AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

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Candidate signature _____

I declare this is my own work.

AS
FURTHER MATHEMATICS

Paper 1

Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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TOTAL	

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Answer **all** questions in the spaces provided.

1 Which of the following exponential expressions is equivalent to $2\sinh x$?

Circle your answer.

[1 mark]

e^{x} $\quad$ $e^{x}+e^{-x}$ $\quad$ $e^{x}-e^{-x}$ $\quad$ e^{-x}

2 The quadratic equation $x^2+px+q=0$ has roots α and β

Which of the following is equal to $\alpha\beta$?

Circle your answer.

[1 mark]

p $\quad$ $-p$ $\quad$ q $\quad$ $-q$

3 Which of the following transformations is represented by the matrix $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$?

Tick (✓) **one** box.

[1 mark]

Rotation of 180° about the x-axis ☐

Reflection in the plane $x=0$ ☐

Rotation of 180° about the y-axis ☐

Reflection in the plane $y=0$ ☑

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4 The complex numbers w and z are defined as

$$w = 2(\cos\alpha + \mathrm{i}\sin\alpha)$$

$$z = 3(\cos\beta + \mathrm{i}\sin\beta)$$

Find the product wz

Tick (✓) **one** box.

[1 mark]

$5(\cos(\alpha\beta) + \mathrm{i}\sin(\alpha\beta))$ ☐

$6(\cos(\alpha\beta) + \mathrm{i}\sin(\alpha\beta))$ ☐

$5(\cos(\alpha + \beta) + \mathrm{i}\sin(\alpha + \beta))$ ☐

$6(\cos(\alpha + \beta) + \mathrm{i}\sin(\alpha + \beta))$ ☑

Turn over for the next question

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5 Show that $(2 + i)^3$ is $2 + 11i$

[3 marks]

$(2 + i)^3 = 2^3 + 3(2^2)(i) + 3(2)(i^2) + i^3$

$= 8 + 12i + 6i^2 + i^3$

$i^2 = -1$ and $i^3 = -i$

$= 8 + 12i - 6 - i$

$= 2 + 11i$ as required

Turn over for the next question

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6 The matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}$$

6 (a) Find $\det \mathbf{A}$

[1 mark]

$\det A = 5 \times 4 - 2 \times (-3) = 20 + 6$

$\det A = 26$

6 (b) Find $\mathbf{A}^{-1}$

[1 mark]

$A^{-1} = \frac{1}{26}\begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$

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6 (c) Given that $\mathbf{AB} = \begin{bmatrix} 9 & 6 \\ 5 & 12 \end{bmatrix}$ and $\mathbf{M} = 2\mathbf{A} + \mathbf{B}$ find the matrix $\mathbf{M}$

[3 marks]

$A^{-1}(AB) = B$, so $B = A^{-1}(AB)$

$B = \frac{1}{26}\begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 9 & 6 \\ 5 & 12 \end{bmatrix}$

$B = \frac{1}{26}\begin{bmatrix} 36-10 & 24-24 \\ 27+25 & 18+60 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 26 & 0 \\ 52 & 78 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

$M = 2A + B = \begin{bmatrix} 10 & 4 \\ -6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

$M = \begin{bmatrix} 11 & 4 \\ -4 & 11 \end{bmatrix}$

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7 The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$l_2 : \mathbf{r} = \begin{bmatrix} -12 \\ a \\ -3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

7 (a) Show that the point $P(-3, 9, -4)$ lies on l_1

[2 marks]

x: $3 + 3\lambda = -3 \Rightarrow 3\lambda = -6 \Rightarrow \lambda = -2$

y: $1 - 4\lambda = 1 - 4(-2) = 9$ ✓

z: $-2 + \lambda = -2 + (-2) = -4$ ✓

The same value $\lambda = -2$ gives all three coordinates of P,

so $P(-3, 9, -4)$ lies on l_1

7 (b) Show that l_1 is perpendicular to l_2

[2 marks]

Direction vectors $(3, -4, 1)$ and $(3, 2, -1)$:

$(3)(3) + (-4)(2) + (1)(-1) = 9 - 8 - 1 = 0$

Scalar product of direction vectors = 0,

so l_1 is perpendicular to l_2

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7 (c) Given that the lines l_1 and l_2 intersect, calculate the value of the constant a

[4 marks]

At the intersection:

x: $3 + 3\lambda = -12 + 3\mu \Rightarrow \lambda = \mu - 5$

z: $-2 + \lambda = -3 - \mu \Rightarrow \lambda = -1 - \mu$

$\mu - 5 = -1 - \mu \Rightarrow 2\mu = 4 \Rightarrow \mu = 2$

$\lambda = -1 - 2 = -3$

y: $1 - 4\lambda = a + 2\mu$

$1 - 4(-3) = a + 2(2)$

$13 = a + 4$

$a = 9$

7 (d) Hence, find the coordinates of the point of intersection of l_1 and l_2

[1 mark]

$\lambda = -3$: $(3 + 3(-3), 1 - 4(-3), -2 + (-3))$

Point of intersection $(-6, 13, -5)$

Turn over ▶

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8 The curve C has the polar equation

$$r = 4 - 2\cos\theta \qquad -\pi < \theta \leq \pi$$

8 (a) Verify that the point with polar coordinates $\left(3, \frac{\pi}{3}\right)$ lies on C

[1 mark]

$\theta = \frac{\pi}{3}$: $r = 4 - 2\cos\frac{\pi}{3} = 4 - 2 \times \frac{1}{2} = 3$

so the point $(3, \frac{\pi}{3})$ lies on C

8 (b) Find the exact polar coordinates of the point on C which is furthest from the pole, O

[3 marks]

$r = 4 - 2\cos\theta$ is greatest when $\cos\theta = -1$

$\cos\theta = -1 \Rightarrow \theta = \pi$ (in the range $-\pi < \theta \leq \pi$)

$r = 4 - 2(-1) = 6$

Furthest point from O: $(6, \pi)$

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8 (c) Find the exact Cartesian coordinates of the point on C where θ is $\frac{\pi}{6}$

[3 marks]

$\theta = \frac{\pi}{6}$: $r = 4 - 2\cos\frac{\pi}{6} = 4 - 2 \times \frac{\sqrt{3}}{2} = 4 - \sqrt{3}$

$x = r\cos\theta = (4 - \sqrt{3}) \times \frac{\sqrt{3}}{2} = 2\sqrt{3} - \frac{3}{2}$

$y = r\sin\theta = (4 - \sqrt{3}) \times \frac{1}{2} = 2 - \frac{\sqrt{3}}{2}$

Cartesian coordinates $\left(2\sqrt{3} - \frac{3}{2},\ 2 - \frac{\sqrt{3}}{2}\right)$

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9 (a) Show that, for $r > 0$,

$$\ln(r+2) - \ln r = \ln\left(1 + \frac{2}{r}\right)$$

[1 mark]

$\ln(r + 2) - \ln r = \ln(\frac{r+2}{r}) = \ln(1 + \frac{2}{r})$ as required

9 (b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \ln\left(1 + \frac{2}{r}\right) = \ln\left(\frac{1}{2}(n+a)(n+b)\right)$$

where a and b are integers to be found.

[4 marks]

$\sum_{r=1}^{n} \ln(1 + \frac{2}{r}) = \sum_{r=1}^{n} (\ln(r + 2) - \ln r)$

$r = 1$: $\ln 3 - \ln 1$

$r = 2$: $\ln 4 - \ln 2$

$r = 3$: $\ln 5 - \ln 3$

$r = 4$: $\ln 6 - \ln 4$

⋮

$r = n - 2$: $\ln n - \ln(n - 2)$

$r = n - 1$: $\ln(n + 1) - \ln(n - 1)$

$r = n$: $\ln(n + 2) - \ln n$

All terms cancel except $\ln(n + 1)$, $\ln(n + 2)$, $-\ln 1$, $-\ln 2$

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$\text{Sum} = \ln(n+1) + \ln(n+2) - \ln 1 - \ln 2$

$= \ln\left(\frac{(n+1)(n+2)}{2}\right)$

$= \ln\left(\frac{1}{2}(n+1)(n+2)\right)$

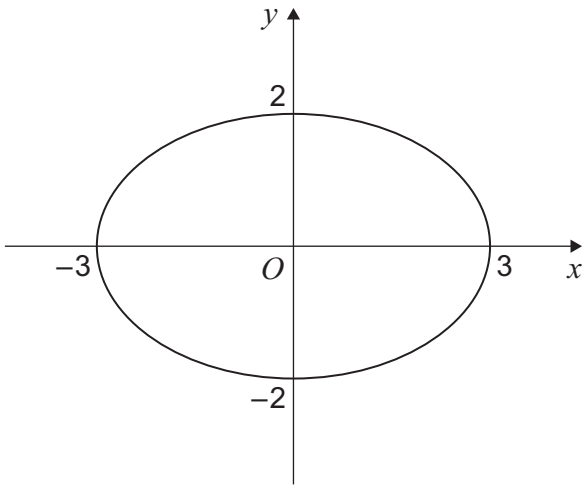
so $a = 1$, $b = 2$

Turn over for the next question

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10 The diagram below shows an ellipse E

The coordinate axes are the lines of symmetry of E



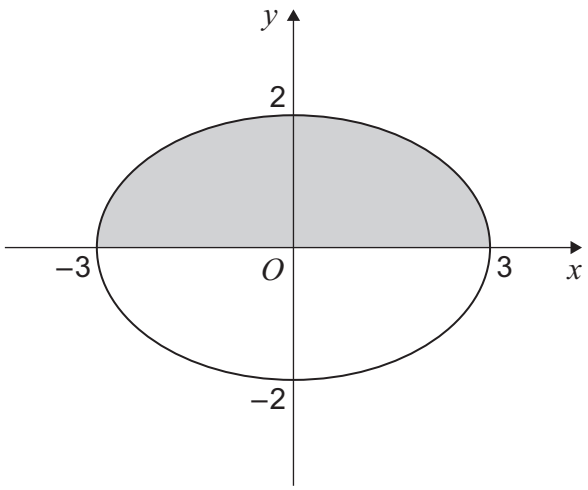
10 (a) Write down an equation of E

[2 marks]

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

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10 (b) The region bounded by the x-axis and the ellipse E for $y \geq 0$ is shaded in the diagram below.



A solid S is formed by rotating the shaded region through 360° about the x-axis.

Show that the volume of S is $a\pi$ where a is an integer to be found.

[4 marks]

$y^2 = 4\left(1 - \frac{x^2}{9}\right) = 4 - \frac{4x^2}{9}$

$V = \pi\int_{-3}^{3} y^2\,dx = \pi\int_{-3}^{3} \left(4 - \frac{4x^2}{9}\right) dx$

$= \pi\left[4x - \frac{4x^3}{27}\right]_{-3}^{3}$

$= \pi\left(12 - \frac{108}{27}\right) - \pi\left(-12 + \frac{108}{27}\right)$

$= \pi(12 - 4) - \pi(-12 + 4)$

$= 8\pi + 8\pi = 16\pi$

Volume of $S = 16\pi$, so $a = 16$

Turn over ▶

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11 Prove by induction that, for all integers $n \geq 1$,

$$(\mathbf{ABA}^{-1})^n = \mathbf{AB}^n\mathbf{A}^{-1}$$

where **A** and **B** are square matrices of equal dimensions, and **A** is non-singular.

[4 marks]

$n = 1$: $(ABA^{-1})^1 = ABA^{-1} = AB^1A^{-1}$ so true for $n = 1$

Assume true for $n = k$: $(ABA^{-1})^k = AB^kA^{-1}$

Then $(ABA^{-1})^{k+1} = (ABA^{-1})^k(ABA^{-1})$

$= AB^kA^{-1}ABA^{-1}$

$= AB^kIBA^{-1}$ $(A^{-1}A = I)$

$= AB^kBA^{-1}$

$= AB^{k+1}A^{-1}$

so if it is true for $n = k$ then it is true for $n = k + 1$

It is true for $n = 1$ and true for $n = k$ implies true for $n = k + 1$,

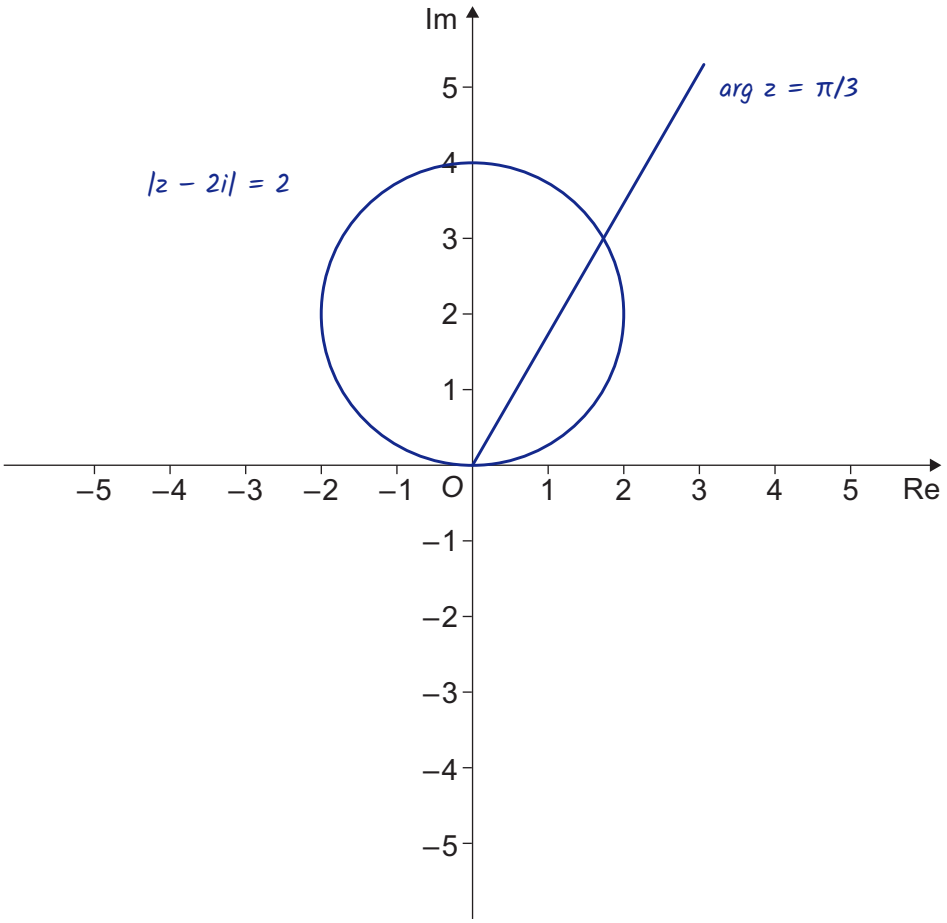
so by induction $(ABA^{-1})^n = AB^nA^{-1}$ for all integers $n \geq 1$

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12 (a) Sketch, on the Argand diagram below, the locus of points satisfying the equation

$$|z - 2i| = 2$$

[2 marks]



12 (b) Sketch, also on the Argand diagram above, the locus of points satisfying the equation

$$\arg z = \frac{\pi}{3}$$

[1 mark]

Do not write outside the box

12 (c) For the complex number w find the maximum value of $|w|$ such that

$$|w - 2i| \leq 2 \qquad \text{and} \qquad 0 \leq \arg w \leq \frac{\pi}{3}$$

[3 marks]

$|w - 2i| = 2$ is a circle centre $2i$, radius 2, through 0 and $4i$

Circle meets $\arg w = \pi/3$ ($y = \sqrt{3}\,x$) where $x^2 + (\sqrt{3}x - 2)^2 = 4$

$4x^2 - 4\sqrt{3}x = 0 \Rightarrow x = \sqrt{3},\ y = 3$

On the circle $|w| = 4\sin(\arg w)$, which increases as $\arg w$

increases from 0 to $\pi/3$, so max $|w|$ is at the point $\sqrt{3} + 3i$

max $|w| = \sqrt{((\sqrt{3})^2 + 3^2)} = \sqrt{12}$

max $|w| = 2\sqrt{3}$

Turn over for the next question

Do not write outside the box

13 A curve C_1 has equation

$$y = \frac{2x + 7}{3x + 5}$$

13 (a) Write down the equations of the asymptotes of curve C_1

[2 marks]

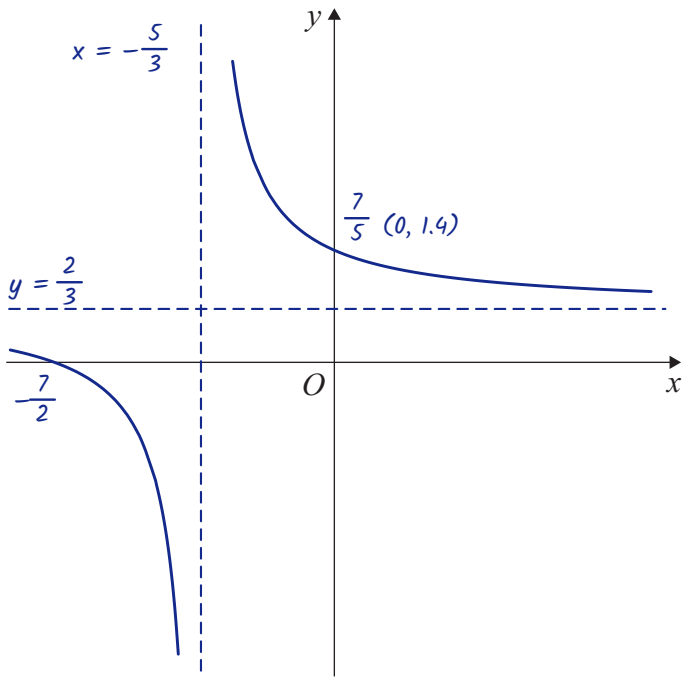
Vertical asymptote: $x = -\frac{5}{3}$

Horizontal asymptote: $y = \frac{2}{3}$

13 (b) On the axes below, sketch the graph of curve C_1

Indicate the values of the intercepts of the curve with the axes.

[3 marks]



13 (c) Hence, or otherwise, solve the inequality

$$\frac{2x + 7}{3x + 5} \geq 0$$

[2 marks]

Critical values $x = -\frac{7}{2}$ and $x = -\frac{5}{3}$

$x \leq -\frac{7}{2}$ or $x > -\frac{5}{3}$

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13 (d) Curve C_2 is a reflection of curve C_1 in the line $y = -x$

Find an equation for curve C_2 in the form $y = \mathrm{f}(x)$

[3 marks]

Reflection in $y = -x$: replace x with $-y$ and y with $-x$

$$-x = \frac{2(-y) + 7}{3(-y) + 5} = \frac{-2y + 7}{-3y + 5}$$

$$-x(-3y + 5) = -2y + 7$$

$$3xy - 5x = -2y + 7$$

$$3xy + 2y = 5x + 7$$

$$y(3x + 2) = 5x + 7$$

$$y = \frac{5x + 7}{3x + 2}$$

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14 The function f is defined by

$$\mathrm{f}(x) = \frac{x^2 - 3}{x^2 + px + 7} \qquad x \in \mathbb{R}$$

where p is a constant.

The graph of $y = \mathrm{f}(x)$ has only one asymptote.

14 (a) Write down the equation of the asymptote.

[1 mark]

$y = 1$

14 (b) Find the set of possible values of p

[4 marks]

Only one asymptote, so no vertical asymptotes:

$x^2 + px + 7 \neq 0$ for all real x, so no real roots

discriminant < 0: $p^2 - 4 \times 1 \times 7 < 0$

$p^2 < 28$

$-2\sqrt{7} < p < 2\sqrt{7}$

$\{ p : -2\sqrt{7} < p < 2\sqrt{7} \}$

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14 (c) Find the coordinates of the points at which the graph of $y = f(x)$ intersects the axes.

[3 marks]

$x = 0$: $y = \frac{0 - 3}{0 + 0 + 7} = -\frac{3}{7}$

$y = 0$: $x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

$(0, -\frac{3}{7})$, $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$

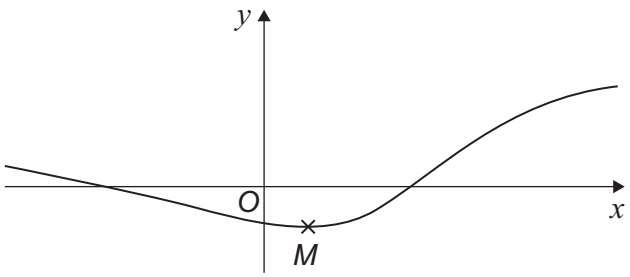
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14 (d) A curve C has equation

$$y = \frac{x^2 - 3}{x^2 - 3x + 7}$$

The curve C has a local minimum at the point M as shown in the diagram.



The line $y = k$ intersects curve C

14 (d) (i) Show that

$$19k^2 - 16k - 12 \leq 0$$

[5 marks]

$y = k$: $k = \frac{x^2 - 3}{x^2 - 3x + 7}$

$k(x^2 - 3x + 7) = x^2 - 3$

$(k - 1)x^2 - 3kx + 7k + 3 = 0$

The line meets C, so this has real roots: $b^2 - 4ac \geq 0$

$(-3k)^2 - 4(k - 1)(7k + 3) \geq 0$

$9k^2 - 4(7k^2 - 4k - 3) \geq 0$

$9k^2 - 28k^2 + 16k + 12 \geq 0$

$-19k^2 + 16k + 12 \geq 0$

$19k^2 - 16k - 12 \leq 0$ as required

(If $k = 1$ the equation is $-3x + 10 = 0$, which has a solution,

and $19 - 16 - 12 = -9 \leq 0$, so $k = 1$ also satisfies it.)

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14 (d) (ii) Hence, find the y-coordinate of point M

[2 marks]

$19k^2 - 16k - 12 = 0$: $k = \frac{16 \pm \sqrt{(256 + 912)}}{38} = \frac{16 \pm \sqrt{1168}}{38}$

$k = \frac{16 \pm 4\sqrt{73}}{38} = \frac{8 \pm 2\sqrt{73}}{19}$ $(k \approx 1.32$ or $k \approx -0.478)$

C only takes values between these roots, so the minimum value of y on C is the smaller root:

y-coordinate of M = $\frac{8 - 2\sqrt{73}}{19}$

Turn over for the next question

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15 The two values of θ that satisfy the equation

$$\sinh^2\theta - \sinh\theta - 2 = 0$$

are θ_1 and θ_2

15 (a) Hamzah is asked to find the value of $\theta_1 + \theta_2$

He writes his answer as follows:

The quadratic coefficients are $a = 1$, $b = -1$, $c = -2$

The sum of the roots is $-\frac{b}{a}$

So $\theta_1 + \theta_2 = -\frac{-1}{1} = 1$

Explain Hamzah's error.

[1 mark]

The roots of the quadratic are sinh θ_1 and sinh θ_2, not θ_1 and θ_2,

so he has only shown that sinh θ_1 + sinh θ_2 = 1

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15 (b) Find the correct value of $\theta_1 + \theta_2$

Give your answer as a single logarithm.

[5 marks]

$\sinh^2\theta - \sinh\theta - 2 = 0$

$(\sinh\theta - 2)(\sinh\theta + 1) = 0$

$\sinh\theta = 2 \quad \text{or} \quad \sinh\theta = -1$

$\theta = \operatorname{arsinh} x = \ln(x + \sqrt{(x^2 + 1)})$

$\theta_1 = \operatorname{arsinh} 2 = \ln(2 + \sqrt{5})$

$\theta_2 = \operatorname{arsinh}(-1) = \ln(-1 + \sqrt{2})$

$\theta_1 + \theta_2 = \ln(2 + \sqrt{5}) + \ln(\sqrt{2} - 1)$

$= \ln((2 + \sqrt{5})(\sqrt{2} - 1))$

$= \ln(2\sqrt{2} - 2 + \sqrt{10} - \sqrt{5})$

END OF QUESTIONS

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