AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 2

Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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9	
10	
11	
12	
13	
TOTAL	

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Answer **all** questions in the spaces provided.

1 Which of the following matrices is singular?

Circle your answer.

[1 mark]

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

2 Find $\arg(-4-7\mathrm{i})$ to the nearest degree.

Circle your answer.

[1 mark]

$-120°$ $\quad$ $-60°$ $\quad$ $30°$ $\quad$ $60°$

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3 The line L has equation $\mathbf{r} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

Which of the following lines is perpendicular to the line L?

Tick (✓) **one** box.

[1 mark]

$\mathbf{r} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ ☐

$\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ ☐

$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ ☐

$\mathbf{r} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ ☑

Turn over for the next question

Do not write outside the box

4 (a) Show that

$$(r+1)^2 - r^2 = 2r + 1$$

[1 mark]

$(r + 1)^2 - r^2 = r^2 + 2r + 1 - r^2$

$= 2r + 1$ as required

4 (b) Use the method of differences to show that

$$\sum_{r=1}^{n} (2r+1) = n^2 + 2n$$

[3 marks]

$\sum_{r=1}^{n} (2r + 1) = \sum_{r=1}^{n} ((r + 1)^2 - r^2)$

$= (2^2 - 1^2)$

$+ (3^2 - 2^2)$

$+ (4^2 - 3^2)$

$+ \ldots$

$+ (n^2 - (n - 1)^2)$

$+ ((n + 1)^2 - n^2)$

All the terms cancel except -1^2 and $(n + 1)^2$:

$\sum_{r=1}^{n} (2r + 1) = (n + 1)^2 - 1$

$= n^2 + 2n + 1 - 1$

$= n^2 + 2n$ as required

Do not write outside the box

4 (c) Verify that using the formula for $\sum_{r=1}^{n} r$ gives the same result as that given in part **(b)**.

[3 marks]

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$\sum_{r=1}^{n} (2r+1) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= 2 \times \frac{1}{2}n(n+1) + n$

$= n^2 + n + n$

$= n^2 + 2n$

which is the same result as in part (b).

Do not write outside the box

5 The equation

$$z^3 + 2z^2 - 5z - 3 = 0$$

has roots α, β and γ

Find a cubic equation with roots

$$\frac{1}{2}\alpha - 1, \ \frac{1}{2}\beta - 1 \ \text{ and } \ \frac{1}{2}\gamma - 1$$

[5 marks]

Let $w = \frac{1}{2}z - 1 \Rightarrow z = 2w + 2$

Substitute into $z^3 + 2z^2 - 5z - 3 = 0$:

$(2w + 2)^3 + 2(2w + 2)^2 - 5(2w + 2) - 3 = 0$

$(2w + 2)^3 = 8w^3 + 24w^2 + 24w + 8$

$2(2w + 2)^2 = 2(4w^2 + 8w + 4) = 8w^2 + 16w + 8$

$-5(2w + 2) = -10w - 10$

$8w^3 + 24w^2 + 24w + 8 + 8w^2 + 16w + 8 - 10w - 10 - 3 = 0$

$8w^3 + 32w^2 + 30w + 3 = 0$

Check: $\alpha + \beta + \gamma = -2$, so the sum of the new roots is

$\frac{1}{2}(\alpha + \beta + \gamma) - 3 = -1 - 3 = -4 = -\frac{32}{8}$ ✓

A cubic equation with the required roots is

$$8w^3 + 32w^2 + 30w + 3 = 0$$

Do not write outside the box

Turn over for the next question

Do not write outside the box

6 The ellipse E_1 has equation

$$x^2 + \frac{y^2}{4} = 1$$

E_1 is translated by the vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ to give the ellipse E_2

6 (a) Write down the equation of E_2

[1 mark]

E_2: $(x - 3)^2 + \frac{y^2}{4} = 1$

6 (b) The ellipse E_3 has equation

$$\frac{x^2}{4} + (y - 3)^2 = 1$$

Describe the transformation that maps E_2 to E_3

[1 mark]

Reflection in the line $y = x$

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6 (c) Each of the lines L_A and L_B is a tangent to both E_2 and E_3

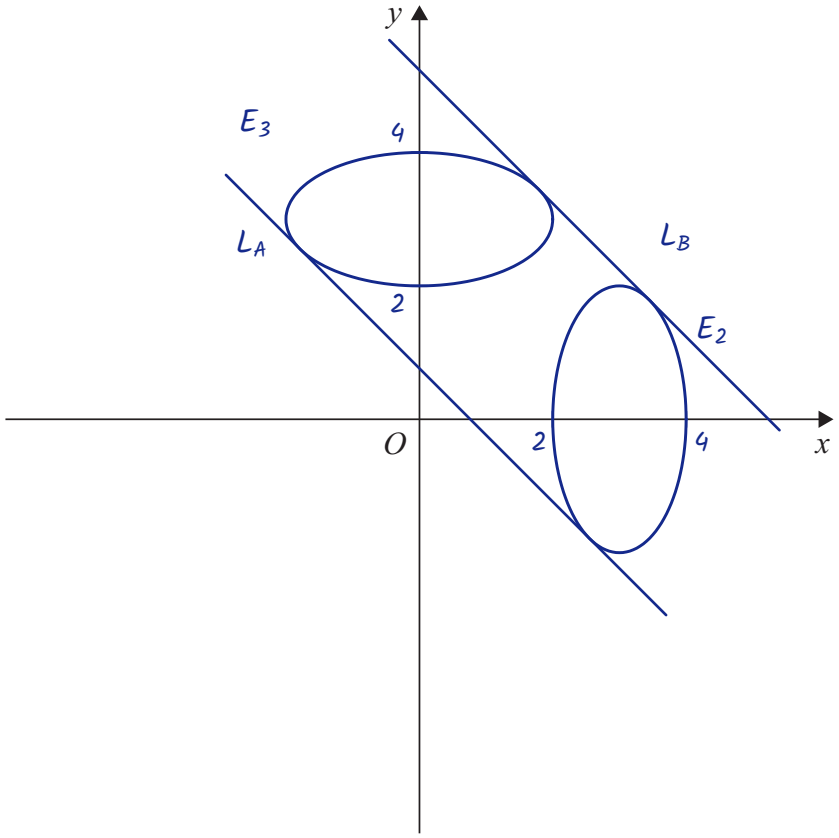
L_A is closer to the origin than L_B

E_2 and E_3 both lie between L_A and L_B

Sketch and label E_2, E_3, L_A and L_B on the axes below.

You do not need to show the values of the axis intercepts for L_A and L_B

[4 marks]



6 (d) Explain, without doing any calculations, why L_A has an equation of the form

$$x + y = c$$

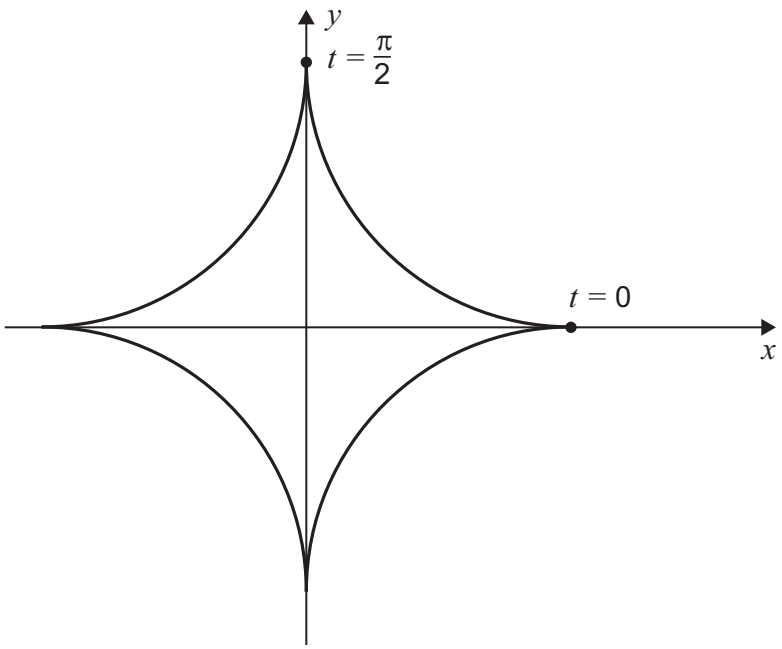
where c is a constant.

[2 marks]

E_3 is the reflection of E_2 in the line $y = x$, so the curves are symmetrical about $y = x$ and L_A touches E_2 and E_3 at points which are reflections of each other in $y = x$. So L_A is perpendicular to $y = x$ (gradient 1), so L_A has gradient -1: $y = -x + c$, i.e. $x + y = c$.

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7



The diagram shows a curve known as an astroid.

The curve has parametric equations

$$x = 4\cos^3 t$$

$$y = 4\sin^3 t$$

$$(0 \leq t < 2\pi)$$

The section of the curve from $t = 0$ to $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

Show that the curved surface area of the shape formed is equal to $\frac{b\pi}{c}$, where b and c are integers.

[7 marks]

$\frac{dx}{dt} = -12\cos^2 t \sin t, \quad \frac{dy}{dt} = 12\sin^2 t \cos t$

$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 144\cos^4 t \sin^2 t + 144\sin^4 t \cos^2 t$

$= 144\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)$

$= 144\sin^2 t \cos^2 t$

$\sqrt{((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)} = 12\sin t \cos t \quad (\geq 0 \text{ for } 0 \leq t \leq \frac{\pi}{2})$

$S = 2\pi \int_0^{\pi/2} y \sqrt{((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)}\, dt$

$= 2\pi \int_0^{\pi/2} 4\sin^3 t \times 12\sin t \cos t\, dt$

$= 96\pi \int_0^{\pi/2} \sin^4 t \cos t\, dt$

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Q7 continued

$= 96\pi\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2}$

$= 96\pi\left(\frac{1}{5} - 0\right)$

$= \frac{96\pi}{5}$

So the curved surface area is $\frac{b\pi}{c}$ with $b = 96$ and $c = 5$, which are integers.

Do not write outside the box

8 The complex number z satisfies the equations

$$|z^* - 1 - 2i| = |z - 3|$$

and

$$|z - a| = 3$$

where a is real.

Show that a must lie in the interval $\left[1 - s\sqrt{t},\ 1 + s\sqrt{t}\right]$, where s and t are prime numbers.

[6 marks]

Let $z = x + iy$, so $z^* = x - iy$

$|x - iy - 1 - 2i| = |x + iy - 3|$

$|(x - 1) - i(y + 2)| = |(x - 3) + iy|$

$(x - 1)^2 + (y + 2)^2 = (x - 3)^2 + y^2$

$x^2 - 2x + 1 + y^2 + 4y + 4 = x^2 - 6x + 9 + y^2$

$4x + 4y = 4 \quad \Rightarrow \quad y = 1 - x$

$|z - a| = 3$ is a circle, centre $(a, 0)$, radius 3: $(x - a)^2 + y^2 = 9$

z satisfies both equations, so the line and the circle must meet. Substitute $y = 1 - x$:

$(x - a)^2 + (1 - x)^2 = 9$

$x^2 - 2ax + a^2 + 1 - 2x + x^2 - 9 = 0$

$2x^2 - (2a + 2)x + (a^2 - 8) = 0$

For real solutions for x the discriminant must be non-negative, $\Delta \geq 0$:

$(2a + 2)^2 - 4(2)(a^2 - 8) \geq 0$

$4a^2 + 8a + 4 - 8a^2 + 64 \geq 0$

$-4a^2 + 8a + 68 \geq 0$

Do not write outside the box

Q8 continued

$a^2 - 2a - 17 \leq 0$

$(a - 1)^2 - 18 \leq 0$

$(a - 1)^2 \leq 18$

$-\sqrt{18} \leq a - 1 \leq \sqrt{18}$

$1 - 3\sqrt{2} \leq a \leq 1 + 3\sqrt{2}$

So a must lie in the interval $[1 - 3\sqrt{2}, 1 + 3\sqrt{2}]$,

i.e. $s = 3$ and $t = 2$, which are both prime numbers.

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9 (a) The line L has polar equation

$$r = \frac{7}{4}\sec\theta \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

Show that L is perpendicular to the initial line.

[2 marks]

$r = \frac{7}{4}\sec\theta \Rightarrow r\cos\theta = \frac{7}{4}$

$x = r\cos\theta$, so L has Cartesian equation $x = \frac{7}{4}$

$x = 7/4$ is a vertical line, perpendicular to the x-axis, which is the initial line. So L is perpendicular to the initial line.

9 (b) The curve C has polar equation

$$r = 3 + \cos\theta \qquad (-\pi < \theta \leq \pi)$$

Find the polar coordinates of the points of intersection of L and C

Fully justify your answer.

[5 marks]

At points of intersection: $\frac{7}{4}\sec\theta = 3 + \cos\theta$

Multiply by $\cos\theta$: $\frac{7}{4} = 3\cos\theta + \cos^2\theta$

$4\cos^2\theta + 12\cos\theta - 7 = 0$

$(2\cos\theta + 7)(2\cos\theta - 1) = 0$

$\cos\theta = -\frac{7}{2}$ or $\cos\theta = \frac{1}{2}$

$\cos\theta = -7/2$ is impossible since $-1 \leq \cos\theta \leq 1$, so reject it.

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$ (both in $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$)

$r = 3 + \cos\theta = 3 + \frac{1}{2} = \frac{7}{2}$

Points of intersection: $\left(\frac{7}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{7}{2}, -\frac{\pi}{3}\right)$

Do not write outside the box

9 (c) The region R is the set of points such that

$$r > \frac{7}{4}\sec\theta$$

and

$$r < 3 + \cos\theta$$

Find the exact area of R

[7 marks]

R is the part of the region inside C that lies to the right of L, between P(7/2, π/3) and Q(7/2, −π/3).

Area of R = (area of C from $\theta = -\frac{\pi}{3}$ to $\frac{\pi}{3}$) − (area of triangle OPQ)

$PQ = 2 \times \frac{7}{2}\sin\frac{\pi}{3} = \frac{7\sqrt{3}}{2}$, distance from O to L = $\frac{7}{4}$

Area of triangle OPQ = $\frac{1}{2} \times \frac{7\sqrt{3}}{2} \times \frac{7}{4} = \frac{49\sqrt{3}}{16}$

Sector area = $2 \times \frac{1}{2}\int_0^{\pi/3} (3 + \cos\theta)^2\, d\theta$ (by symmetry)

$= \int_0^{\pi/3} (9 + 6\cos\theta + \cos^2\theta)\, d\theta$

$= \int_0^{\pi/3} (9 + 6\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2})\, d\theta$

$= [\frac{19}{2}\theta + 6\sin\theta + \frac{1}{4}\sin 2\theta]_0^{\pi/3}$

$= \frac{19\pi}{6} + 6 \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2} - 0$

$= \frac{19\pi}{6} + 3\sqrt{3} + \frac{\sqrt{3}}{8} = \frac{19\pi}{6} + \frac{25\sqrt{3}}{8}$

Area of R = $\frac{19\pi}{6} + \frac{25\sqrt{3}}{8} - \frac{49\sqrt{3}}{16}$

$= \frac{19\pi}{6} + \frac{\sqrt{3}}{16}$

Do not write outside the box

10 In a colony of seabirds, there are y birds at time t years.

10 (a) The rate of reduction in the number of birds due to birds dying or leaving the colony is proportional to the number of birds.

In one year the reduction in the number of birds due to birds dying or leaving the colony is equal to 16% of the number of birds at the start of the year.

If no birds are born or join the colony, find the constant k such that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky$$

Give your answer to three significant figures.

[4 marks]

$\frac{dy}{dt} = -ky \quad \Rightarrow \quad \int \frac{1}{y}\, dy = \int -k\, dt$

$\ln y = -kt + C \quad \Rightarrow \quad y = y_0 e^{-kt}$

After 1 year the number is 84% of the start: $y = 0.84y_0$

$0.84y_0 = y_0 e^{-k} \quad \Rightarrow \quad e^{-k} = 0.84$

$k = -\ln 0.84 = 0.174353\ldots$

$k = 0.174 \quad (3\text{ s.f.})$

10 (b) A wildlife protection group takes measures to support the colony.

The rate of reduction in the number of birds due to birds dying or leaving the colony is the same as in part **(a)**, but in addition:

- The rate of increase in the number of birds due to births is $20t$ per year.
- The wildlife protection group brings 45 birds into the colony each year.

Write down a first-order differential equation for y and t

[2 marks]

$\frac{dy}{dt} = -0.174y + 20t + 45$

Do not write outside the box

10 (c) The initial number of birds is 340

Solve your differential equation from part **(b)** to find y in terms of t

[5 marks]

$\frac{dy}{dt} + 0.174y = 20t + 45$

CF: $y = Ae^{-0.174t}$

PI: $y = p + qt$, $\frac{dy}{dt} = q$

$q + 0.174(p + qt) = 20t + 45$

t terms: $0.174q = 20 \Rightarrow q = 114.94... \approx 115$

constants: $q + 0.174p = 45 \Rightarrow p = \frac{45 - 114.94...}{0.174} = -401.97... \approx -402$

General solution: $y = Ae^{-0.174t} + 115t - 402$

$t = 0, y = 340$: $340 = A - 402 \Rightarrow A = 742$

$y = 742e^{-0.174t} + 115t - 402$

10 (d) Describe two limitations of the model you have used.

[2 marks]

Limitation 1 Births happen at one time of year (the breeding season), not continuously at a rate of 20t per year.

Limitation 2 The model predicts the number of birds grows without limit as t increases, which is unrealistic as food and nesting space are limited.

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11 The Cartesian equation of the line L_1 is

$$\frac{x+1}{3} = \frac{-y+5}{2} = \frac{2z+5}{3}$$

The Cartesian equation of the line L_2 is

$$\frac{2x-1}{2} = \frac{y-14}{m} = \frac{z+12}{p}$$

The non-singular matrix $\mathbf{N} = \begin{bmatrix} -0.5 & 1 & 2 \\ 1 & b & 4 \\ -3 & -2 & c \end{bmatrix}$ maps the line L_1 onto the line L_2

Calculate the values of the constants b, c, m and p

Fully justify your answers.

[9 marks]

L_1: let each ratio = 2λ: $x = -1 + 6\lambda$, $y = 5 - 4\lambda$, $z = -\frac{5}{2} + 3\lambda$

$$r = \begin{bmatrix} -1 \\ 5 \\ -5/2 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$$

L_2: let each ratio = μ: $x = \frac{1}{2} + \mu$, $y = 14 + m\mu$, $z = -12 + p\mu$

$$r = \begin{bmatrix} 1/2 \\ 14 \\ -12 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ m \\ p \end{bmatrix}$$

N maps each point of L_1 to a point of L_2:

$$\begin{bmatrix} -0.5 & 1 & 2 \\ 1 & b & 4 \\ -3 & -2 & c \end{bmatrix} \begin{bmatrix} -1 + 6\lambda \\ 5 - 4\lambda \\ -2.5 + 3\lambda \end{bmatrix} = \begin{bmatrix} 0.5 + \mu \\ 14 + m\mu \\ -12 + p\mu \end{bmatrix}$$

Row 1: $0.5 - 3\lambda + 5 - 4\lambda - 5 + 6\lambda = 0.5 - \lambda$

Row 2: $-1 + 6\lambda + 5b - 4b\lambda - 10 + 12\lambda = (5b - 11) + (18 - 4b)\lambda$

Row 3: $3 - 18\lambda - 10 + 8\lambda - 2.5c + 3c\lambda = (-7 - 2.5c) + (3c - 10)\lambda$

Row 1: $0.5 - \lambda = 0.5 + \mu \Rightarrow \mu = -\lambda$

So $\lambda = 0$, the point $(-1, 5, -2.5)$ on L_1, maps to $\mu = 0$, the point $(0.5, 14, -12)$ on L_2. Comparing constant terms:

$5b - 11 = 14 \Rightarrow b = 5$

$-7 - 2.5c = -12 \Rightarrow c = 2$

Do not write outside the box

Q11 continued

Comparing λ terms, using $\mu = -\lambda$:

Row 2: $(18 - 4b)\lambda = m\mu \Rightarrow (18 - 20)\lambda = -m\lambda \Rightarrow m = 2$

Row 3: $(3c - 10)\lambda = p\mu \Rightarrow (6 - 10)\lambda = -p\lambda \Rightarrow p = 4$

These hold for every value of λ, so with these values every point of L_1 is mapped onto L_2.

$b = 5, \quad c = 2, \quad m = 2, \quad p = 4$

Do not write outside the box

12 The integral S_n is defined by

$$S_n = \int_0^a x^n \sinh x \, \mathrm{d}x \qquad (n \geq 0)$$

12 (a) Show that for $n \geq 2$

$$S_n = n(n-1)S_{n-2} + a^n \cosh a - na^{n-1} \sinh a$$

[7 marks]

$S_n = \int_0^a x^n \sinh x \, dx$

$u = x^n, \ \frac{dv}{dx} = \sinh x \ \Rightarrow \ \frac{du}{dx} = nx^{n-1}, \ v = \cosh x$

$S_n = [x^n \cosh x]_0^a - \int_0^a nx^{n-1} \cosh x \, dx$

$= a^n \cosh a - 0 - n\int_0^a x^{n-1} \cosh x \, dx$

$u = x^{n-1}, \ \frac{dv}{dx} = \cosh x \ \Rightarrow \ \frac{du}{dx} = (n-1)x^{n-2}, \ v = \sinh x$

$\int_0^a x^{n-1} \cosh x \, dx = [x^{n-1} \sinh x]_0^a - \int_0^a (n-1)x^{n-2} \sinh x \, dx$

$= a^{n-1} \sinh a - 0 - (n-1)S_{n-2}$

$S_n = a^n \cosh a - n(a^{n-1} \sinh a - (n-1)S_{n-2})$

$= a^n \cosh a - na^{n-1} \sinh a + n(n-1)S_{n-2}$

$S_n = n(n-1)S_{n-2} + a^n \cosh a - na^{n-1} \sinh a$ as required

Do not write outside the box

12 (b) Hence show that

$$\int_0^1 x^4 \sinh x \, \mathrm{d}x = \frac{9}{2}\mathrm{e} + \frac{65}{2}\mathrm{e}^{-1} - 24$$

[5 marks]

The required integral is S_4 with $a = 1$:

$S_n = n(n-1)S_{n-2} + \cosh 1 - n \sinh 1$

$S_0 = \int_0^1 \sinh x \, dx = [\cosh x]_0^1 = \cosh 1 - 1$

$S_2 = 2(1)S_0 + \cosh 1 - 2\sinh 1$

$= 2\cosh 1 - 2 + \cosh 1 - 2\sinh 1 = 3\cosh 1 - 2\sinh 1 - 2$

$S_4 = 4(3)S_2 + \cosh 1 - 4\sinh 1$

$= 36\cosh 1 - 24\sinh 1 - 24 + \cosh 1 - 4\sinh 1$

$= 37\cosh 1 - 28\sinh 1 - 24$

$= \frac{37}{2}(e + e^{-1}) - 14(e - e^{-1}) - 24$

$= (\frac{37}{2} - 14)e + (\frac{37}{2} + 14)e^{-1} - 24$

$= \frac{9}{2}e + \frac{65}{2}e^{-1} - 24$

So $\int_0^1 x^4 \sinh x \, dx = \frac{9}{2}e + \frac{65}{2}e^{-1} - 24$ as required

Do not write outside the box

13 (a) Two of the solutions to the equation $\cos 6\theta = 0$ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

Find the other solutions to the equation $\cos 6\theta = 0$ for $0 \leq \theta \leq \pi$

[2 marks]

$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$ $\quad (0 \leq 6\theta \leq 6\pi)$

$\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}$

Other solutions: $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

13 (b) Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

[5 marks]

$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$ (de Moivre)

Let $c = \cos\theta$, $s = \sin\theta$:

$(c + is)^6 = c^6 + 6c^5(is) + 15c^4(is)^2 + 20c^3(is)^3$

$+ 15c^2(is)^4 + 6c(is)^5 + (is)^6$

$= c^6 + 6ic^5s - 15c^4s^2 - 20ic^3s^3 + 15c^2s^4 + 6ics^5 - s^6$

Equating real parts:

$\cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$ and $s^2 = 1 - c^2$:

$= c^6 - 15c^4(1 - c^2) + 15c^2(1 - c^2)^2 - (1 - c^2)^3$

$= c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6$

$- 1 + 3c^2 - 3c^4 + c^6$

$= 32c^6 - 48c^4 + 18c^2 - 1$

$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

Do not write outside the box

13 (c) Use the fact that $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ are solutions to the equation $\cos 6\theta = 0$ to find a factor of $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ in the form $(a\cos^2\theta + b)$, where a and b are integers.

[4 marks]

$\theta = \frac{\pi}{4} \Rightarrow \cos 6\theta = \cos\frac{3\pi}{2} = 0$

so $32c^6 - 48c^4 + 18c^2 - 1 = 0$ when $c = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

By the factor theorem $(\cos\theta - \frac{1}{\sqrt{2}})$ is a factor.

$\theta = \frac{3\pi}{4} \Rightarrow \cos 6\theta = \cos\frac{9\pi}{2} = 0$ and $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

so $(\cos\theta + \frac{1}{\sqrt{2}})$ is also a factor.

$(\cos\theta - \frac{1}{\sqrt{2}})(\cos\theta + \frac{1}{\sqrt{2}}) = \cos^2\theta - \frac{1}{2}$

So $(2\cos^2\theta - 1)$ is a factor $(a = 2, b = -1)$

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13 (d) Hence show that

$$\cos\left(\frac{11\pi}{12}\right) = -\sqrt{\frac{2+\sqrt{3}}{4}}$$

[5 marks]

Let $c = \cos\theta$. Dividing by $(2c^2 - 1)$:

$32c^6 - 48c^4 + 18c^2 - 1 = (2c^2 - 1)(16c^4 - 16c^2 + 1)$

(check: $32c^6 - 32c^4 + 2c^2 - 16c^4 + 16c^2 - 1$ ✓)

$2c^2 - 1 = 0$ gives $\theta = \pi/4$ and $3\pi/4$, so the cosines of the other four solutions from part (a), $\theta = \pi/12, 5\pi/12, 7\pi/12, 11\pi/12$, are the roots of

$16c^4 - 16c^2 + 1 = 0$

$c^2 = \frac{16 \pm \sqrt{(256 - 64)}}{32} = \frac{16 \pm 8\sqrt{3}}{32} = \frac{2 \pm \sqrt{3}}{4}$

$c = \pm\sqrt{\left(\frac{2 \pm \sqrt{3}}{4}\right)}$

$\cos\theta$ is decreasing for $0 \leq \theta \leq \pi$, so of $\pi/12, 5\pi/12, 7\pi/12$ and $11\pi/12$ the angle $11\pi/12$ has the negative cosine of greatest magnitude, so it takes the root $-\sqrt{((2 + \sqrt{3})/4)}$.

$\cos\left(\frac{11\pi}{12}\right) = -\sqrt{\left(\frac{2 + \sqrt{3}}{4}\right)}$

END OF QUESTIONS

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