AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level MATHEMATICS

Paper 1

Time allowed: 2 hours

Materials
- You must have the AQA Formulae for A-level Mathematics booklet.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you need extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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12	
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15	
TOTAL	

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Answer **all** questions in the spaces provided.

1 A curve is defined by the parametric equations

$$x = \cos\theta \text{ and } y = \sin\theta \qquad \text{where } 0 \leq \theta \leq 2\pi$$

Which of the options shown below is a Cartesian equation for this curve?

Circle your answer.

[1 mark]

$\frac{y}{x} = \tan\theta$ $\quad$ (circled) $x^2 + y^2 = 1$ $\quad$ $x^2 - y^2 = 1$ $\quad$ $x^2 y^2 = 1$

2 A periodic sequence is defined by

$$U_n = (-1)^n$$

State the period of the sequence.

Circle your answer.

[1 mark]

-1 $\quad$ 0 $\quad$ 1 $\quad$ (circled) 2

3 The curve

$$y = \log_4 x$$

is transformed by a stretch, scale factor 2, parallel to the y-axis.

State the equation of the curve after it has been transformed.

Circle your answer.

[1 mark]

$y = \frac{1}{2}\log_4 x$ $\quad$ (circled) $y = 2\log_4 x$ $\quad$ $y = \log_4 2x$ $\quad$ $y = \log_8 x$

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4 The graph of

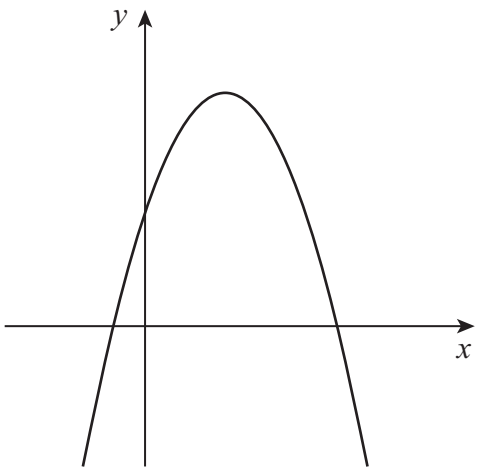
$$y = \mathrm{f}(x)$$

where

$$\mathrm{f}(x) = ax^2 + bx + c$$

is shown in **Figure 1**.

Figure 1

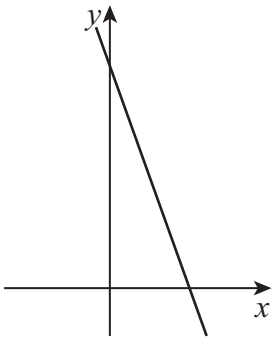


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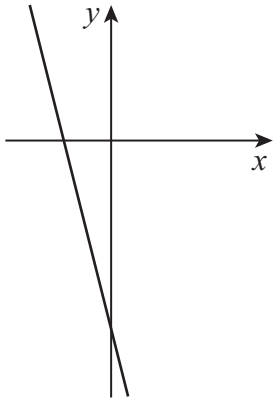
Which of the following shows the graph of $y = f'(x)$?

Tick (✓) **one** box.

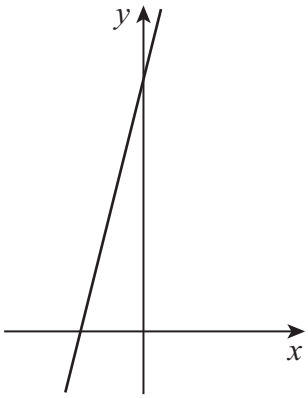
[1 mark]



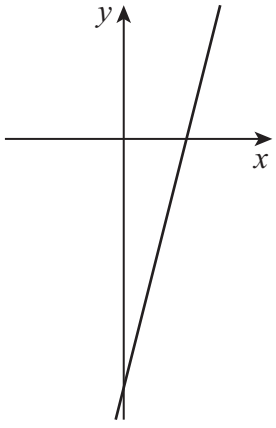
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5 Find an equation of the tangent to the curve

$$y = (x - 2)^4$$

at the point where $x = 0$

[3 marks]

$\frac{dy}{dx} = 4(x - 2)^3$

At $x = 0$: $y = (0 - 2)^4 = 16$

$\frac{dy}{dx} = 4(0 - 2)^3 = 4(-8) = -32$

Tangent: $y - 16 = -32(x - 0)$

$y = -32x + 16$

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6 (a) Find the first two terms, in ascending powers of x, of the binomial expansion of

$$\left(1-\frac{x}{2}\right)^{\frac{1}{2}}$$

[2 marks]

$(1-\frac{x}{2})^{1/2} \approx 1 + (\frac{1}{2})(-\frac{x}{2}) + \ldots$

$\approx 1 - \frac{1}{4}x$

6 (b) Hence, for small values of x, show that

$$\sin 4x + \sqrt{\cos x} \approx A + Bx + Cx^2$$

where A, B and C are constants to be found.

[4 marks]

For small x: $\sin 4x \approx 4x$ and $\cos x \approx 1 - \frac{x^2}{2}$

$\sqrt{(\cos x)} \approx (1 - \frac{x^2}{2})^{1/2}$

Replace x by x^2 in (a): $(1 - \frac{x^2}{2})^{1/2} \approx 1 - \frac{1}{4}x^2$

$\sin 4x + \sqrt{(\cos x)} \approx 4x + (1 - \frac{1}{4}x^2)$

$\approx 1 + 4x - \frac{1}{4}x^2$

$A = 1, \quad B = 4, \quad C = -\frac{1}{4}$

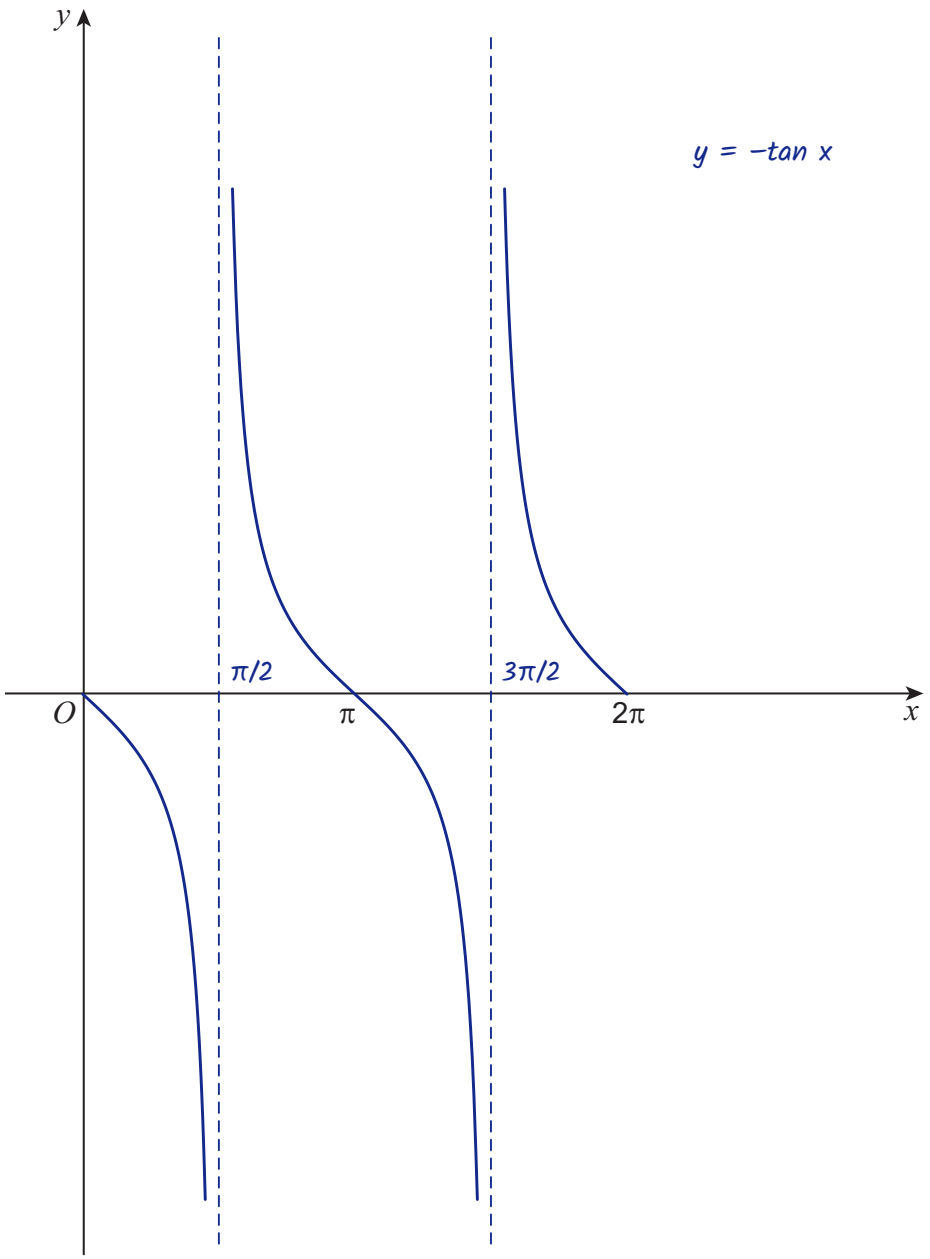
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7 Sketch the graph of

$$y = \cot\left(x - \frac{\pi}{2}\right)$$

for $0 \leq x \leq 2\pi$

[3 marks]



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8 The lines L_1 and L_2 are parallel.

L_1 has equation

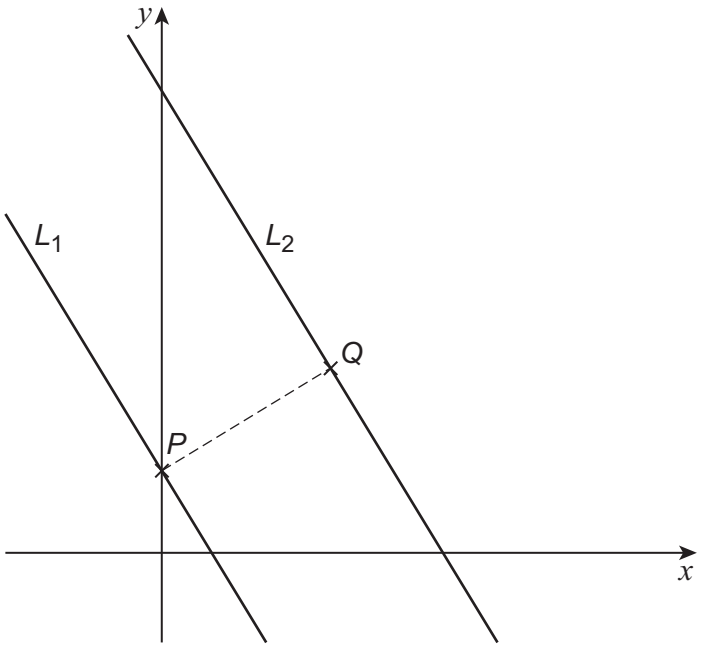
$$5x + 3y = 15$$

and L_2 has equation

$$5x + 3y = 83$$

L_1 intersects the y-axis at the point P.

The point Q is the point on L_2 closest to P, as shown in the diagram.



8 (a) (i) Find the coordinates of Q.

[5 marks]

P: $x = 0 \Rightarrow 3y = 15 \Rightarrow y = 5$, so $P(0, 5)$

L_1: $y = 5 - \frac{5}{3}x$, gradient $-\frac{5}{3}$

PQ is perpendicular to L_1, so gradient of $PQ = \frac{3}{5}$

PQ: $y = \frac{3}{5}x + 5 \Rightarrow 3x - 5y = -25$

Solve with L_2: $5x + 3y = 83$

$\times 3$: $9x - 15y = -75$

$\times 5$: $25x + 15y = 415$

Add: $34x = 340 \Rightarrow x = 10$

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$y = \frac{3}{5}(10) + 5 = 11$

$Q(10, 11)$

8 (a) (ii) Hence show that $PQ = k\sqrt{34}$, where k is an integer to be found.

[2 marks]

$PQ^2 = (10 - 0)^2 + (11 - 5)^2 = 100 + 36 = 136$

$PQ = \sqrt{136} = \sqrt{(4 \times 34)} = 2\sqrt{34}$

so $k = 2$

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8 (b) A circle, C, has centre $(a, -17)$.

L_1 and L_2 are both tangents to C.

8 (b) (i) Find a.

[2 marks]

L_1 and L_2 are parallel tangents, so the centre lies on the line midway

Midway line: $5x + 3y = \frac{15 + 83}{2} = 49$

Centre $(a, -17)$: $5a + 3(-17) = 49$

$5a = 100 \Rightarrow a = 20$

8 (b) (ii) Find the equation of C.

[2 marks]

Radius $= \frac{1}{2}PQ = \sqrt{34}$, so $r^2 = 34$

$(x - 20)^2 + (y + 17)^2 = 34$

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9 The first three terms of an arithmetic sequence are given by

$$2x + 5 \qquad 5x + 1 \qquad 6x + 7$$

9 (a) Show that $x = 5$ is the only value which gives an arithmetic sequence.

[3 marks]

Equal differences: $(5x + 1) - (2x + 5) = (6x + 7) - (5x + 1)$

$3x - 4 = x + 6$

$2x = 10 \Rightarrow x = 5$

The equation is linear so it has exactly one solution,

therefore $x = 5$ is the only value which gives an arithmetic sequence

9 (b) (i) Write down the value of the first term of the sequence.

[1 mark]

$a = 2(5) + 5 = 15$

9 (b) (ii) Find the value of the common difference of the sequence.

[1 mark]

Terms 15, 26, 37: $d = 26 - 15 = 11$

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9 (c) The sum of the first N terms of the arithmetic sequence is S_N where

$$S_N < 100\,000$$

$$S_{N+1} > 100\,000$$

Find the value of N.

[4 marks]

$S_N = \frac{N}{2}(2 \times 15 + 11(N-1)) = \frac{N}{2}(11N + 19)$

$\frac{N}{2}(11N + 19) = 100000$

$11N^2 + 19N - 200000 = 0$

$N = \frac{-19 + \sqrt{(19^2 + 4 \times 11 \times 200000)}}{2 \times 11} = \frac{-19 + \sqrt{8800361}}{22}$

$N = 133.98...$ (taking the positive root)

$S_N < 100000$ and $S_{N+1} > 100000$, so $N = 133$

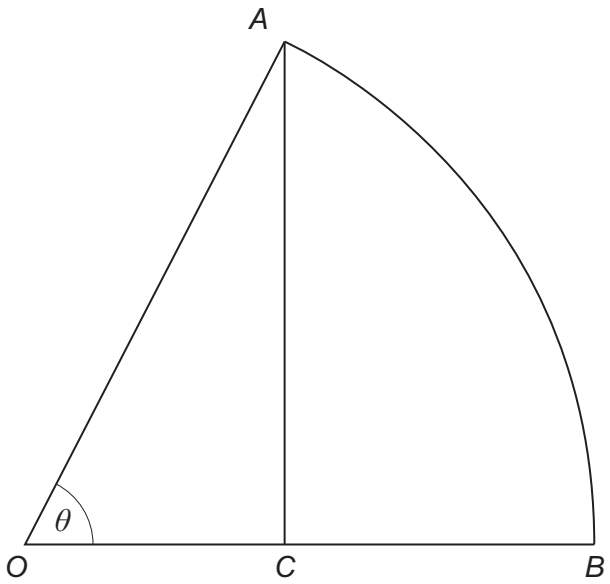
Check: $S_{133} = \frac{133}{2}(11 \times 133 + 19) = 98553 < 100000$

$S_{134} = \frac{134}{2}(11 \times 134 + 19) = 100031 > 100000$

$N = 133$

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10 The diagram shows a sector of a circle OAB.



The point C lies on OB such that AC is perpendicular to OB.

Angle AOB is θ radians.

10 (a) Given the area of the triangle OAC is half the area of the sector OAB, show that

$$\theta = \sin 2\theta$$

[4 marks]

Let $OA = OB = r$

$OC = r\cos\theta$, $AC = r\sin\theta$

Area of triangle $OAC = \frac{1}{2} \times r\cos\theta \times r\sin\theta = \frac{1}{2}r^2\sin\theta\cos\theta$

Area of sector $OAB = \frac{1}{2}r^2\theta$

$\frac{1}{2}r^2\sin\theta\cos\theta = \frac{1}{2} \times \frac{1}{2}r^2\theta$

$2\sin\theta\cos\theta = \theta$

Using $\sin 2\theta = 2\sin\theta\cos\theta$: $\theta = \sin 2\theta$ as required

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10 (b) Use a suitable **change of sign** to show that a solution to the equation

$$\theta = \sin 2\theta$$

lies in the interval given by $\theta \in \left[\frac{\pi}{5}, \frac{2\pi}{5}\right]$

[2 marks]

$\theta = \sin 2\theta \Rightarrow \theta - \sin 2\theta = 0$. Let $f(\theta) = \theta - \sin 2\theta$

$f(\frac{\pi}{5}) = 0.6283... - \sin(\frac{2\pi}{5}) = -0.3227... < 0$

$f(\frac{2\pi}{5}) = 1.2566... - \sin(\frac{4\pi}{5}) = 0.6688... > 0$

Change of sign and f is continuous, so a solution of $\theta = \sin 2\theta$ lies in the interval $\theta \in [\frac{\pi}{5}, \frac{2\pi}{5}]$

Question 10 continues on the next page

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10 (c) The Newton-Raphson method is used to find an approximate solution to the equation

$$\theta = \sin 2\theta$$

10 (c) (i) Using $\theta_1 = \frac{\pi}{5}$ as a first approximation for θ apply the Newton-Raphson method twice to find the value of θ_3

Give your answer to three decimal places.

[3 marks]

$f(\theta) = \theta - \sin 2\theta, \quad f'(\theta) = 1 - 2\cos 2\theta$

$\theta_{n+1} = \theta_n - \frac{\theta_n - \sin 2\theta_n}{1 - 2\cos 2\theta_n}$

$\theta_2 = \frac{\pi}{5} - \frac{\pi/5 - \sin(2\pi/5)}{1 - 2\cos(2\pi/5)} = 1.4732575...$

$\theta_3 = 1.4732575... - \frac{1.4732575... - \sin(2.946515...)}{1 - 2\cos(2.946515...)} = 1.0413241...$

$\theta_3 = 1.041 \quad (3\text{ d.p.})$

10 (c) (ii) Explain how a more accurate approximation for θ can be found using the Newton-Raphson method.

[1 mark]

Use more iterations: keep applying the Newton-Raphson formula

$(\theta_4, \theta_5, ...)$ until the values agree to the required accuracy

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10 (c) (iii) Explain why using $\theta_1 = \frac{\pi}{6}$ as a first approximation in the Newton-Raphson method does not lead to a solution for θ.

[2 marks]

$f\left(\frac{\pi}{6}\right) = 1 - 2\cos\left(\frac{\pi}{3}\right) = 1 - 2 \times \frac{1}{2} = 0$

So $\theta_1 = \frac{\pi}{6}$ is at a stationary point of f: the tangent there is horizontal and does not cross the θ-axis, so $\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$ is undefined (division by zero) and the method fails

Turn over for the next question

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11 The polynomial $\mathrm{p}(x)$ is given by

$$\mathrm{p}(x) = x^3 + (b+2)x^2 + 2(b+2)x + 8$$

where b is a constant.

11 (a) Use the factor theorem to prove that $(x+2)$ is a factor of $\mathrm{p}(x)$ for **all** values of b.

[3 marks]

$p(-2) = (-2)^3 + (b + 2)(-2)^2 + 2(b + 2)(-2) + 8$

$= -8 + 4b + 8 - 4b - 8 + 8$

$= 0$

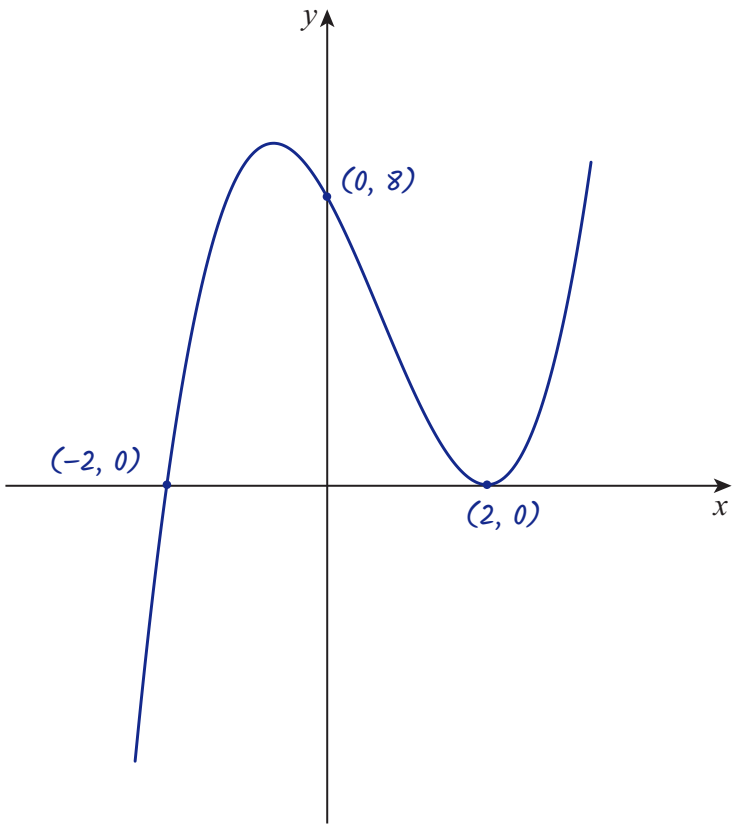
$p(-2) = 0$ for all values of b, so by the factor theorem

$(x + 2)$ is a factor of $p(x)$ for all values of b

11 (b) The graph of $y = \mathrm{p}(x)$ meets the x-axis at exactly two points.

11 (b) (i) Sketch a possible graph of $y = \mathrm{p}(x)$

[3 marks]



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11 (b) (ii) Given $\mathrm{p}(x)$ can be written as

$$\mathrm{p}(x) = (x + 2)(x^2 + bx + 4)$$

find the value of b.

Fully justify your answer.

[4 marks]

$x = -2$ is one root, and $p(x)$ meets the x-axis at exactly two points,

so $x^2 + bx + 4$ must have a repeated root:

$b^2 - 4ac = 0 \quad \Rightarrow \quad b^2 - 16 = 0 \quad \Rightarrow \quad b = \pm 4$

$b = 4$: $x^2 + 4x + 4 = (x + 2)^2$, so $p(x) = (x + 2)^3$

which meets the x-axis at only one point, so reject $b = 4$

$b = -4$: $x^2 - 4x + 4 = (x - 2)^2$, so $p(x) = (x + 2)(x - 2)^2$

which meets the x-axis at exactly two points, $x = -2$ and $x = 2$

(If instead -2 were a root of the quadratic: $4 - 2b + 4 = 0$, $b = 4$, rejected)

$\therefore \; b = -4$

Turn over for the next question

Do not write outside the box

12 (a) A geometric sequence has first term 1 and common ratio $\frac{1}{2}$

12 (a) (i) Find the sum to infinity of the sequence.

[2 marks]

$S_\infty = \frac{a}{1-r} = \frac{1}{1-1/2} = 2$

12 (a) (ii) Hence, or otherwise, evaluate

$$\sum_{n=1}^{\infty} (\sin 30°)^n$$

[2 marks]

$\sin 30° = \frac{1}{2}$

$\Sigma (\sin 30°)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ (from $n = 1$)

Geometric with $a = \frac{1}{2}$ and $r = \frac{1}{2}$:

Sum $= \frac{1/2}{1-1/2} = 1$

Do not write outside the box

12 (b) Find the smallest positive exact value of θ, in **radians**, which satisfies the equation

$$\sum_{n=0}^{\infty} (\cos\theta)^n = 2 - \sqrt{2}$$

[4 marks]

Geometric series with $a = 1$ and $r = \cos\theta$ $(|\cos\theta| < 1)$

$$\frac{1}{1 - \cos\theta} = 2 - \sqrt{2}$$

$$1 - \cos\theta = \frac{1}{2 - \sqrt{2}} = \frac{2 + \sqrt{2}}{(2 - \sqrt{2})(2 + \sqrt{2})} = \frac{2 + \sqrt{2}}{2}$$

$$\cos\theta = 1 - \frac{2 + \sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

Smallest positive value: $\theta = \frac{3\pi}{4}$

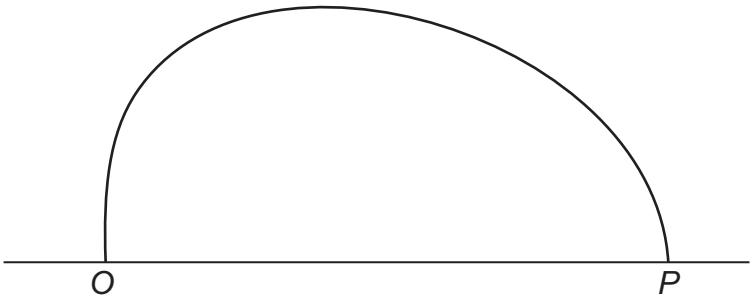
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13 **Figure 2** shows the approximate shape of the vertical cross section of the entrance to a cave. The cave has a horizontal floor.

The entrance to the cave joins the floor at the points O and P.

Figure 2



Garry models the shape of the cross section of the entrance to the cave using the equation

$$x^2 + y^2 = a\sqrt{x} - y$$

where a is a constant, and x and y are the horizontal and vertical distances respectively, in metres, measured from O.

13 (a) The distance OP is 16 metres.

Find the value of a that Garry should use in the model.

[2 marks]

At P: $x = 16$, $y = 0$

$16^2 + 0^2 = a\sqrt{16} - 0$

$256 = 4a$

$a = 64$

Do not write outside the box

13 (b) Show that the maximum height of the cave above *OP* is approximately 10.5 metres.

[6 marks]

$x^2 + y^2 = 64x^{1/2} - y$

Differentiate implicitly: $2x + 2y\frac{dy}{dx} = 32x^{-1/2} - \frac{dy}{dx}$

At the maximum height $\frac{dy}{dx} = 0$:

$2x = 32x^{-1/2} = \frac{32}{\sqrt{x}}$

$x^{3/2} = 16 \Rightarrow x = 16^{2/3} = 6.3496...$

Substitute: $(6.3496...)^2 + y^2 = 64\sqrt{6.3496...} - y$

$y^2 + y - 120.952... = 0$

$y = \frac{-1 + \sqrt{(1 + 4 \times 120.952...)}}{2} = 10.509...$ $(y > 0)$

Maximum height = 10.51 m, which is approximately 10.5 metres

13 (c) Suggest one limitation of the model Garry has used.

[1 mark]

The entrance to the cave is unlikely to be a perfectly smooth curve: real rock will have dents and bumps.

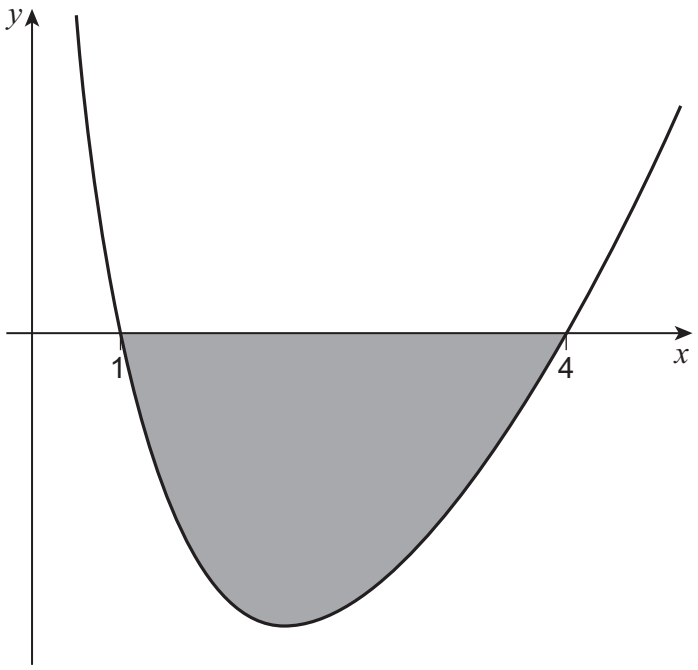
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14 The region bounded by the curve

$$y = (2x - 8)\ln x$$

and the x-axis is shaded in the diagram below.



14 (a) Use the trapezium rule with **5 ordinates** to find an estimate for the area of the shaded region.

Give your answer correct to three significant figures.

[3 marks]

$h = \frac{4-1}{4} = 0.75$

x: 1 1.75 2.5 3.25 4

y: 0 −2.51827 −2.74887 −1.76798 0

$\int \approx \frac{0.75}{2}[0 + 0 + 2(-2.51827 - 2.74887 - 1.76798)]$

$= -5.27634...$

Area ≈ 5.28 (3 s.f.)

Do not write outside the box

14 (b) Show that the exact area is given by

$$32\ln 2 - \frac{33}{2}$$

Fully justify your answer.

[6 marks]

$u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}; \qquad \frac{dv}{dx} = 2x - 8, \quad v = x^2 - 8x$

$\int (2x - 8)\ln x\, dx = (x^2 - 8x)\ln x - \int \frac{x^2 - 8x}{x}\, dx$

$= (x^2 - 8x)\ln x - \int (x - 8)\, dx$

$= (x^2 - 8x)\ln x - \frac{x^2}{2} + 8x$

$\int_1^4 (2x - 8)\ln x\, dx = \left[(x^2 - 8x)\ln x - \frac{x^2}{2} + 8x\right]_1^4$

$= ((16 - 32)\ln 4 - 8 + 32) - ((1 - 8)\ln 1 - \frac{1}{2} + 8)$

$= -16\ln 4 + 24 - \frac{15}{2}$

$= \frac{33}{2} - 32\ln 2$

The shaded region is below the x-axis, so this integral is negative

Area $= 32\ln 2 - \frac{33}{2}$

Turn over ▶

Do not write outside the box

15 (a) Given that

$$y = \text{cosec}\,\theta$$

15 (a) (i) Express y in terms of $\sin\theta$.

[1 mark]

$y = \frac{1}{\sin\theta} = (\sin\theta)^{-1}$

15 (a) (ii) Hence, prove that

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\,\text{cosec}\,\theta\,\cot\theta$$

[3 marks]

$y = (\sin\theta)^{-1}$

Chain rule: $\frac{dy}{d\theta} = -1 \times (\sin\theta)^{-2} \times \cos\theta$

$= -\frac{\cos\theta}{\sin^2\theta}$

$= -\frac{1}{\sin\theta} \times \frac{\cos\theta}{\sin\theta}$

$= -\text{cosec}\,\theta\,\cot\theta$ as required

Do not write outside the box

15 (a) (iii) Show that

$$\frac{\sqrt{y^2-1}}{y} = \cos\theta \qquad \text{for } 0 < \theta < \frac{\pi}{2}$$

[3 marks]

$y = \operatorname{cosec}\theta$

$\frac{\sqrt{(y^2-1)}}{y} = \frac{\sqrt{(\operatorname{cosec}^2\theta - 1)}}{\operatorname{cosec}\theta}$

$= \frac{\sqrt{(\cot^2\theta)}}{\operatorname{cosec}\theta}$ using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$= \frac{\cot\theta}{\operatorname{cosec}\theta}$ ($\cot\theta > 0$ for $0 < \theta < \pi/2$)

$= \frac{\cos\theta}{\sin\theta} \times \sin\theta$

$= \cos\theta$ as required

Turn over ▶

Do not write outside the box

15 (b) (i) Use the substitution

$$x = 2\,\text{cosec}\,u$$

to show that

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,\mathrm{d}x \qquad \text{for } x > 2$$

can be written as

$$k\int \sin u\,\mathrm{d}u$$

where k is a constant to be found.

[6 marks]

$x = 2\,\text{cosec}\,u \Rightarrow \frac{dx}{du} = -2\,\text{cosec}\,u\,\cot u, \quad dx = -2\,\text{cosec}\,u\,\cot u\,du$

$x^2 - 4 = 4\,\text{cosec}^2 u - 4 = 4(\text{cosec}^2 u - 1) = 4\cot^2 u$

$\sqrt{(x^2-4)} = 2\cot u \qquad (x > 2 \text{ so } 0 < u < \pi/2,\ \cot u > 0)$

$\int \frac{1}{x^2\sqrt{(x^2-4)}}\,dx = \int \frac{-2\,\text{cosec}\,u\,\cot u}{4\,\text{cosec}^2 u\,\sqrt{(4\,\text{cosec}^2 u - 4)}}\,du$

$= \int \frac{-2\,\text{cosec}\,u\,\cot u}{4\,\text{cosec}^2 u \times 2\cot u}\,du$

$= \int -\frac{1}{4\,\text{cosec}\,u}\,du$

$= -\frac{1}{4}\int \sin u\,du$

$k = -\frac{1}{4}$

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15 (b) (ii) Hence, show

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,\mathrm{d}x = \frac{\sqrt{x^2-4}}{4x} + c \qquad \text{for } x > 2$$

where c is a constant.

[3 marks]

$-\frac{1}{4}\int \sin u\, du = -\frac{1}{4}(-\cos u) + c = \frac{1}{4}\cos u + c$

Here $\operatorname{cosec} u = \frac{x}{2}$, so by (a)(iii) with $y = \frac{x}{2}$ $(0 < u < \pi/2)$:

$\cos u = \frac{\sqrt{((x/2)^2 - 1)}}{x/2} = \frac{\sqrt{((x^2-4)/4)}}{x/2} = \frac{(1/2)\sqrt{(x^2-4)}}{x/2} = \frac{\sqrt{(x^2-4)}}{x}$

$\int \frac{1}{x^2\sqrt{(x^2-4)}}\,dx = \frac{1}{4} \times \frac{\sqrt{(x^2-4)}}{x} + c$

$= \frac{\sqrt{(x^2-4)}}{4x} + c$ as required

END OF QUESTIONS

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