+1)^2 - (r-1)^2r^2 = pr^3$$

where p is an integer to be found.

[1 mark]

$r^2(r+1)^2 - (r-1)^2r^2 = r^2(r^2+2r+1) - r^2(r^2-2r+1)$

$= r^4 + 2r^3 + r^2 - r^4 + 2r^3 - r^2$

$= 4r^3$

so $p = 4$

Do not write outside the box

5 (b) Hence use the method of differences to show that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

[3 marks]

$\sum_{r=1}^{n} 4r^3 = \sum_{r=1}^{n} [\, r^2(r+1)^2 - (r-1)^2r^2 \,]$

$r = 1$: $1^2 \times 2^2 - 0^2 \times 1^2$

$r = 2$: $2^2 \times 3^2 - 1^2 \times 2^2$

$r = 3$: $3^2 \times 4^2 - 2^2 \times 3^2$

$\vdots$

$r = n - 1$: $(n-1)^2n^2 - (n-2)^2(n-1)^2$

$r = n$: $n^2(n+1)^2 - (n-1)^2n^2$

Adding, all the terms cancel except $n^2(n+1)^2$ and $-0^2 \times 1^2$:

$\sum_{r=1}^{n} 4r^3 = n^2(n+1)^2 - 0 = n^2(n+1)^2$

$4\sum_{r=1}^{n} r^3 = n^2(n+1)^2$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ as required

Do not write outside the box

6 Anna has been asked to describe the transformation given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

She writes her answer as follows:

The transformation is a rotation about the x-axis through an angle of θ, where

$$\sin\theta = \frac{1}{2} \quad \text{and} \quad -\sin\theta = -\frac{1}{2}$$

$$\theta = 30°$$

Identify and correct the error in Anna's work.

[2 marks]

Anna has only used the sine entries and has ignored

cos θ = $-\frac{\sqrt{3}}{2}$, which is negative, so θ cannot be 30°.

sin θ > 0 and cos θ < 0, so θ is obtuse (second quadrant):

θ = 180° − 30° = 150°

The transformation is a rotation about the x-axis through 150°

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7 Prove by induction that, for all integers $n \geq 1$, the expression $7^n - 3^n$ is divisible by 4 **[4 marks]**

$n = 1$: $7^1 - 3^1 = 4$, which is divisible by 4,

so the result is true for $n = 1$

Assume the result is true for $n = k$, i.e.

$7^k - 3^k = 4m$ for some integer m, so $7^k = 4m + 3^k$

Then $7^{k+1} - 3^{k+1} = 7 \times 7^k - 3 \times 3^k$

$= 7(4m + 3^k) - 3 \times 3^k$

$= 28m + 7 \times 3^k - 3 \times 3^k$

$= 28m + 4 \times 3^k$

$= 4(7m + 3^k)$

$7m + 3^k$ is an integer, so $7^{k+1} - 3^{k+1}$ is divisible by 4,

i.e. the result is true for $n = k + 1$.

The result is true for $n = 1$, and if it is true for $n = k$ then

it is true for $n = k + 1$, so by induction $7^n - 3^n$ is

divisible by 4 for all integers $n \geq 1$.

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8 (a) Prove that

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

[5 marks]

Let $y = \tanh^{-1} x$, so $x = \tanh y$

$x = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$ (× e^y top and bottom)

$x(e^{2y} + 1) = e^{2y} - 1$

$xe^{2y} + x = e^{2y} - 1$

$1 + x = e^{2y} - xe^{2y}$

$1 + x = e^{2y}(1 - x)$

$e^{2y} = \frac{1+x}{1-x}$

$2y = \ln\left(\frac{1+x}{1-x}\right)$

$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\therefore \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as required

Do not write outside the box

8 (b) Prove that the graphs of

$$y = \sinh x \quad \text{and} \quad y = \cosh x$$

do **not** intersect.

[3 marks]

Assume, for contradiction, that the graphs do intersect.

Then sinh x = cosh x for some real x:

$\frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x})$

$e^x - e^{-x} = e^x + e^{-x}$

$2e^{-x} = 0 \quad \Rightarrow \quad e^{-x} = 0$

But $e^{-x} > 0$ for all real x, which is a contradiction.

$\therefore$ the graphs of y = sinh x and y = cosh x do not intersect.

Do not write outside the box

9 The quadratic equation $2x^2 + px + 3 = 0$ has two roots, α and β, where $\alpha > \beta$.

9 (a) (i) Write down the value of $\alpha\beta$.

[1 mark]

$\alpha\beta = \frac{3}{2}$

9 (a) (ii) Express $\alpha + \beta$ in terms of p.

[1 mark]

$\alpha + \beta = -\frac{p}{2}$

9 (b) Hence find $(\alpha - \beta)^2$ in terms of p.

[2 marks]

$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

$= (\alpha + \beta)^2 - 4\alpha\beta$

$= \left(-\frac{p}{2}\right)^2 - 4 \times \frac{3}{2}$

$= \frac{p^2}{4} - 6$

Do not write outside the box

9 (c) **Hence** find, in terms of p, a quadratic equation with roots $\alpha - 1$ and $\beta + 1$

[4 marks]

New roots: $\alpha - 1$ and $\beta + 1$

Sum $= (\alpha - 1) + (\beta + 1) = \alpha + \beta = -\frac{p}{2}$

Product $= (\alpha - 1)(\beta + 1) = \alpha\beta + \alpha - \beta - 1$

$(\alpha - \beta)^2 = \frac{p^2}{4} - 6 = \frac{p^2 - 24}{4}$

$\alpha > \beta$ so $\alpha - \beta > 0$: $\alpha - \beta = \frac{1}{2}\sqrt{(p^2 - 24)}$

Product $= \frac{3}{2} + \frac{1}{2}\sqrt{(p^2 - 24)} - 1 = \frac{1}{2} + \frac{1}{2}\sqrt{(p^2 - 24)}$

Equation: $x^2 - (\text{sum})x + (\text{product}) = 0$

$x^2 + \frac{p}{2}x + \frac{1}{2} + \frac{1}{2}\sqrt{(p^2 - 24)} = 0$

or $2x^2 + px + 1 + \sqrt{(p^2 - 24)} = 0$

Do not write outside the box

10 (a) Show that the equation

$$y = \frac{3x - 5}{2x + 4}$$

can be written in the form

$$(x + a)(y + b) = c$$

where a, b and c are integers to be found.

[3 marks]

$y(2x + 4) = 3x - 5$

$2xy + 4y - 3x + 5 = 0$

$xy + 2y - \frac{3}{2}x + \frac{5}{2} = 0$

$(x + 2)(y - \frac{3}{2}) = xy - \frac{3}{2}x + 2y - 3$

so $xy + 2y - \frac{3}{2}x = (x + 2)(y - \frac{3}{2}) + 3$

$(x + 2)(y - \frac{3}{2}) + 3 + \frac{5}{2} = 0$

$(x + 2)(y - \frac{3}{2}) = -\frac{11}{2}$

$a = 2, \quad b = -\frac{3}{2}, \quad c = -\frac{11}{2}$

10 (b) Write down the equations of the asymptotes of the graph of

$$y = \frac{3x - 5}{2x + 4}$$

[2 marks]

$x = -2$

$y = \frac{3}{2}$

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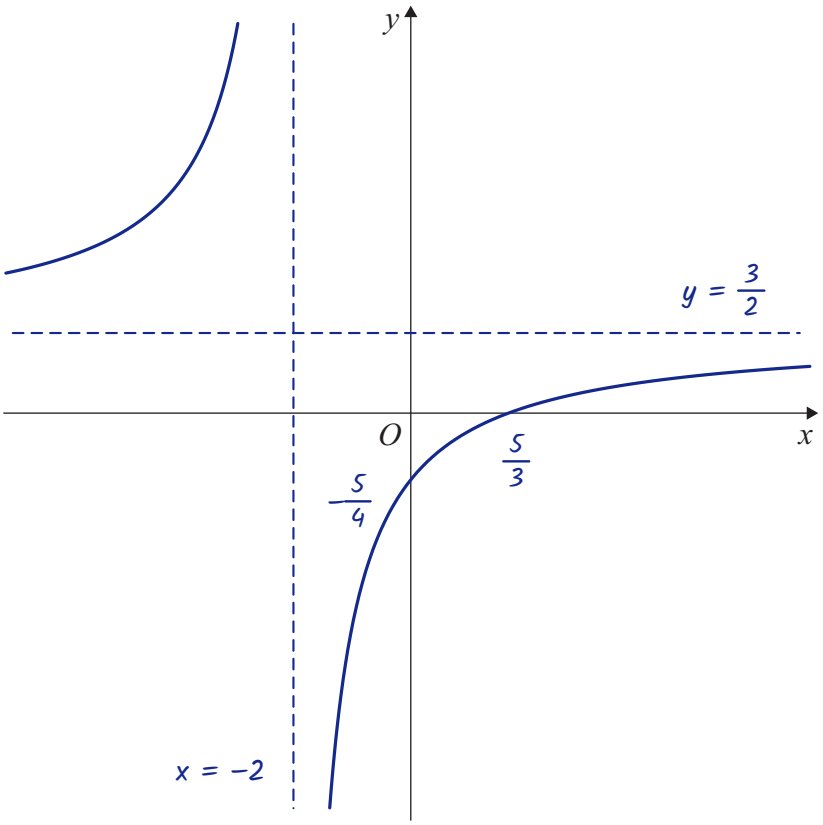
10 (c) Sketch, on the axes provided, the graph of

$$y = \frac{3x - 5}{2x + 4}$$

[3 marks]

Asymptotes $x = -2$ and $y = 3/2$

$x = 0$: $y = \frac{0 - 5}{0 + 4} = -\frac{5}{4}$ $\quad$ $y = 0$: $3x - 5 = 0$, $x = \frac{5}{3}$



Turn over ▶

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11 Sketch the polar graph of

$$r = \sinh\theta + \cosh\theta$$

for $0 \leq \theta \leq 2\pi$

[3 marks]

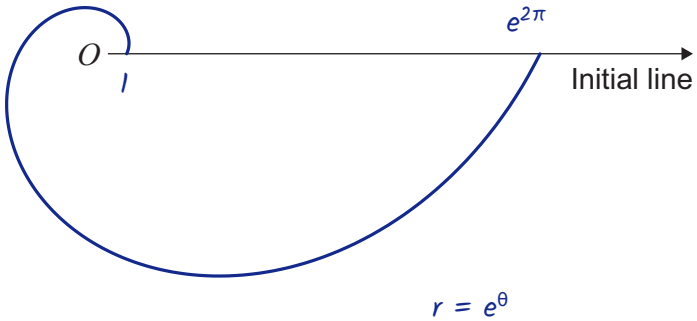
$\sinh\theta + \cosh\theta = \frac{e^{\theta} - e^{-\theta}}{2} + \frac{e^{\theta} + e^{-\theta}}{2} = e^{\theta}$

so $r = e^{\theta}$

$\theta = 0$: $r = 1$; $\theta = \pi/2$: $r \approx 4.81$; $\theta = \pi$: $r \approx 23.1$

$\theta = 3\pi/2$: $r \approx 111$; $\theta = 2\pi$: $r = e^{2\pi} \approx 535$

r increases as θ increases, so the graph is a spiral that starts at $r = 1$ on the initial line and ends at $r = e^{2\pi}$ on the initial line (not to scale).



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12 The mean value of the function f over the interval $1 \leq x \leq 5$ is m.

The graph of $y = \mathrm{g}(x)$ is a reflection in the x-axis of $y = \mathrm{f}(x)$.

The graph of $y = \mathrm{h}(x)$ is a translation of $y = \mathrm{g}(x)$ by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Determine, in terms of m, the mean value of the function h over the interval $4 \leq x \leq 8$

[2 marks]

Reflection in the x-axis: g(x) = −f(x)

so the mean value of g over 1 ≤ x ≤ 5 is −m

Translation 3 right and 7 up: h(x) = g(x − 3) + 7

so the interval 1 ≤ x ≤ 5 becomes 4 ≤ x ≤ 8 and every

value of the function is increased by 7

Mean value of h over 4 ≤ x ≤ 8 = −m + 7 = 7 − m

Turn over for the next question

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13 Line l_1 has equation

$$\frac{x-2}{3} = \frac{1-2y}{4} = -z$$

and line l_2 has equation

$$\mathbf{r} = \begin{bmatrix} -7 \\ 4 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} 12 \\ a+3 \\ 2b \end{bmatrix}$$

13 (a) In the case when l_1 and l_2 are parallel, show that $a = -11$ and find the value of b.

[4 marks]

$\frac{1-2y}{4} = \frac{-2(y-0.5)}{4} = \frac{y-0.5}{-2}$ so l_1: $\frac{x-2}{3} = \frac{y-0.5}{-2} = \frac{z-0}{-1}$

Direction of l_1: $3i - 2j - k$

Direction of l_2: $12i + (a+3)j + 2bk$

Parallel: $12i + (a+3)j + 2bk = \lambda(3i - 2j - k)$

i: $12 = 3\lambda \Rightarrow \lambda = 4$

j: $a + 3 = -2\lambda = -8 \Rightarrow a = -11$

k: $2b = -\lambda = -4 \Rightarrow b = -2$

$\therefore$ $a = -11$ (as required) and $b = -2$

Do not write outside the box

13 (b) In a **different** case, the lines l_1 and l_2 intersect at exactly one point, and the value of b is 3

Find the value of a.

[5 marks]

$b = 3$: l_2: $r = (-7 + 12\mu)i + (4 + \mu(a + 3))j + (-2 + 6\mu)k$

l_1: $r = (2 + 3\lambda)i + (0.5 - 2\lambda)j - \lambda k$

At the point of intersection:

i: $-7 + 12\mu = 2 + 3\lambda$

j: $4 + \mu(a + 3) = 0.5 - 2\lambda$

k: $-2 + 6\mu = -\lambda \Rightarrow \lambda = 2 - 6\mu$

Substituting into i: $-7 + 12\mu = 2 + 3(2 - 6\mu) = 8 - 18\mu$

$30\mu = 15 \Rightarrow \mu = \frac{1}{2}$

$\lambda = 2 - 6 \times \frac{1}{2} = -1$

j: $4 + \frac{1}{2}(a + 3) = 0.5 - 2(-1) = 2.5$

$\frac{1}{2}(a + 3) = -1.5$

$a + 3 = -3$

$a = -6$

Turn over ▶

Do not write outside the box

14 (a) Given

$$\frac{x+7}{x+1} \leq x+1$$

show that

$$\frac{(x+a)(x+b)}{x+c} \geq 0$$

where a, b, and c are integers to be found.

[4 marks]

$\frac{x+7}{x+1} \leq x+1$

$0 \leq x+1-\frac{x+7}{x+1} = \frac{(x+1)^2}{x+1} - \frac{x+7}{x+1}$

$0 \leq \frac{x^2+2x+1-x-7}{x+1}$

$0 \leq \frac{x^2+x-6}{x+1}$

$\frac{(x+3)(x-2)}{x+1} \geq 0$

$a = 3, \quad b = -2, \quad c = 1$

14 (b) Briefly explain why this statement is incorrect.

$$\frac{(x+p)(x+q)}{x+r} \geq 0 \Leftrightarrow (x+p)(x+q)(x+r) \geq 0$$

[1 mark]

$x = -r$ is a solution of the inequality on the right (it gives 0),

but the left-hand side is undefined at $x = -r$ (division by zero),

so $x = -r$ is not a solution of the inequality on the left.

Do not write outside the box

14 (c) Solve

$$\frac{x+7}{x+1} \leq x+1$$

[2 marks]

$\frac{(x+3)(x-2)}{x+1} \geqslant 0$ critical values $x = -3,\ -1,\ 2$

$x < -3$: negative $-3 < x < -1$: positive

$-1 < x < 2$: negative $x > 2$: positive

$x = -1$ is excluded (denominator 0); $x = -3$ and $x = 2$ give 0

$-3 \leqslant x < -1$ or $x \geqslant 2$

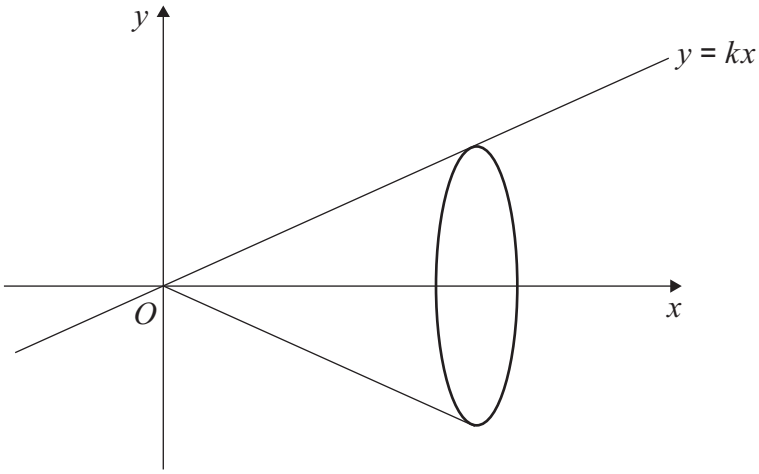
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15 A segment of the line $y = kx$ is rotated about the x-axis to generate a cone with vertex O.

The distance of O from the centre of the base of the cone is h.

The radius of the base of the cone is r.



15 (a) Find k in terms of r and h.

[1 mark]

The line passes through (h, r), so $k = \frac{r}{h}$

Do not write outside the box

15 (b) Use calculus to prove that the volume of the cone is

$$\frac{1}{3}\pi r^2 h$$

[3 marks]

$V = \pi\int_0^h y^2\,dx = \pi\int_0^h \left(\frac{rx}{h}\right)^2 dx$

$= \pi\int_0^h \frac{r^2x^2}{h^2}\,dx$

$= \pi\left[\frac{r^2x^3}{3h^2}\right]_0^h$

$= \pi\left(\frac{r^2h^3}{3h^2} - 0\right)$

$= \frac{1}{3}\pi r^2 h$ as required

Do not write outside the box

16 **A** and **B** are non-singular square matrices.

16 (a) Write down the product $\mathbf{AA}^{-1}$ as a single matrix.

[1 mark]

$AA^{-1} = I$

16 (b) **M** is a matrix such that $\mathbf{M} = \mathbf{AB}$.

Prove that $\mathbf{M}^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

[3 marks]

$M = AB$

Pre-multiply by $B^{-1}A^{-1}$:

$B^{-1}A^{-1}M = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

Post-multiply by M^{-1}:

$B^{-1}A^{-1}MM^{-1} = IM^{-1}$

$B^{-1}A^{-1}I = M^{-1}$

$\therefore \quad M^{-1} = B^{-1}A^{-1}$

Do not write outside the box

17 The polar equation of the circle C is

$$r = a(\cos\theta + \sin\theta)$$

Find, in terms of a, the radius of C.

Fully justify your answer.

[4 marks]

$r = a(\cos\theta + \sin\theta)$

$r^2 = a(r\cos\theta + r\sin\theta)$

Using $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$:

$x^2 + y^2 = a(x + y)$

$x^2 - ax + y^2 - ay = 0$

$\left(x - \frac{a}{2}\right)^2 - \frac{a^2}{4} + \left(y - \frac{a}{2}\right)^2 - \frac{a^2}{4} = 0$

$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{2}$

Circle with centre $\left(\frac{a}{2}, \frac{a}{2}\right)$ and radius $\sqrt{\left(\frac{a^2}{2}\right)}$

radius $= \frac{a}{\sqrt{2}}$

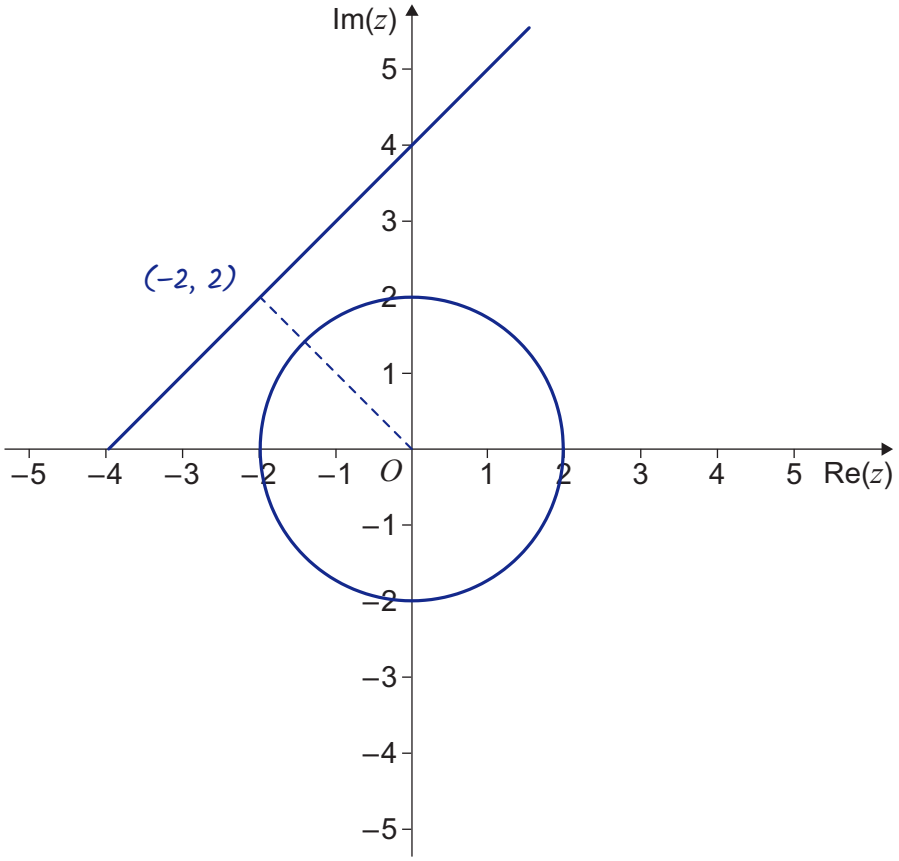
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18 The locus of points L_1 satisfies the equation $|z| = 2$

The locus of points L_2 satisfies the equation $\arg(z + 4) = \frac{\pi}{4}$

18 (a) Sketch L_1 on the Argand diagram below.

[1 mark]



18 (b) Sketch L_2 on the Argand diagram above.

[1 mark]

Do not write outside the box

18 (c) The complex number $a + ib$, where a and b are real, lies on L_1

The complex number $c + id$, where c and d are real, lies on L_2

Calculate the least possible value of the expression

$$(c - a)^2 + (d - b)^2$$

[3 marks]

$(c - a)^2 + (d - b)^2$ is the square of the distance between the point $a + ib$ on L_1 and the point $c + id$ on L_2, so the least value is (shortest distance between the circle and the half-line)^2

L_2: $y = x + 4$ (from $(-4, 0)$ with gradient 1)

Perpendicular from O to L_2: $y = -x$, meets L_2 at $(-2, 2)$

Distance from O to $(-2, 2) = \sqrt{((-2 - 0)^2 + (2 - 0)^2)} = \sqrt{8} = 2\sqrt{2}$

$2\sqrt{2} > 2$, so shortest distance $= \sqrt{8} - 2 = 2\sqrt{2} - 2$

Least value $= (2\sqrt{2} - 2)^2 = 8 - 8\sqrt{2} + 4$

$= 12 - 8\sqrt{2}$

END OF QUESTIONS

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ANSWER IN THE SPACES PROVIDED

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