AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

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Candidate signature _____

AS
FURTHER MATHEMATICS

Paper 1

Monday 13 May 2019 Afternoon Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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14	
TOTAL	

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Answer **all** questions in the spaces provided.

1 Which of the following matrices is an identity matrix?

Circle your answer.

[1 mark]

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 Which of the following expressions is the determinant of the matrix $\begin{bmatrix} a & 2 \\ b & 5 \end{bmatrix}$?

Circle your answer.

[1 mark]

$5a - 2b$ $2a - 5b$ $5b - 2a$ $2b - 5a$

3 Point P has polar coordinates $\left(2, \frac{2\pi}{3}\right)$.

Which of the following are the Cartesian coordinates of P?

Circle your answer.

[1 mark]

$(1, -\sqrt{3})$ $(-\sqrt{3}, 1)$ $(\sqrt{3}, -1)$ $(-1, \sqrt{3})$

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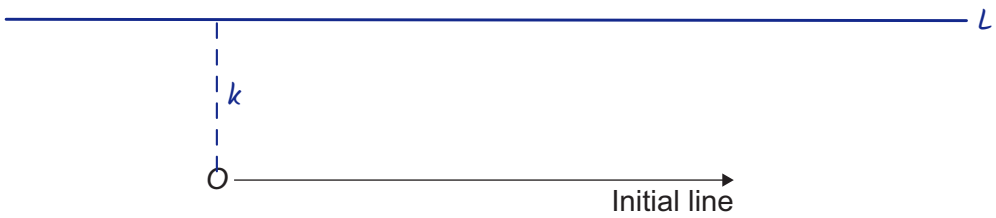
4 The line L has polar equation

$$r = \frac{k}{\sin\theta}$$

where k is a positive constant.

4 (a) Sketch L.

[1 mark]



4 (b) State the minimum distance between L and the point O.

[1 mark]

Minimum distance = k

Turn over for the next question

Do not write outside the box

5 A hyperbola H has the equation

$$\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$$

where a is a positive constant.

5 (a) Write down the equations of the asymptotes of H.

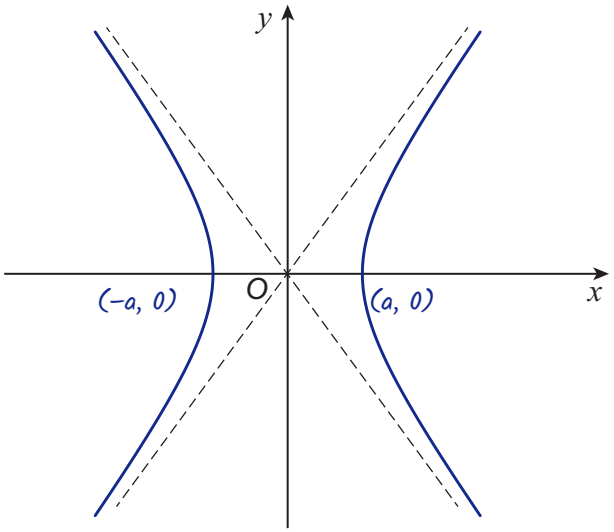
[1 mark]

$y = 2x$ and $y = -2x$ $(y = \pm 2x)$

5 (b) Sketch the hyperbola H on the axes below, indicating the coordinates of any points of intersection with the coordinate axes.

The asymptotes have already been drawn.

[2 marks]



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5 (c) The finite region bounded by *H*, the positive x-axis, the positive y-axis and the line $y = a$ is rotated through 360° about the y-axis.

Show that the volume of the solid generated is ma^3, where $m = 3.40$ correct to three significant figures.

[5 marks]

Rotating about the y-axis: $V = \pi \int x^2 \, dy$

$x^2 = a^2\left(1 + \frac{y^2}{4a^2}\right) = a^2 + \frac{y^2}{4}$

$V = \pi \int_0^a \left(\frac{y^2}{4} + a^2\right) dy$

$= \pi\left[\frac{y^3}{12} + a^2 y\right]_0^a$

$= \pi\left(\frac{a^3}{12} + a^3 - 0\right)$

$= \frac{13\pi}{12} a^3$

$= 3.4033\ldots a^3$

$= 3.40a^3$ (3 s.f.)

So $V = ma^3$ with $m = \frac{13\pi}{12} = 3.40$ correct to 3 s.f.

Turn over for the next question

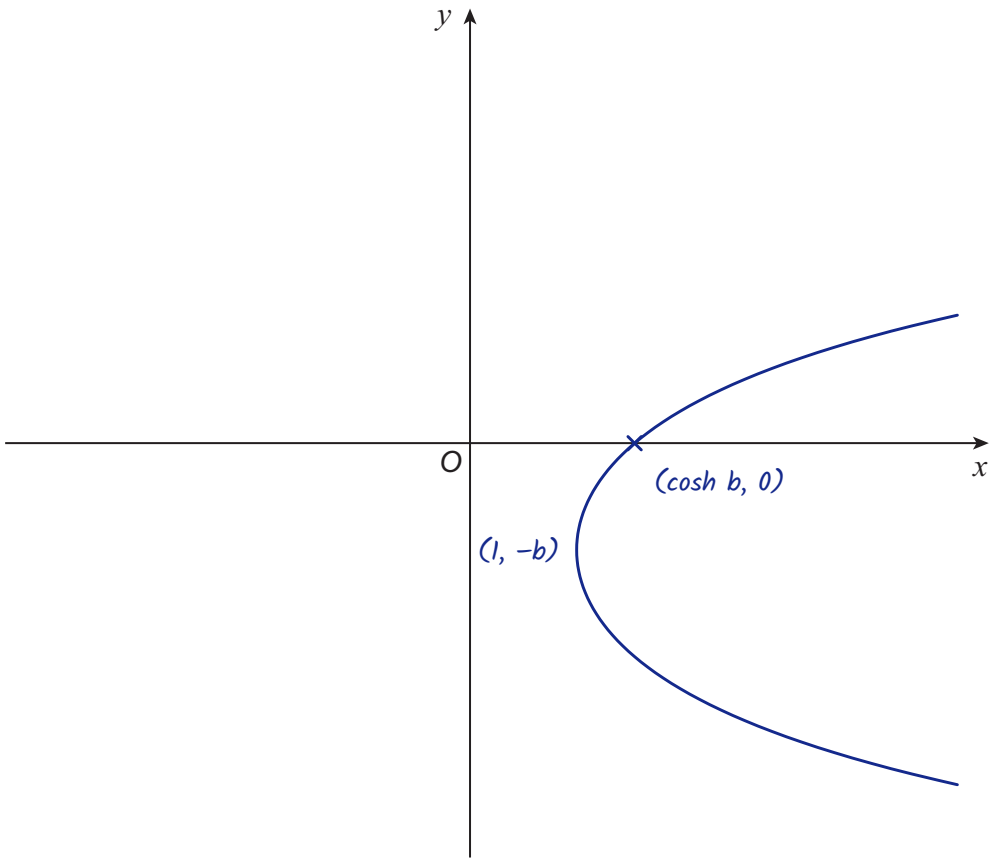
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6 (a) On the axes provided, sketch the graph of

$$x = \cosh(y + b)$$

where b is a positive constant.

[4 marks]



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6 (b) Determine the minimum distance between the graph of $x = \cosh(y + b)$ and the y-axis.

[1 mark]

Minimum value of x = cosh(y + b) is 1 (when y = −b)

Minimum distance = 1

Turn over for the next question

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7 (a) Show that

$$\frac{1}{r-1}-\frac{1}{r+1}\equiv\frac{A}{r^2-1}$$

where A is a constant to be found.

[1 mark]

$\frac{1}{r-1}-\frac{1}{r+1}\equiv\frac{(r+1)-(r-1)}{(r-1)(r+1)}\equiv\frac{2}{r^2-1}$ $\quad A=2$

7 (b) Hence use the method of differences to show that

$$\sum_{r=2}^{n}\frac{1}{r^2-1}\equiv\frac{an^2+bn+c}{4n(n+1)}$$

where a, b and c are integers to be found.

[4 marks]

$\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$

$\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$:

$r=2$: $\frac{1}{1}-\frac{1}{3}$

$r=3$: $\frac{1}{2}-\frac{1}{4}$

$r=4$: $\frac{1}{3}-\frac{1}{5}$

$\vdots$

$r=n-2$: $\frac{1}{n-3}-\frac{1}{n-1}$

$r=n-1$: $\frac{1}{n-2}-\frac{1}{n}$

$r=n$: $\frac{1}{n-1}-\frac{1}{n+1}$

Terms cancel diagonally, leaving four terms:

$\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)=\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}$

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$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} - \frac{(n+1)+n}{n(n+1)}\right)$$

$$= \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

$$= \frac{3n(n+1) - 2(2n+1)}{4n(n+1)}$$

$$= \frac{3n^2 + 3n - 4n - 2}{4n(n+1)}$$

$$= \frac{3n^2 - n - 2}{4n(n+1)}$$

$a = 3$, $b = -1$, $c = -2$

Turn over for the next question

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8 Given that $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

8 (a) Find the value of $|z_1 z_2|$

[1 mark]

$|z_1 z_2| = |z_1| \times |z_2| = 2 \times 2 = 4$

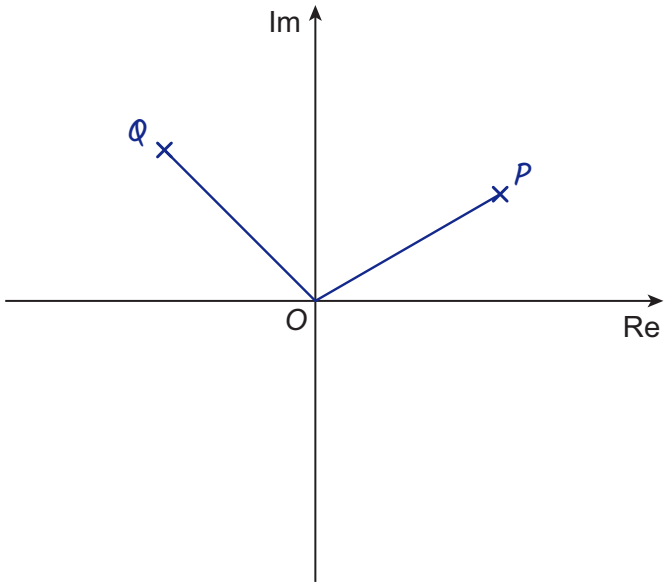
8 (b) Find the value of $\arg\left(\frac{z_1}{z_2}\right)$

[1 mark]

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6} - \frac{3\pi}{4} = -\frac{7\pi}{12}$

8 (c) Sketch z_1 and z_2 on the Argand diagram below, labelling the points as P and Q respectively.

[2 marks]



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8 (d) A third complex number w satisfies both $|w| = 2$ and $-\pi < \arg w < 0$

Given that w is represented on the Argand diagram as the point R, find the angle $P\hat{R}Q$.

Fully justify your answer.

[3 marks]

$|z_1| = |z_2| = 2$ and $|w| = 2$, so P, Q and R all lie on the circle with centre O and radius 2.

Angle $POQ = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$ (angle at the centre)

$-\pi < \arg w < 0$, so R lies below the real axis, on the major arc PQ.

The angle subtended by chord PQ at the circumference is half the angle subtended at the centre, for any position of R on that arc.

Angle $PRQ = \frac{1}{2} \times \frac{7\pi}{12} = \frac{7\pi}{24}$

Turn over for the next question

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9 (a) Saul is solving the equation

$$2\cosh x + \sinh^2 x = 1$$

He writes his steps as follows:

$$2\cosh x + \sinh^2 x = 1$$

$$2\cosh x + 1 - \cosh^2 x = 1$$

$$2\cosh x - \cosh^2 x = 0$$

$$\cosh x \neq 0 \ \therefore \ 2 - \cosh x = 0$$

$$\cosh x = 2$$

$$x = \pm \cosh^{-1}(2)$$

Identify and explain the error in Saul's method.

[2 marks]

The error is in the 2nd line: $\sinh^2 x$ has been replaced by $1 - \cosh^2 x$.

The identity is $\cosh^2 x - \sinh^2 x = 1$, so $\sinh^2 x = \cosh^2 x - 1$,

not $1 - \cosh^2 x$. Line 2 should read $2\cosh x + \cosh^2 x - 1 = 1$.

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9 (b) Anna is solving the different equation

$$\sinh^2(2x) - 2\cosh(2x) = 1$$

and finds the correct answers in the form $x = \frac{1}{p}\cosh^{-1}(q + \sqrt{r})$, where p, q and r are integers.

Find the possible values of p, q and r.

Fully justify your answer.

[5 marks]

$\cosh^2(2x) - \sinh^2(2x) = 1 \Rightarrow \sinh^2(2x) = \cosh^2(2x) - 1$

$\cosh^2(2x) - 1 - 2\cosh(2x) = 1$

$\cosh^2(2x) - 2\cosh(2x) - 2 = 0$

$\cosh(2x) = \frac{2 \pm \sqrt{(4 + 8)}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

But $\cosh(2x) \geq 1$ for all real x, and $1 - \sqrt{3} < 1$, so reject $1 - \sqrt{3}$

$\therefore \cosh(2x) = 1 + \sqrt{3}$

$2x = \pm\cosh^{-1}(1 + \sqrt{3})$

$x = \pm\frac{1}{2}\cosh^{-1}(1 + \sqrt{3})$

$p = 2$ or $p = -2$, $q = 1$, $r = 3$

Turn over for the next question

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10 (a) Using the definition of $\cosh x$ and the Maclaurin series expansion of e^x, find the first three non-zero terms in the Maclaurin series expansion of $\cosh x$.

[3 marks]

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$

$\cosh x = \frac{1}{2}\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots\right)$

$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots$

10 (b) Hence find a trigonometric function for which the first three terms of its Maclaurin series are the same as the first three terms of the Maclaurin series for $\cosh(ix)$.

[3 marks]

$\cosh(ix) = 1 + \frac{(ix)^2}{2} + \frac{(ix)^4}{24} + \ldots$

$= 1 + \frac{i^2x^2}{2} + \frac{i^4x^4}{24} + \ldots$

$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots \quad (i^2 = -1,\ i^4 = 1)$

This is the Maclaurin series for cos x, so the function is cos x

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11 (a) Curve C has equation

$$y = \frac{x^2 + px - q}{x^2 - r}$$

where p, q and r are positive constants.

Write down the equations of its asymptotes.

[2 marks]

Vertical asymptotes where $x^2 - r = 0$: $x = \sqrt{r}$ and $x = -\sqrt{r}$

As $x \to \pm\infty$, $y \to 1$: horizontal asymptote $y = 1$

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11 (b) Find the set of possible y-coordinates for the graph of

$$y = \frac{x^2 + x - 6}{x^2 - 1}, \qquad x \neq \pm 1$$

giving your answer in exact form.

No credit will be given for solutions based on differentiation.

[6 marks]

Let $k = \frac{x^2 + x - 6}{x^2 - 1}$

$k(x^2 - 1) = x^2 + x - 6$

$(k - 1)x^2 - x + (6 - k) = 0$

For real x: $b^2 - 4ac \geq 0$

$(-1)^2 - 4(k - 1)(6 - k) \geq 0$

$1 - 4(6k - k^2 - 6 + k) \geq 0$

$4k^2 - 28k + 25 \geq 0$

Critical values: $k = \frac{28 \pm \sqrt{(784 - 400)}}{8} = \frac{28 \pm 8\sqrt{6}}{8} = \frac{7 \pm 2\sqrt{6}}{2}$

Positive quadratic, so outside the roots:

$y \leq \frac{7 - 2\sqrt{6}}{2}$ or $y \geq \frac{7 + 2\sqrt{6}}{2}$

Turn over for the next question

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12 The matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

12 (a) Prove by induction that, for all integers $n \geq 1$,

$$\mathbf{A}^n = \begin{bmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{bmatrix}$$

[4 marks]

$n = 1$: $A^1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3^1 - 1 \\ 0 & 3^1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\therefore$ true for $n = 1$

Assume true for $n = k$:

$A^k = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}$

$A^{k+1} = A^k A = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 + 3(3^k - 1) \\ 0 & 3 \times 3^k \end{bmatrix} = \begin{bmatrix} 1 & 3^{k+1} - 1 \\ 0 & 3^{k+1} \end{bmatrix}$

$\therefore$ it is also true for $n = k + 1$

True for $n = 1$, and true for $n = k \Rightarrow$ true for $n = k + 1$,

so by induction it is true for all integers $n \geq 1$

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12 (b) Find all invariant lines under the transformation matrix **A**.

Fully justify your answer.

[6 marks]

$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \Rightarrow x' = x + 2y\ ,\ y' = 3y$

Invariant line $y = mx + c$, so $y' = mx' + c$:

$3(mx + c) = m(x + 2(mx + c)) + c$

$3mx + 3c \equiv mx + 2m^2x + 2mc + c$

Compare x: $3m = m + 2m^2 \Rightarrow 2m(m - 1) = 0 \Rightarrow m = 0$ or $m = 1$

Constants: $3c = 2mc + c \Rightarrow 2c(1 - m) = 0$

$m = 0$: $c = 0$, giving $y = 0$

$m = 1$: $0 = 0$ for all c, giving $y = x + c$

Vertical lines $x = d$: $x' = d + 2y \neq d$ $(y \neq 0)$, so not invariant

Invariant lines are $y = 0$ and $y = x + c$ (c any constant)

12 (c) Find a line of invariant points under the transformation matrix **A**.

[2 marks]

Invariant points: $x = x + 2y$ and $y = 3y$

$\Rightarrow y = 0$, x any value

Line of invariant points: $y = 0$

Turn over ▶

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13 Line l_1 has Cartesian equation

$$x - 3 = \frac{2y + 2}{3} = 2 - z$$

13 (a) Write the equation of line l_1 in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a parameter and **a** and **b** are vectors to be found.

[2 marks]

$x - 3 = \frac{y + 1}{1.5} = \frac{z - 2}{-1} = \lambda$

$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1.5 \\ -1 \end{pmatrix}$

13 (b) Line l_2 passes through the points $P(3, 2, 0)$ and $Q(n, 5, n)$, where n is a constant.

13 (b) (i) Show that the lines l_1 and l_2 are **not** perpendicular.

[3 marks]

Direction of l_2: $\overrightarrow{PQ} = \begin{pmatrix} n - 3 \\ 5 - 2 \\ n - 0 \end{pmatrix} = \begin{pmatrix} n - 3 \\ 3 \\ n \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1.5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} n - 3 \\ 3 \\ n \end{pmatrix} = n - 3 + 4.5 - n = 1.5$

The scalar product is $1.5 \neq 0$ (for all n)

$\therefore l_1$ and l_2 are not perpendicular

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13 (b) (ii) Explain briefly why lines l_1 and l_2 cannot be parallel.

[2 marks]

Parallel only if $\begin{pmatrix} n-3 \\ 3 \\ n \end{pmatrix} = t\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ (l_1 direction $\parallel$ (2, 3, −2))

y-components give $t = 1$, so $n - 3 = 2 \Rightarrow n = 5$, and $n = -2$

n cannot be both 5 and −2, so the direction vectors are never multiples of each other $\therefore$ l_1 and l_2 cannot be parallel

13 (b) (iii) Given that θ is the acute angle between lines l_1 and l_2, show that

$$\cos\theta = \frac{p}{\sqrt{34n^2 + qn + 306}}$$

where p and q are constants to be found.

[3 marks]

$\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} n-3 \\ 3 \\ n \end{pmatrix} = 2(n-3) + 9 - 2n = 3$

$\sqrt{(2^2 + 3^2 + 2^2)} = \sqrt{17}$, $\sqrt{((n-3)^2 + 3^2 + n^2)} = \sqrt{(2n^2 - 6n + 18)}$

$3 = \sqrt{17} \times \sqrt{(2n^2 - 6n + 18)} \times \cos\theta$

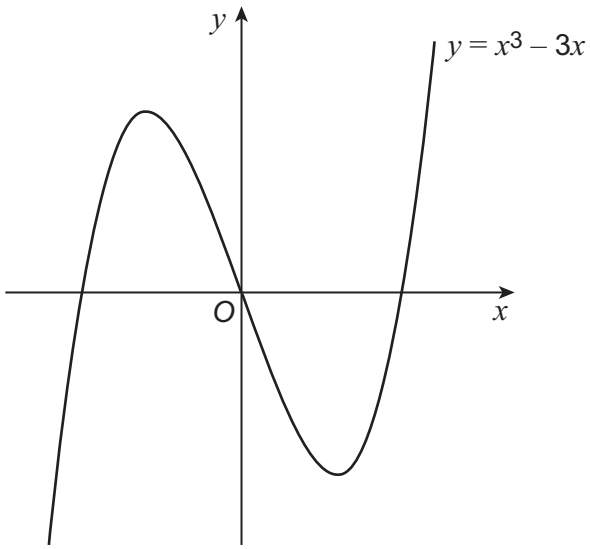
$\cos\theta = \dfrac{3}{\sqrt{17}\sqrt{(2n^2 - 6n + 18)}} = \dfrac{3}{\sqrt{(34n^2 - 102n + 306)}}$

$p = 3$, $q = -102$

Turn over ▶

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14 The graph of $y = x^3 - 3x$ is shown below.



The two stationary points have x-coordinates of -1 and 1

The cubic equation

$$x^3 - 3x + p = 0$$

where p is a real constant, has the roots α, β and γ.

The roots α and β are **not** real.

14 (a) Explain why $\alpha + \beta = -\gamma$

[1 mark]

The coefficient of x^2 is 0, so $\alpha + \beta + \gamma = 0 \ \therefore \ \alpha + \beta = -\gamma$

14 (b) Find the set of possible values for the real constant p.

[2 marks]

$x^3 - 3x = -p$ has only one real root, so $y = -p$ meets the curve once

Max point $(-1, 2)$, min point $(1, -2)$

So $-p > 2$ or $-p < -2$

$p < -2$ or $p > 2$

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14 (c) $\mathrm{f}(x) = 0$ is a cubic equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$

14 (c) (i) Show that the constant term of $\mathrm{f}(x)$ is $p + 2$

[3 marks]

Let $w = x + 1$, so $x = w - 1$:

$(w - 1)^3 - 3(w - 1) + p = 0$

$w^3 - 3w^2 + 3w - 1 - 3w + 3 + p = 0$

$w^3 - 3w^2 + (p + 2) = 0$

So $f(x) = x^3 - 3x^2 + p + 2$, and the constant term is $p + 2$

14 (c) (ii) Write down the x-coordinates of the stationary points of $y = \mathrm{f}(x)$

[1 mark]

$y = f(x)$ is $y = x^3 - 3x + p$ translated 1 unit right

$x = 0$ and $x = 2$

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