AQA

Please write clearly in block capitals.

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Surname StemUp Model Solution

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I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 3 Discrete

Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)
- You must ensure you have the other optional Question Paper/Answer Book for which you are entered (**either** Mechanics **or** Statistics). You will have 2 hours to complete **both** papers.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 50.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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5	
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7	
TOTAL	

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Answer **all** questions in the spaces provided.

1 Which of the following statements about critical path analysis is always true?

Tick (✓) **one** box.

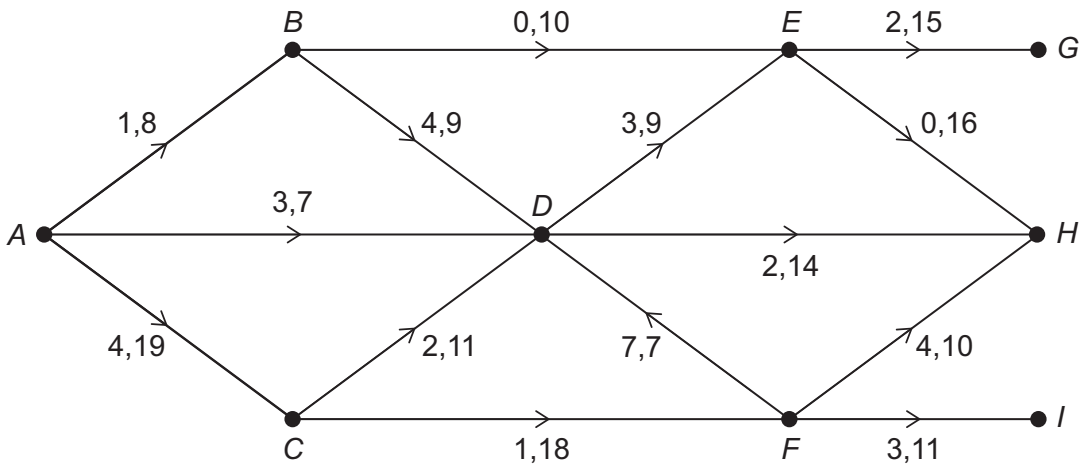
[1 mark]

- All activity networks have exactly one critical path. ☐
- All critical activities have a non-zero float. ☐
- The first activity in a critical path has an earliest start time of zero. ☑
- A delay on a critical activity may not delay the project. ☐

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2 The network below represents a system of pipes.

The numbers on each arc represent the lower and upper capacity for each pipe.



Find the value of the cut $\{A, B, C, D, E\}$ $\{F, G, H, I\}$.

Circle your answer.

[1 mark]

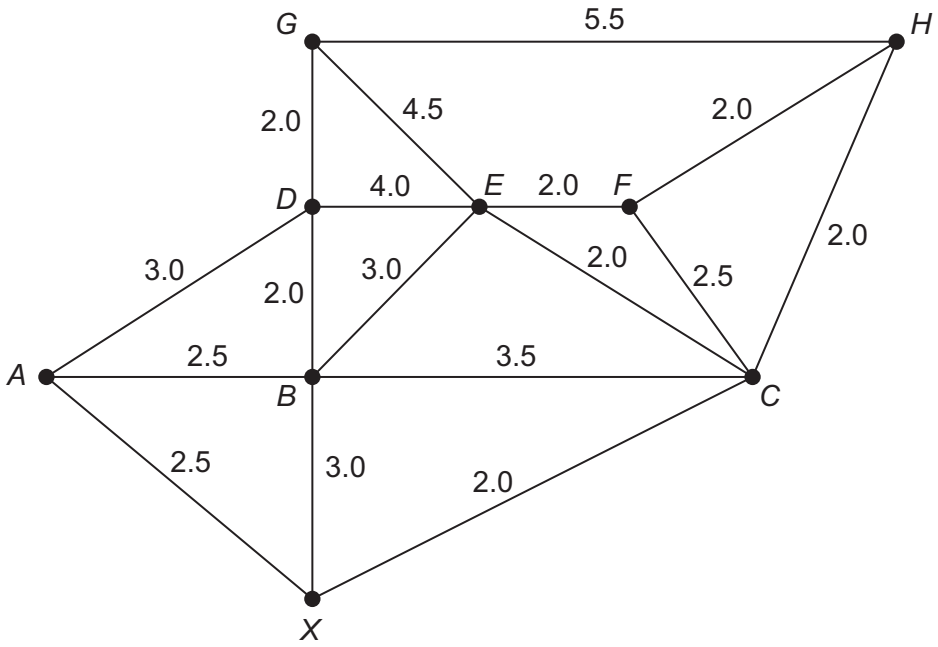
(56) 58 63 70

Turn over for the next question

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3 A mining company wants to open a new mine in an area where the ground contains a precious metal.

The mining company has carried out a survey of the area. The network below shows nodes which represent the entrance to the new mine, *X*, and the 8 ventilation shafts, *A*, *B*, …, *H*, which have been installed to prevent the build up of dangerous gases underground.



Each arc represents a possible underground tunnel which could be mined.

The weight on each arc represents the estimated amount of precious metal in that possible underground tunnel in tonnes.

Due to geological reasons, the mining company can only create 8 underground tunnels.

All 8 ventilation shafts must be accessible from the entrance of the mine.

3 (a) (i) The mining company wants to **maximise** the amount of precious metal it can extract from the new mine.

Determine the tunnels the mining company should use.

[2 marks]

Maximum spanning tree (Kruskal, largest arcs first):

GH 5.5, EG 4.5, DE 4.0, BC 3.5, BX 3.0, AD 3.0, BE 3.0,

then XA 2.5 and AB 2.5 rejected (cycles), CF 2.5 added.

8 arcs connect all 9 nodes. Tunnels to use:

GH, EG, DE, BC, BX, AD, BE and CF

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3 (a) (ii) Estimate the maximum amount of precious metal the mining company can extract from the new mine.

[2 marks]

Total = 5.5 + 4.5 + 4.0 + 3.5 + 3.0 + 3.0 + 3.0 + 2.5

= 29

Maximum amount of precious metal ≈ 29 tonnes

3 (b) Comment on why the maximum amount of precious metal the mining company can extract from the new mine may be different from your answer to part **(a)(ii)**.

[2 marks]

The weights on the arcs are only estimates of the amount of precious metal in each tunnel, so 29 tonnes is itself only an estimate of the maximum amount.

The true maximum may be more than or less than 29 tonnes.

3 (c) Before the mining company begins work on the new mine, a government survey prevents the mining company drilling the tunnel represented by *CF*.

Determine the effect, if any, the government survey has on your answers to part **(a)(i)** and part **(a)(ii)**.

[2 marks]

CF (2.5) is in the maximum spanning tree, so F must now be joined by another arc: EF or FH (both weight 2.0).

(a)(i): CF is replaced by EF (or FH); other 7 tunnels the same.

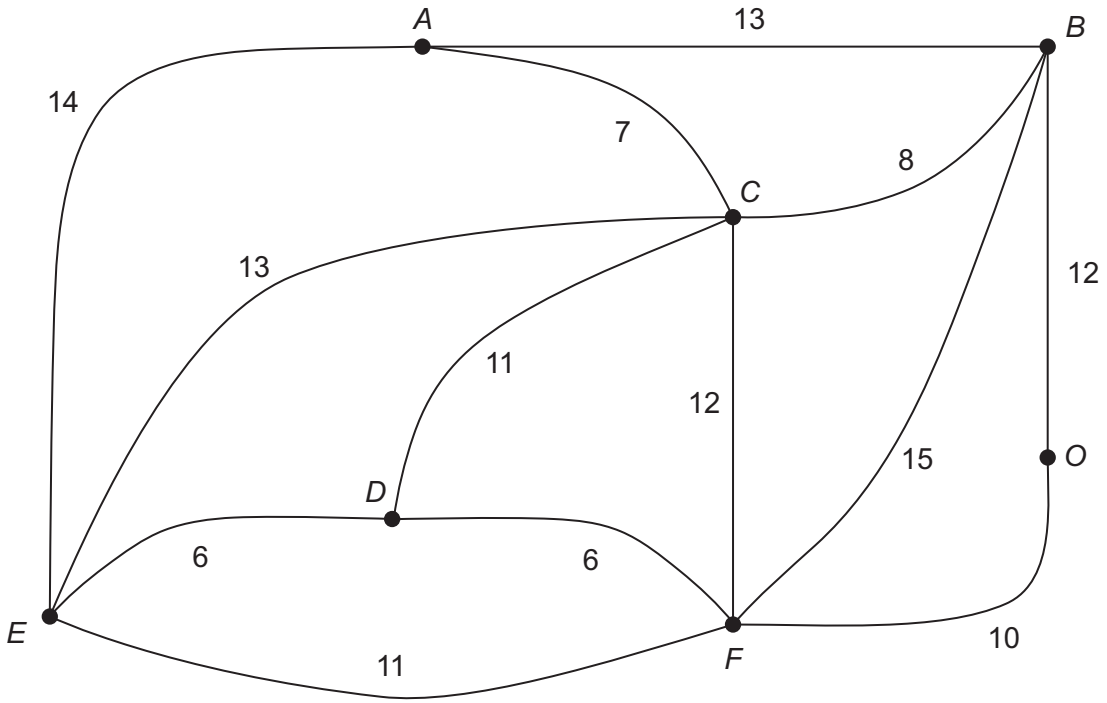
(a)(ii): new estimate = 29 − 2.5 + 2.0 = 28.5 tonnes, so the estimate decreases by 0.5 tonnes to 28.5 tonnes.

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4 Derrick, a tanker driver, is required to deliver fuel to 6 different service stations *A*, *B*, *C*, *D*, *E* and *F*.

Derrick needs to begin and finish his delivery journey at the refinery *O*.

The distances, in miles, between the 7 locations which have a direct road between them are shown in the network below.



Derrick spends 30 minutes at each service station to complete the fuel delivery.

When driving, the tanker travels at an average speed of 40 miles per hour.

The minimum total time that it takes Derrick to travel to and deliver fuel to all 6 service stations, starting and finishing at the refinery, is T minutes.

4 (a) Using the nearest neighbour algorithm starting from the refinery, find an upper bound for T

[5 marks]

Nearest neighbour from O: O → F (10) (OF = 10 < OB = 12)

F → D (6) D → E (6)

E → C (13) (EC = 13 < EA = 14)

C → A (7) A → B (13) B → O (12)

Hamiltonian cycle: O F D E C A B O

Distance = 10 + 6 + 6 + 13 + 7 + 13 + 12 = 67 miles

Driving time = (67 ÷ 40) × 60 = 100.5 minutes

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Delivery time = 6 × 30 = 180 minutes

Upper bound = 100.5 + 180 = 280.5

T ≤ 280.5 minutes

4 (b) Before setting off to make his fuel deliveries, Derrick is notified that, due to a low bridge, the road represented by *CE* is not suitable for tankers to travel along.

State, with a reason, the effect this new information has on your answer to part **(a)**.

[2 marks]

CE was used in the cycle in (a), so at E the nearest unvisited vertex is now A (14); then A → C (7), C → B (8), B → O (12).

O F D E A C B O = 10 + 6 + 6 + 14 + 7 + 8 + 12 = 63 miles

Time = (63 ÷ 40) × 60 + 180 = 94.5 + 180 = 274.5 minutes

The upper bound for T reduces by 6 minutes to 274.5 minutes.

Turn over for the next question

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5 (a) Describe the conditions necessary for a set of elements, S, under a binary operation * to form a group.

[3 marks]

Closure: for all $a, b \in S$, $a * b \in S$

Identity: there exists $e \in S$ such that $e * a = a * e = a$

for all $a \in S$

Inverse: for each $a \in S$ there exists $a^{-1} \in S$ such that

$a * a^{-1} = a^{-1} * a = e$

Associativity: for all $a, b, c \in S$,

$(a * b) * c = a * (b * c)$

5 (b) In the multiplicative group of integers modulo 13, the group G is defined as

$$G = (\langle 10 \rangle, \times_{13})$$

5 (b) (i) Explain why G is an abelian group.

[1 mark]

Multiplication modulo 13 is commutative, $a \times_{13} b = b \times_{13} a$

for all $a, b \in G$, so G is an abelian group.

5 (b) (ii) Find the order of G.

[2 marks]

$10^1 \equiv 10, \quad 10^2 = 100 \equiv 9, \quad 10^3 \equiv 90 \equiv 12 \pmod{13}$

$10^4 \equiv 120 \equiv 3, \quad 10^5 \equiv 30 \equiv 4, \quad 10^6 \equiv 40 \equiv 1 \pmod{13}$

$G = \{1, 3, 4, 9, 10, 12\}$

The order of G is 6

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5 (c) State the identity element of G and prove it is an identity element.

Fully justify your answer.

[3 marks]

The identity element of G is 1 $(1 = 10^6 \in G)$

Let g be any element of G. Then

$1 \times_{13} g = g$ and $g \times_{13} 1 = g$

since multiplying any integer by 1 leaves it unchanged.

Both left- and right-multiplication by 1 return g, so

1 is the identity element of G.

5 (d) Find all the proper non-trivial subgroups of G, giving your answers in the form $(\langle g \rangle, \times_{13})$, where g is an integer less than 13

[2 marks]

Orders of subgroups divide 6, so proper non-trivial: 2 or 3

Order 2: $\{1, 12\} = (\langle 12 \rangle, \times_{13})$ $(12^2 = 144 \equiv 1)$

Order 3: $\{1, 3, 9\} = (\langle 9 \rangle, \times_{13})$ $(9^2 = 81 \equiv 3, 9^3 \equiv 1)$

Turn over for the next question

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6 (a) A connected planar graph has $(x+1)^2$ vertices, $(25+2x-2y)$ edges and $(y-1)^2$ faces, where $x>0$ and $y>0$

Find the possible values for the number of vertices, edges and faces for the graph.

[6 marks]

Euler's formula for a connected planar graph: $V - E + F = 2$

$(x + 1)^2 - (25 + 2x - 2y) + (y - 1)^2 = 2$

$x^2 + 2x + 1 - 25 - 2x + 2y + y^2 - 2y + 1 = 2$

$x^2 + y^2 - 23 = 2$

$x^2 + y^2 = 25$

V, E and F are whole numbers, so x and y are positive integers:

$x = 3, y = 4$ or $x = 4, y = 3$

$x = 3, y = 4$: $V = 4^2 = 16$, $E = 25 + 6 - 8 = 23$, $F = 3^2 = 9$

$x = 4, y = 3$: $V = 5^2 = 25$, $E = 25 + 8 - 6 = 27$, $F = 2^2 = 4$

Check: $16 - 23 + 9 = 2$ and $25 - 27 + 4 = 2$

Either 16 vertices, 23 edges and 9 faces

or 25 vertices, 27 edges and 4 faces.

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6 (b) Explain why K_6, the complete graph with 6 vertices, is not planar.

Fully justify your answer.

[2 marks]

K_6 contains K_5 as a subgraph (remove any one vertex and the 5 edges joined to it).

Kuratowski's theorem: a graph which contains a subgraph that is a subdivision of K_5 or $K_{3,3}$ is not planar.

Hence K_6 is not planar.

Turn over for the next question

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7 Avon and Roj play a zero-sum game.

The game is represented by the following pay-off matrix for Avon.

		Roj		
	Strategy	**R_1**	**R_2**	**R_3**
Avon	**A_1**	3	1	4
	A_2	2	3	2
	A_3	6	2	1

The zero-sum game does not have a stable solution.

7 (a) Avon wants to know the optimal mixed strategy he should play in order to maximise his winnings from the game.

Avon defines the following variables:

$p_1 =$ the probability of Avon playing strategy **A_1**

$p_2 =$ the probability of Avon playing strategy **A_2**

$p_3 =$ the probability of Avon playing strategy **A_3**

$v =$ the value of the game for Avon

Formulate Avon's situation as a linear programming problem.

[4 marks]

No strategy is dominated, so all three rows are used.

Maximise $P = v$

subject to $3p_1 + 2p_2 + 6p_3 \geq v$

$p_1 + 3p_2 + 2p_3 \geq v$

$4p_1 + 2p_2 + p_3 \geq v$

$p_1 + p_2 + p_3 \leq 1$

$p_1, p_2, p_3 \geq 0$

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7 (b) (i) Using your answer to part **(a)**, complete the initial Simplex tableau for Avon's situation in the grid below.

[3 marks]

P	v	p_1	p_2	p_3	s_1	s_2	s_3	s_4	**value**
1	−1	0	0	0	0	0	0	0	0
0	1	−3	−2	−6	1	0	0	0	0
0	1	−1	−3	−2	0	1	0	0	0
0	1	−4	−2	−1	0	0	1	0	0
0	0	1	1	1	0	0	0	1	1

7 (b) (ii) Hence, perform **one** iteration of the Simplex algorithm, showing your answer on the grid below.

[3 marks]

P	v	p_1	p_2	p_3	s_1	s_2	s_3	s_4	**value**
1	0	−3	−2	−6	1	0	0	0	0
0	1	−3	−2	−6	1	0	0	0	0
0	0	2	−1	4	−1	1	0	0	0
0	0	−1	0	5	−1	0	1	0	0
0	0	1	1	1	0	0	0	1	1

Pivot column v (only negative in P row); ratios 0, 0, 0 (tie), pivot on row 2.
Row 2 unchanged; row 1 + row 2, row 3 − row 2, row 4 − row 2, row 5 unchanged

Question 7 continues on the next page

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7 (c) Avon performs further iterations of the Simplex algorithm until an optimal solution is reached.

The grid below shows part of Avon's final Simplex tableau.

p_1	p_2	p_3	**value**
1	0	0	0.25
0	1	0	0.70
0	0	1	0.05

7 (c) (i) Find the optimal mixed strategy for Avon.

[1 mark]

Avon should play A_1 with probability 0.25, A_2 with probability 0.70 and A_3 with probability 0.05

7 (c) (ii) Find the value of the game for Avon.

[1 mark]

Against R_1: $3 \times 0.25 + 2 \times 0.70 + 6 \times 0.05$

$= 0.75 + 1.40 + 0.30 = 2.45$

The value of the game for Avon is 2.45

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7 (d) Roj thinks that his best outcome from the game is to play strategy $\mathbf{R_2}$ each time.

Avon notices that Roj always plays strategy $\mathbf{R_2}$ and Avon wants to use this knowledge to maximise his expected pay-off from the game.

Explain how your answer to part **(c)(i)** should change and find Avon's maximum expected pay-off from the game.

[2 marks]

If Roj always plays R_2, Avon's pay-offs are A_1: 1, A_2: 3, A_3: 2

Avon should no longer use the mixed strategy in (c)(i);

he should play strategy A_2 every time (probability 1),

as this gives the largest pay-off against R_2.

Avon's maximum expected pay-off = 3

END OF QUESTIONS

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