AQA

Please write clearly in block capitals.

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I declare this is my own work.

AS
FURTHER MATHEMATICS

Paper 1

Monday 13 May 2024 Afternoon Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
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TOTAL	

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Answer **all** questions in the spaces provided.

1 Express $\cosh^2 x$ in terms of $\sinh x$

Circle your answer.

[1 mark]

$1 + \sinh^2 x$ $\quad$ $1 - \sinh^2 x$ $\quad$ $\sinh^2 x - 1$ $\quad$ $-1 - \sinh^2 x$

2 The function f is defined by

$$\mathrm{f}(x) = 2x + 3 \qquad 0 \leq x \leq 5$$

The region *R* is enclosed by $y = \mathrm{f}(x)$, $x = 5$, the x-axis and the y-axis.

The region *R* is rotated through 2π radians about the x-axis.

Give an expression for the volume of the solid formed.

Tick (✓) **one** box.

[1 mark]

$\pi\int_0^5 (2x + 3)\,\mathrm{d}x$ ☐

$\pi\int_0^5 (2x + 3)^2\,\mathrm{d}x$ ☑

$2\pi\int_0^5 (2x + 3)\,\mathrm{d}x$ ☐

$2\pi\int_0^5 (2x + 3)^2\,\mathrm{d}x$ ☐

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3 The matrix **A** is such that $\det(\mathbf{A}) = 2$

Determine the value of $\det(\mathbf{A}^{-1})$

Circle your answer.

[1 mark]

-2 $\quad -\frac{1}{2}$ $\quad \frac{1}{2}$ $\quad 2$

4 The line L has vector equation

$$\mathbf{r} = \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -9 \\ 1 \\ 3 \end{bmatrix}$$

Give the equation of L in Cartesian form.

Tick (✓) **one** box.

[1 mark]

$\frac{x+4}{-9} = \frac{y-7}{1} = \frac{z}{3}$ ☐

$\frac{x-4}{-9} = \frac{y+7}{1} = \frac{z}{3}$ ✓

$\frac{x+9}{4} = \frac{y-1}{-7}$, $z = 3$ ☐

$\frac{x-9}{4} = \frac{y+1}{-7}$, $z = 3$ ☐

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5 The vectors **a** and **b** are given by

$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$

5 (a) Calculate **a.b**

[1 mark]

$a.b = 3 \times 2 + 4 \times (-1) + (-2) \times (-5) = 6 - 4 + 10 = 12$

5 (b) Calculate $|\mathbf{a}|$ and $|\mathbf{b}|$

[2 marks]

$|a| = \sqrt{(3^2 + 4^2 + (-2)^2)} = \sqrt{(9 + 16 + 4)} = \sqrt{29}$

$|b| = \sqrt{(2^2 + (-1)^2 + (-5)^2)} = \sqrt{(4 + 1 + 25)} = \sqrt{30}$

$|\mathbf{a}| =$ $\sqrt{29}$ $|\mathbf{b}| =$ $\sqrt{30}$

5 (c) Calculate the acute angle between **a** and **b**

Give your answer to the nearest degree.

[2 marks]

$\cos\theta = \frac{a.b}{|a||b|} = \frac{12}{\sqrt{29} \times \sqrt{30}} = 0.40683...$

$\theta = \cos^{-1}(0.40683...) = 65.9936...°$

Acute angle = 66° (nearest degree)

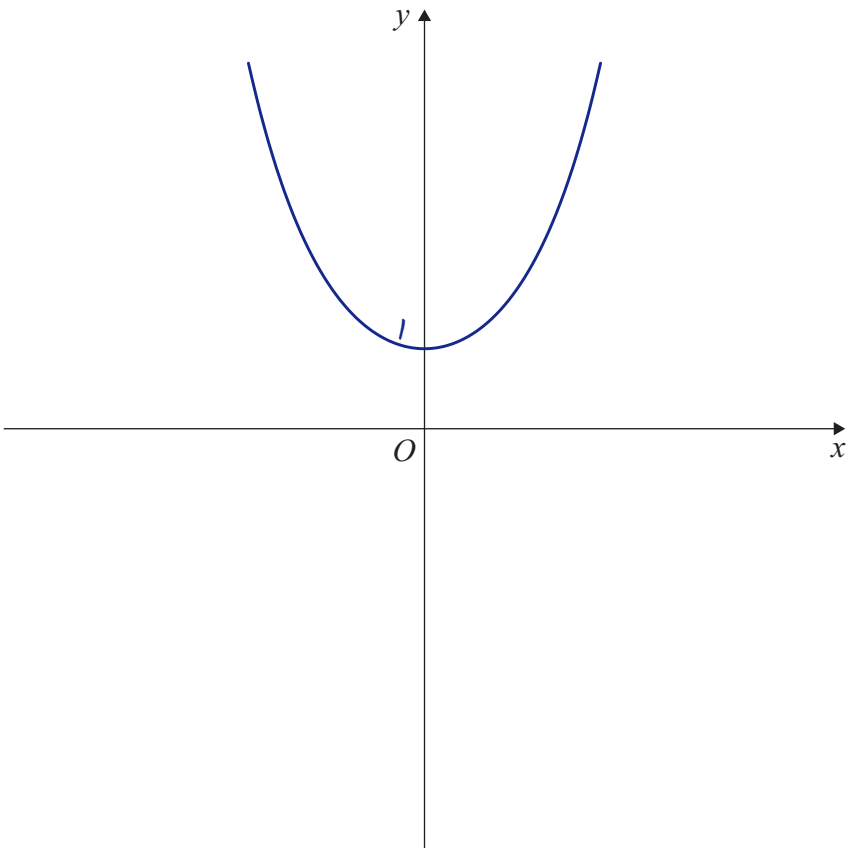
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6 (a) On the axes below, sketch the graph of

$$y = \cosh x$$

Indicate the value of any intercept of the curve with the axes.

[2 marks]



6 (b) Solve the equation

$$\cosh x = 2$$

Give your answers to three significant figures.

[2 marks]

$\cosh x = 2 \Rightarrow x = \pm\cosh^{-1}2 = \pm\ln(2 + \sqrt{(2^2 - 1)})$

$x = \pm\ln(2 + \sqrt{3}) = \pm 1.3169...$

$x = 1.32$ or $x = -1.32$ (3 s.f.)

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7 The function f is defined by

$$f(x) = \frac{1}{\sqrt{x}} \qquad 4 \leq x \leq 7$$

Find the mean value of f over the interval $4 \leq x \leq 7$

Give your answer in exact form.

[3 marks]

Mean value $= \frac{1}{7-4}\int_4^7 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\int_4^7 x^{-\frac{1}{2}}\,dx$

$= \frac{1}{3}\left[2x^{\frac{1}{2}}\right]_4^7$

$= \frac{1}{3}\left(2\sqrt{7} - 2\sqrt{4}\right) = \frac{1}{3}\left(2\sqrt{7} - 4\right)$

$= \frac{2}{3}\left(\sqrt{7} - 2\right)$

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8 (a) The complex number z is given by $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$

8 (a) (i) Write down the complex conjugate z^* in terms of x and y

[1 mark]

$z^* = x - iy$

8 (a) (ii) Hence prove that zz^* is real for all $z \in \mathbb{C}$

[2 marks]

$zz^* = (x + iy)(x - iy)$

$= x^2 - ixy + ixy - i^2y^2$

$= x^2 + y^2$ (since $i^2 = -1$)

x and y are both real, so $x^2 + y^2$ is real

$\therefore zz^*$ is real for all $z \in \mathbb{C}$

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8 (b) The complex number w satisfies the equation

$$3w + 10\mathrm{i} = 2w^* + 5$$

8 (b) (i) Find w

[3 marks]

Let $w = x + iy$, where $x, y \in \mathbb{R}$, so $w^* = x - iy$

$3(x + iy) + 10i = 2(x - iy) + 5$

$3x + 3iy + 10i = 2x - 2iy + 5$

Real parts: $3x = 2x + 5 \Rightarrow x = 5$

Imaginary parts: $3y + 10 = -2y \Rightarrow 5y = -10 \Rightarrow y = -2$

$w = 5 - 2i$

8 (b) (ii) Calculate the value of $w^2(w^*)^2$

[1 mark]

$w^2(w^*)^2 = (ww^*)^2 = (5^2 + 2^2)^2 = 29^2 = 841$

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9 (a) Show that, for all positive integers r,

$$\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{1}{(r+1)(r+2)}$$

[1 mark]

$\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{(r+1)^2 - r(r+2)}{(r+1)(r+2)}$

$= \frac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)}$

$= \frac{1}{(r+1)(r+2)}$

9 (b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{an+b}$$

where a and b are integers to be determined.

[3 marks]

$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{r+1}{r+2} - \frac{r}{r+1} \right)$

$r = 1$: $\frac{2}{3} - \frac{1}{2}$

$r = 2$: $\frac{3}{4} - \frac{2}{3}$

$r = 3$: $\frac{4}{5} - \frac{3}{4}$

$\vdots$

$r = n-1$: $\frac{n}{n+1} - \frac{n-1}{n}$

$r = n$: $\frac{n+1}{n+2} - \frac{n}{n+1}$

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Adding, all terms cancel except the first and the last:

$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n+1}{n+2} - \frac{1}{2}$

$= \frac{2(n+1) - (n+2)}{2(n+2)}$

$= \frac{n}{2n+4}$

$a = 2$, $b = 4$

9 (c) Hence find the exact value of

$$\sum_{r=1001}^{2000} \frac{1}{(r+1)(r+2)}$$

[3 marks]

$\sum_{r=1001}^{2000} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{2000} - \sum_{r=1}^{1000}$

$\sum_{r=1}^{2000} = \frac{2000}{2 \times 2000 + 4} = \frac{2000}{4004} = \frac{500}{1001}$

$\sum_{r=1}^{1000} = \frac{1000}{2 \times 1000 + 4} = \frac{1000}{2004} = \frac{250}{501}$

$\sum_{r=1001}^{2000} \frac{1}{(r+1)(r+2)} = \frac{500}{1001} - \frac{250}{501} = \frac{250500 - 250250}{501501} = \frac{250}{501501}$

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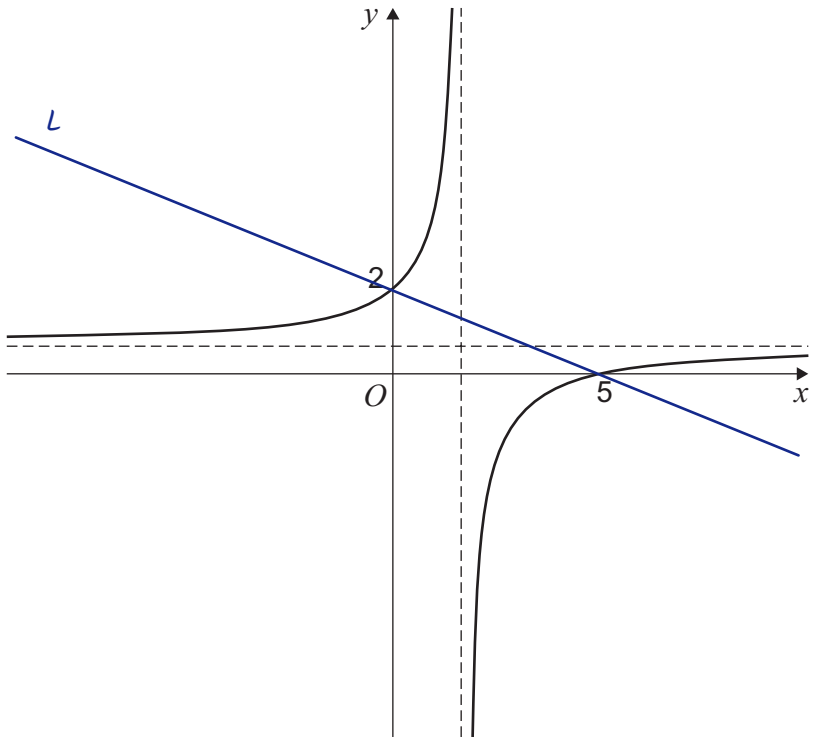
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10 The curve C has equation

$$y = \frac{2x - 10}{3x - 5}$$

Figure 1 shows the curve C with its asymptotes.

Figure 1



10 (a) Write down the equations of the asymptotes of C

[2 marks]

Vertical asymptote: $3x - 5 = 0 \Rightarrow x = \frac{5}{3}$

Horizontal asymptote: as $x \to \pm\infty$, $y \to \frac{2}{3} \Rightarrow y = \frac{2}{3}$

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10 (b) The line L has equation

$$y = -\frac{2}{5}x + 2$$

10 (b) (i) Draw the line L on **Figure 1**

[2 marks]

Line L drawn on Figure 1, through (0, 2) and (5, 0)

10 (b) (ii) Hence, or otherwise, solve the inequality

$$\frac{2x - 10}{3x - 5} \leq -\frac{2}{5}x + 2$$

[2 marks]

C and L meet at (0, 2) and (5, 0); C is on or below L when

$x \leq 0$, $\frac{5}{3} < x \leq 5$

Turn over for the next question

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11 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 3\mathrm{i} & -2 \\ a & -\mathrm{i} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 5 \\ -2\mathrm{i} & -1 \end{bmatrix}$$

where a is a real number.

Calculate the product **AB** in terms of a

Give your answer in its simplest form.

[3 marks]

$AB = \begin{bmatrix} 3i & -2 \\ a & -i \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -2i & -1 \end{bmatrix}$

$= \begin{bmatrix} 3i \times 4 + (-2)(-2i) & 3i \times 5 + (-2)(-1) \\ 4a + (-i)(-2i) & 5a + (-i)(-1) \end{bmatrix}$

$= \begin{bmatrix} 12i + 4i & 15i + 2 \\ 4a + 2i^2 & 5a + i \end{bmatrix}$

$= \begin{bmatrix} 16i & 2 + 15i \\ 4a - 2 & 5a + i \end{bmatrix}$

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12 Prove by induction that, for all $n \in \mathbb{N}$, the expression

$$5^n - 2^n$$

is divisible by 3

[4 marks]

$n = 1$: $5^1 - 2^1 = 3$, and 3 is divisible by 3, so true for $n = 1$

Assume true for $n = k$: $5^k - 2^k = 3m$ for some integer m

$\Rightarrow 5^k = 3m + 2^k$

$n = k + 1$: $5^{k+1} - 2^{k+1} = 5 \times 5^k - 2 \times 2^k$

$= 5(3m + 2^k) - 2 \times 2^k$

$= 15m + 5 \times 2^k - 2 \times 2^k$

$= 15m + 3 \times 2^k$

$= 3(5m + 2^k)$

$5m + 2^k$ is an integer, so $5^{k+1} - 2^{k+1}$ is divisible by 3

$\therefore$ true for $n = k + 1$

If $5^n - 2^n$ is divisible by 3 when $n = k$, then it is also divisible by 3 when $n = k + 1$. Since it is divisible by 3 when $n = 1$, by induction $5^n - 2^n$ is divisible by 3 for all $n \in \mathbb{N}$

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13 The cubic equation $x^3 - x - 7 = 0$ has roots α, β and γ

The cubic equation $\mathrm{p}(x) = 0$ has roots $\alpha - 1, \beta - 1$ and $\gamma - 1$

The coefficient of x^3 in $\mathrm{p}(x)$ is 1

13 (a) Describe fully the transformation that maps the graph of $y = x^3 - x - 7$

onto the graph of $y = \mathrm{p}(x)$

[2 marks]

Each root is reduced by 1, so the graph moves 1 unit to the left:

Translation by vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

13 (b) Find $\mathrm{p}(x)$

[3 marks]

Let $y = x - 1 \Rightarrow x = y + 1$

$(y + 1)^3 - (y + 1) - 7 = 0$

$y^3 + 3y^2 + 3y + 1 - y - 1 - 7 = 0$

$y^3 + 3y^2 + 2y - 7 = 0$

So $p(x) = x^3 + 3x^2 + 2x - 7$

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Turn over for the next question

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14 The matrix **M** represents the transformation T, and is given by

$$\mathbf{M} = \begin{bmatrix} 3 & -1 \\ -2 & 6 \end{bmatrix}$$

14 (a) The point A has coordinates $(4, -5)$

Find the coordinates of the image of A under T

[2 marks]

$\begin{bmatrix} 3 & -1 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \times 4 + (-1)(-5) \\ -2 \times 4 + 6 \times (-5) \end{bmatrix} = \begin{bmatrix} 17 \\ -38 \end{bmatrix}$

Image of A is $(17, -38)$

14 (b) Show that the only invariant point under T is the origin.

[3 marks]

For an invariant point (x, y): $\begin{bmatrix} 3 & -1 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$3x - y = x$ and $-2x + 6y = y$

$\Rightarrow y = 2x$ and $2x = 5y$

Substituting: $2x = 5(2x) \Rightarrow 8x = 0 \Rightarrow x = 0$, and then $y = 0$

So $(0, 0)$, the origin, is the only invariant point under T

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14 (c) The line L_1 has equation $y = x + 1$

The transformation T maps the line L_1 onto the line L_2

Find the equation of L_2 in the form $y = mx + c$

[5 marks]

Any point on L_1 is $(x, x + 1)$. Its image (X, Y) lies on L_2: $Y = mX + c$

$$\begin{bmatrix} 3 & -1 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ x + 1 \end{bmatrix} = \begin{bmatrix} X \\ mX + c \end{bmatrix}$$

$3x - (x + 1) = X \Rightarrow X = 2x - 1$

$-2x + 6(x + 1) = mX + c \Rightarrow 4x + 6 = mX + c$

So $4x + 6 \equiv m(2x - 1) + c$ for all x

Compare x terms: $4 = 2m \Rightarrow m = 2$

Compare constants: $6 = -m + c = -2 + c \Rightarrow c = 8$

L_2: $y = 2x + 8$

Turn over ▶

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15 (a) Use Maclaurin's series expansion for $\ln(1+x)$ to show that the first three terms of the Maclaurin's series expansion of $\ln(1+3x)$ are

$$3x - \frac{9}{2}x^2 + 9x^3$$

[1 mark]

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ replace x with $3x$:

$\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \ldots = 3x - \frac{9x^2}{2} + \frac{27x^3}{3} - \ldots$

First three terms: $3x - \frac{9}{2}x^2 + 9x^3$

15 (b) Julia attempts to use the series expansion found in part **(a)** to find an approximation for $\ln 4$

Julia's incorrect working is shown below.

$$\begin{aligned} \text{Let } 1 + 3x &= 4 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} \text{So } \ln 4 &\approx 3 \times 1 - \frac{9}{2} \times 1^2 + 9 \times 1^3 \\ &\approx 3 - 4.5 + 9 \\ &\approx 7.5 \end{aligned}$$

Explain the error in Julia's working.

[2 marks]

The expansion of $\ln(1+3x)$ is only valid for $-1 < 3x \leq 1$,

i.e. for $-\frac{1}{3} < x \leq \frac{1}{3}$

$x = 1$ is not in this valid range, so the series does not converge

to $\ln 4$ there: Julia should not have substituted $x = 1$

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15 (c) Use $x = -\frac{1}{6}$ in the series expansion found in part **(a)** to find an approximation for $\ln 4$

Fully justify your answer.

[4 marks]

$-\frac{1}{3} < -\frac{1}{6} \leq \frac{1}{3}$, so $x = -\frac{1}{6}$ is in the range of validity

$\ln(1 + 3(-\frac{1}{6})) \approx 3(-\frac{1}{6}) - \frac{9}{2}(-\frac{1}{6})^2 + 9(-\frac{1}{6})^3$

$\ln \frac{1}{2} \approx -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} = -\frac{16}{24} = -\frac{2}{3}$

But $\ln 4 = 2 \ln 2 = -2 \ln \frac{1}{2}$

$\therefore \ln 4 \approx -2 \times (-\frac{2}{3})$

$\ln 4 \approx \frac{4}{3}$

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16 The curve C has the polar equation

$$r = \frac{2}{\sqrt{\cos^2\theta + 4\sin^2\theta}} \qquad -\pi < \theta \leq \pi$$

16 (a) Show that the Cartesian equation of C can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive integers to be determined.

[4 marks]

$r = \frac{2}{\sqrt{(\cos^2\theta + 4\sin^2\theta)}} \Rightarrow r^2 = \frac{4}{\cos^2\theta + 4\sin^2\theta}$

$r^2\cos^2\theta + 4r^2\sin^2\theta = 4$

$(r\cos\theta)^2 + 4(r\sin\theta)^2 = 4$

Using $x = r\cos\theta$ and $y = r\sin\theta$: $x^2 + 4y^2 = 4$

$\frac{x^2}{4} + y^2 = 1$

$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

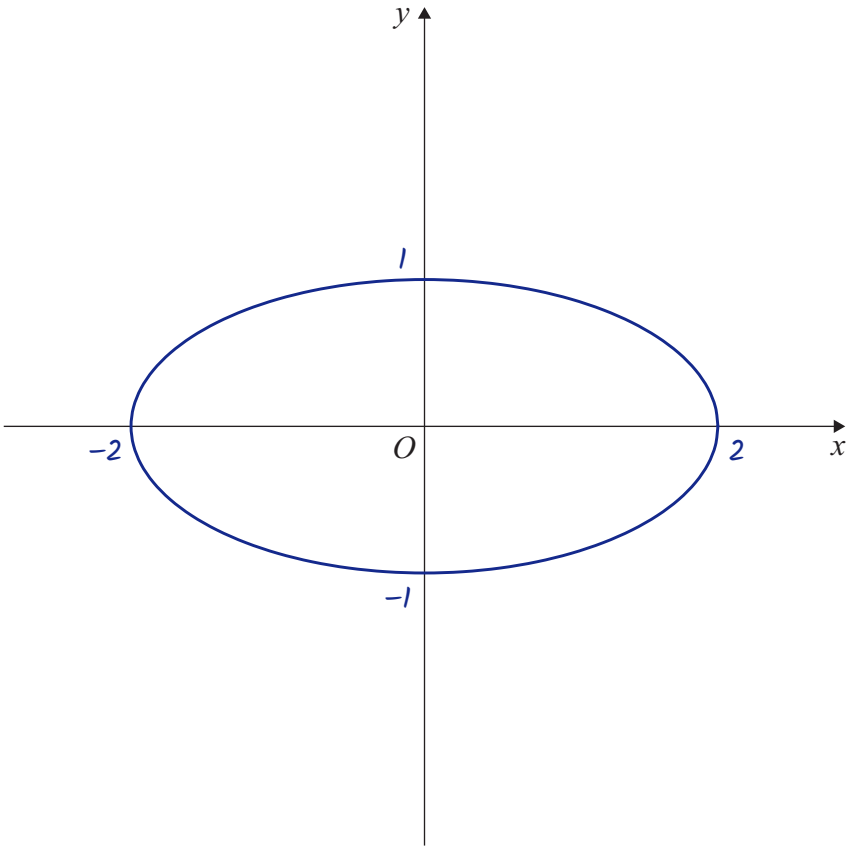
$a = 2$, $b = 1$

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16 (b) Hence sketch the graph of C on the axes below.

Indicate the value of any intercepts of the curve with the axes.

[2 marks]



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17 The circle C represents the locus of points satisfying the equation

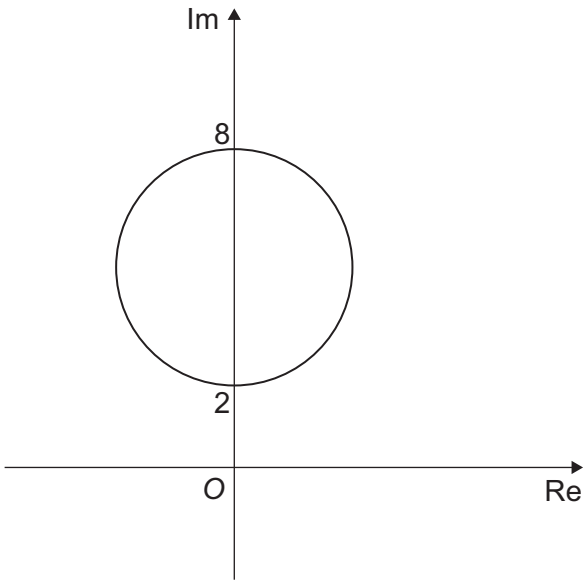
$$|z - a\mathrm{i}| = b$$

where a and b are real constants.

The circle C intersects the imaginary axis at $2\mathrm{i}$ and $8\mathrm{i}$

The circle C is shown on the Argand diagram in **Figure 2**

Figure 2



17 (a) (i) Write down the value of a

[1 mark]

$a = 5$ (centre is 5i, midway between 2i and 8i)

17 (a) (ii) Write down the value of b

[1 mark]

$b = 3$ (radius = (8 − 2) ÷ 2)

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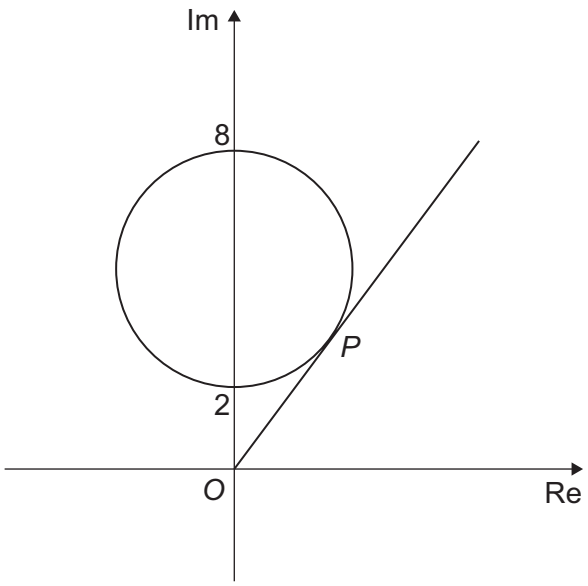
17 (b) The half-line L represents the locus of points satisfying the equation

$$\arg(z) = \tan^{-1}(k)$$

where k is a positive constant.

The point P is the **only** point which lies on both C and L, as shown in **Figure 3**

Figure 3



17 (b) (i) The point O represents the number $0 + 0\mathrm{i}$

Calculate the length OP

[2 marks]

Centre $C = 5i$, radius 3. OP is a tangent, so angle $OPC = 90°$

$OP^2 + CP^2 = OC^2$

$OP^2 = 5^2 - 3^2 = 16$

$OP = 4$

Turn over ▶

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17 (b) (ii) Calculate the exact value of k

[2 marks]

Triangle OPC is a 3, 4, 5 right-angled triangle.

Let $\theta = \arg(z)$ = angle between OP and the positive real axis.

Angle COP = $90° - \theta$, and $\tan(\text{angle COP}) = \frac{CP}{OP} = \frac{3}{4}$

So $\tan\theta = \frac{1}{\tan(90° - \theta)} = \frac{4}{3}$

$k = \frac{4}{3}$

17 (b) (iii) Find the complex number represented by point *P*

Give your answer in the form $x + yi$ where x and y are real.

[3 marks]

$\tan\theta = \frac{4}{3} \Rightarrow \cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$, and $|OP| = 4$

$x = 4\cos\theta = \frac{3}{5} \times 4 = 2.4$

$y = 4\sin\theta = \frac{4}{5} \times 4 = 3.2$

P represents $2.4 + 3.2i$

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