AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 1

Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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12	
13	
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15	
TOTAL	

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Answer **all** questions in the spaces provided.

1 Find

$$\sum_{r=1}^{20} (r^2 - 2r)$$

Circle your answer.

[1 mark]

2450 2660 5320 43680

$\Sigma r^2 - 2\Sigma r = \frac{1}{6}(20)(21)(41) - 2 \times \frac{1}{2}(20)(21) = 2870 - 420 = 2450$

2 Given that $z = 1 - 3i$ is one root of the equation $z^2 + pz + r = 0$, where p and r are real, find the value of r.

Circle your answer.

[1 mark]

-8 -2 6 10

Other root is $z = 1 + 3i$ (real coefficients)

r = product of roots $= (1 - 3i)(1 + 3i) = 1 + 9 = 10$

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3 The curve C has polar equation

$$r^2 \sin 2\theta = 4$$

Find a Cartesian equation for C.

Circle your answer.

[1 mark]

$y = 2x$ $\quad$ $y = \frac{x}{2}$ $\quad$ $y = \frac{2}{x}$ $\quad$ $y = 4x$

$r^2 \sin 2\theta = 2(r\cos\theta)(r\sin\theta) = 2xy = 4 \Rightarrow y = \frac{2}{x}$

Turn over for the next question

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4 Show that the solutions to the equation

$$3\tanh^2 x - 2\operatorname{sech} x = 2$$

can be expressed in the form

$$x = \pm \ln(a + \sqrt{b})$$

where a and b are integers to be found.

You may use without proof the result $\cosh^{-1} y = \ln(y + \sqrt{y^2 - 1})$

[5 marks]

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$3(1 - \operatorname{sech}^2 x) - 2\operatorname{sech} x = 2$

$3\operatorname{sech}^2 x + 2\operatorname{sech} x - 1 = 0$

$(3\operatorname{sech} x - 1)(\operatorname{sech} x + 1) = 0$

$\operatorname{sech} x = \frac{1}{3}$ or $\operatorname{sech} x = -1$

But $\operatorname{sech} x > 0$ for all x, so $\operatorname{sech} x = \frac{1}{3}$

$\cosh x = 3$

$x = \pm\cosh^{-1} 3$ (cosh x is even, so both signs)

$x = \pm\ln(3 + \sqrt{(3^2 - 1)})$

$x = \pm\ln(3 + \sqrt{8})$

$a = 3, \quad b = 8$

Do not write outside the box

Turn over for the next question

Do not write outside the box

5 The matrix $\mathbf{M}$ is defined by $\mathbf{M} = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Prove by induction that $\mathbf{M}^n = \begin{bmatrix} 3^n & 3^n - 1 & -3^n + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for all integers $n \geq 1$

[5 marks]

$n = 1$: $M^1 = \begin{bmatrix} 3^1 & 3^1 - 1 & -3^1 + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M$

so the result is true for $n = 1$

Assume true for $n = k$: $M^k = \begin{bmatrix} 3^k & 3^k - 1 & -3^k + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$M^{k+1} = MM^k = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 3^k - 1 & -3^k + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 \times 3^k & 3(3^k - 1) + 2 & 3(-3^k + 1) - 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 & -3^{k+1} + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

which is the result for $n = k + 1$, so true for $n = k + 1$

The result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$. Hence, by induction, it is true for all integers $n \geq 1$.

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Turn over for the next question

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6 (a) Show that the equation

$$(2z - z^*)^* = z^2$$

has exactly **four** solutions and state these solutions.

[7 marks]

Let $z = x + iy$ $(x, y$ real$)$, so $z^* = x - iy$

$2z - z^* = 2x + 2iy - x + iy = x + 3iy$

$(2z - z^*)^* = x - 3iy$

$z^2 = x^2 - y^2 + 2xyi$

Real parts: $x = x^2 - y^2$ ①

Imaginary parts: $-3y = 2xy$ ②

② $\Rightarrow$ $y(2x + 3) = 0$ $\Rightarrow$ $y = 0$ or $x = -\frac{3}{2}$

If $y = 0$: ① $\Rightarrow x^2 - x = 0 \Rightarrow x = 0$ or $x = 1$

If $x = -\frac{3}{2}$: ① $\Rightarrow -\frac{3}{2} = \frac{9}{4} - y^2 \Rightarrow y^2 = \frac{15}{4} \Rightarrow y = \pm\frac{\sqrt{15}}{2}$

These are the only possible cases, so the only solutions are

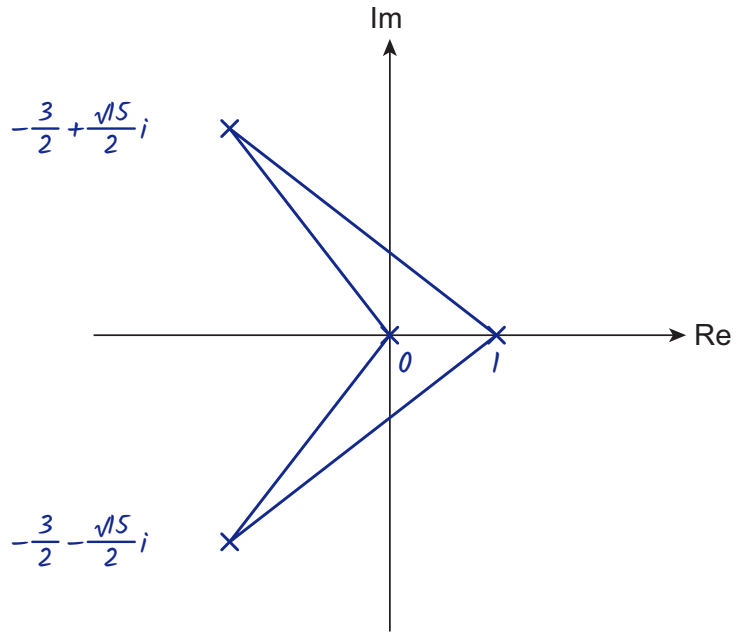
$z = 0$, $z = 1$, $z = -\frac{3}{2} + \frac{\sqrt{15}}{2}i$, $z = -\frac{3}{2} - \frac{\sqrt{15}}{2}i$

Hence the equation has exactly four solutions.

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6 (b) (i) Plot the four solutions to the equation in part **(a)** on the Argand diagram below and join them together to form a quadrilateral with one line of symmetry.

[2 marks]



6 (b) (ii) Show that the area of this quadrilateral is $\frac{\sqrt{15}}{2}$ square units.

[1 mark]

The quadrilateral is symmetrical about the real axis (a dart).

Upper triangle: vertices 0, 1, $-\frac{3}{2}+\frac{\sqrt{15}}{2}i$; base 1 on the real axis, height $\frac{\sqrt{15}}{2}$

Area of upper triangle = $\frac{1}{2}\times 1\times\frac{\sqrt{15}}{2}=\frac{\sqrt{15}}{4}$ = area of lower triangle

Total area = $2\times\frac{\sqrt{15}}{4}=\frac{\sqrt{15}}{2}$ square units, as required

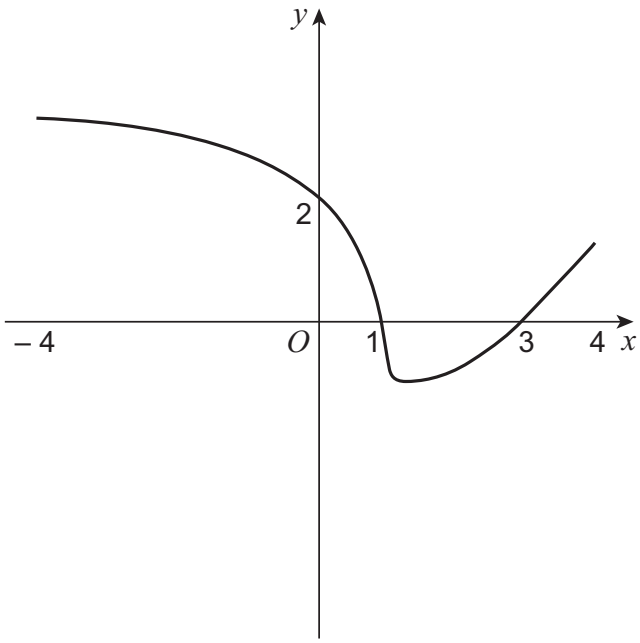
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7 The diagram below shows the graph of $y = f(x)$ $(-4 \leq x \leq 4)$

The graph meets the x-axis at $x = 1$ and $x = 3$

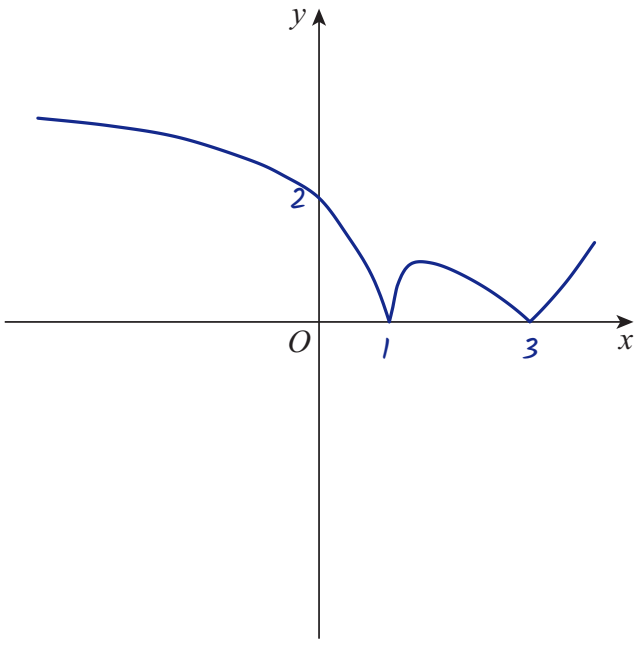
The graph meets the y-axis at $y = 2$



7 (a) Sketch the graph of $y = |f(x)|$ on the axes below.

Show any axis intercepts.

[2 marks]

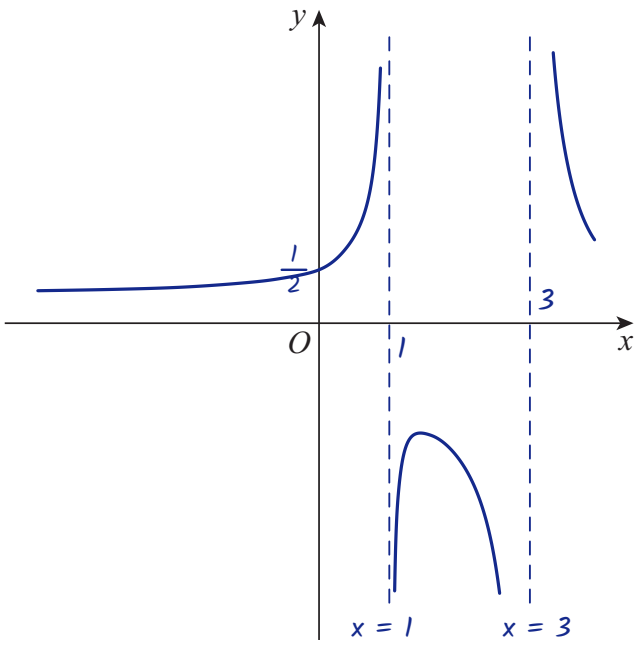


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7 (b) Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$ on the axes below.

Show any axis intercepts and asymptotes.

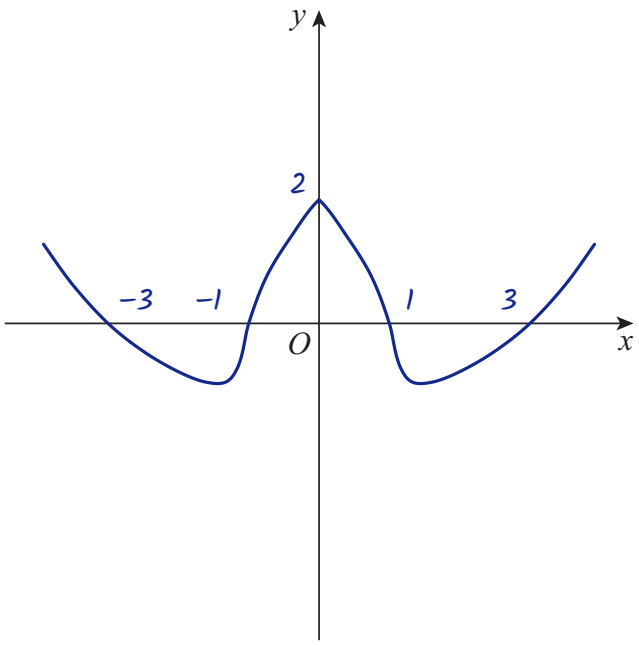
[3 marks]



7 (c) Sketch the graph of $y = \mathrm{f}(|x|)$ on the axes below.

Show any axis intercepts.

[2 marks]



Turn over ▶

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8 A particle of mass 4 kg moves horizontally in a straight line.

At time t seconds the velocity of the particle is v m s^{-1}

The following horizontal forces act on the particle:

- a constant driving force of magnitude 1.8 newtons
- another driving force of magnitude $30\sqrt{t}$ newtons
- a resistive force of magnitude $0.08v^2$ newtons

When $t = 70$, $v = 54$

Use Euler's method with a step length of 0.5 to estimate the velocity of the particle after 71 seconds.

Give your answer to **four** significant figures.

[6 marks]

Newton's second law: $4\frac{dv}{dt} = 1.8 + 30\sqrt{t} - 0.08v^2$

$\frac{dv}{dt} = 0.45 + 7.5\sqrt{t} - 0.02v^2$

Euler: $v_{n+1} = v_n + h\frac{dv}{dt}\Big|_n$, $h = 0.5$

$t = 70$, $v = 54$: $\frac{dv}{dt} = 0.45 + 7.5\sqrt{70} - 0.02 \times 54^2$

$= 0.45 + 62.7495... - 58.32 = 4.87950...$

$v(70.5) \approx 54 + 0.5 \times 4.87950... = 56.43975...$

$t = 70.5$: $\frac{dv}{dt} = 0.45 + 7.5\sqrt{70.5} - 0.02 \times 56.43975...^2$

$= 0.45 + 62.97320... - 63.70890... = -0.28570...$

$v(71) \approx 56.43975... + 0.5 \times (-0.28570...) = 56.29690...$

Velocity after 71 seconds ≈ 56.30 m s^{-1} (4 s.f.)

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Turn over for the next question

Do not write outside the box

9 Use l'Hôpital's rule to show that

$$\lim_{x \to \infty} (xe^{-x}) = 0$$

Fully justify your answer.

[4 marks]

Let $f(x) = x$ and $g(x) = e^x$, so $xe^{-x} = \frac{f(x)}{g(x)}$

$f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, and both are differentiable,

so the limit is of the form $\frac{\infty}{\infty}$ and l'Hôpital's rule can be used.

$f'(x) = 1, \quad g'(x) = e^x$

$\lim_{x\to\infty} (xe^{-x}) = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1}{e^x}$

$= \lim_{x\to\infty} e^{-x} = 0$ since $e^{-x} \to 0$ as $x \to \infty$

$\therefore \lim_{x\to\infty} (xe^{-x}) = 0$ as required

Do not write outside the box

10 Evaluate the improper integral

$$\int_0^8 \ln x \, \mathrm{d}x$$

showing the limiting process.

[6 marks]

ln x is not defined at x = 0, so

$\int_0^8 \ln x\, dx = \lim_{h\to 0} \int_h^8 \ln x\, dx \quad (h > 0)$

By parts: $u = \ln x$, $\frac{dv}{dx} = 1 \Rightarrow \frac{du}{dx} = \frac{1}{x}$, $v = x$

$\int \ln x\, dx = x \ln x - \int x \times \frac{1}{x}\, dx = x \ln x - x$

$\int_h^8 \ln x\, dx = [x \ln x - x]_h^8 = (8 \ln 8 - 8) - (h \ln h - h)$

l'Hôpital: $\lim_{h\to 0} h \ln h = \lim_{h\to 0} \frac{\ln h}{1/h} = \lim_{h\to 0} \frac{1/h}{-1/h^2} = \lim_{h\to 0}(-h) = 0$

so $\lim_{h\to 0} (h \ln h - h) = 0$

$\int_0^8 \ln x\, dx = \lim_{h\to 0} [(8 \ln 8 - 8) - (h \ln h - h)] = 8 \ln 8 - 8$

$= 24 \ln 2 - 8$

Turn over ▶

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11 The line L_1 has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

The line L_2 has equation $\mathbf{r} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

11 (a) Find the acute angle between the lines L_1 and L_2, giving your answer to the nearest 0.1°

[3 marks]

$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -4 + 3 - 1 = -2$, moduli $\sqrt{14}$ and $\sqrt{6}$

$\cos\alpha = \frac{-2}{\sqrt{14}\sqrt{6}} = \frac{-1}{\sqrt{21}} \Rightarrow \alpha = 102.604...°$

Acute angle between the lines $= 180° - 102.604...° = 77.4°$

11 (b) The lines L_1 and L_2 lie in the plane Π_1

11 (b) (i) Find the equation of Π_1, giving your answer in the form $\mathbf{r}.\mathbf{n} = d$

[4 marks]

$\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \times 1 - (-1) \times 1 \\ (-1)(-2) - 2 \times 1 \\ 2 \times 1 - 3 \times (-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Point (2, 2, 3) on L_1: $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 2 + 0 + 6 = 8$

Π_1: $\mathbf{r} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 8$

Do not write outside the box

11 (b) (ii) Hence find the shortest distance of the plane Π_1 from the origin.

[1 mark]

$|n| = \sqrt{(1 + 0 + 4)} = \sqrt{5}$, distance $= \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}$ $(= 3.58)$

11 (c) The points $A(4, -1, -1)$, $B(1, 5, -7)$ and $C(3, 4, -8)$ lie in the plane Π_2

Find the angle between the planes Π_1 and Π_2, giving your answer to the nearest 0.1°

[4 marks]

$$AB = \begin{bmatrix} 1-4 \\ 5+1 \\ -7+1 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \qquad AC = \begin{bmatrix} 3-4 \\ 4+1 \\ -8+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$$

$$n_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 14-10 \\ -2+7 \\ 5-2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

Angle between planes = angle between normals n_1 and n_2:

$\cos\beta = \frac{(1)(4) + (0)(5) + (2)(3)}{\sqrt{5} \times \sqrt{50}} = \frac{10}{\sqrt{250}} = \frac{2}{\sqrt{10}}$

$\beta = 50.768...°$

Angle between the planes = 50.8°

Do not write outside the box

12 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 4 & -2 & p \\ 8 & 5 & -11 \end{bmatrix}$, where p is a constant.

12 (a) Given that $\mathbf{A}$ is a non-singular matrix, find $\mathbf{A}^{-1}$ in terms of p.

State any restrictions on the value of p.

[6 marks]

$\det A = 1(22 - 5p) - 5(-44 - 8p) + 3(20 + 16)$

$= 22 - 5p + 220 + 40p + 108 = 350 + 35p = 35(10 + p)$

Matrix of cofactors:

$$\begin{bmatrix} 22 - 5p & 44 + 8p & 36 \\ 70 & -35 & 35 \\ 5p + 6 & 12 - p & -22 \end{bmatrix}$$

Transpose and divide by det A:

$$A^{-1} = \frac{1}{35(10 + p)} \begin{bmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & 12 - p \\ 36 & 35 & -22 \end{bmatrix}$$

A non-singular requires $\det A \neq 0$, so $p \neq -10$

Do not write outside the box

12 (b) The equations below represent three planes.

$$\begin{aligned} x + 5y + 3z &= 5 \\ 4x - 2y + pz &= 24 \\ 8x + 5y - 11z &= -30 \end{aligned}$$

12 (b) (i) Find, in terms of p, the coordinates of the point of intersection of the three planes.

[4 marks]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 5 \\ 24 \\ -30 \end{bmatrix} = \frac{1}{35(10+p)}\begin{bmatrix} 22-5p & 70 & 5p+6 \\ 44+8p & -35 & 12-p \\ 36 & 35 & -22 \end{bmatrix}\begin{bmatrix} 5 \\ 24 \\ -30 \end{bmatrix}$$

$$= \frac{1}{35(10+p)}\begin{bmatrix} 110-25p+1680-150p-180 \\ 220+40p-840-360+30p \\ 180+840+660 \end{bmatrix} = \frac{1}{35(10+p)}\begin{bmatrix} 1610-175p \\ -980+70p \\ 1680 \end{bmatrix}$$

$$= \frac{1}{10+p}\begin{bmatrix} 46-5p \\ -28+2p \\ 48 \end{bmatrix}$$

Point of intersection: $\left(\frac{46-5p}{10+p}, \frac{2p-28}{10+p}, \frac{48}{10+p}\right)$

Turn over ▶

Do not write outside the box

12 (b) (ii) In the case where $p = 2$, show that the planes are mutually perpendicular.

[4 marks]

$p = 2$: normals $n_1 = (1, 5, 3)$, $n_2 = (4, -2, 2)$, $n_3 = (8, 5, -11)$

$n_1 \cdot n_2 = 4 - 10 + 6 = 0$

$n_1 \cdot n_3 = 8 + 25 - 33 = 0$

$n_2 \cdot n_3 = 32 - 10 - 22 = 0$

All three scalar products are zero, so each pair of normal vectors is perpendicular, hence each pair of planes is perpendicular:

the three planes are mutually perpendicular.

Do not write outside the box

13 The transformation S is represented by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

The transformation T is a translation by the vector $\begin{bmatrix} 0 \\ -5 \end{bmatrix}$

Kamla transforms the graphs of various functions by applying first S, then T.

Leo says that, for some graphs, Kamla would get a different result if she applied first T, then S.

Kamla disagrees.

State who is correct.

Fully justify your answer.

[3 marks]

S then T: $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ y \end{bmatrix}$, $\begin{bmatrix} 3x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3x \\ y-5 \end{bmatrix}$

T then S: $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ y-5 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y-5 \end{bmatrix} = \begin{bmatrix} 3x \\ y-5 \end{bmatrix}$

Both orders map every point (x, y) to $(3x, y-5)$

(S is a stretch parallel to the x-axis, so it only changes x; T only changes y)

so every graph has the same image whichever order is used.

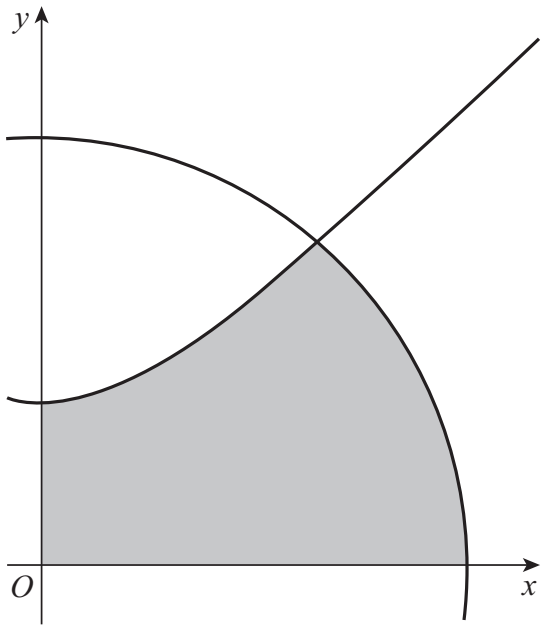
Kamla is correct.

Do not write outside the box

14 The hyperbola H has equation $y^2 - x^2 = 16$

The circle C has equation $x^2 + y^2 = 32$

The diagram below shows part of the graph of H and part of the graph of C.



Show that the shaded region in the first quadrant enclosed by H, C, the x-axis and the y-axis has area

$$\frac{16\pi}{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$$

[12 marks]

At P: $y^2 - x^2 = 16$ and $x^2 + y^2 = 32 \Rightarrow 2x^2 = 16 \Rightarrow x = 2\sqrt{2}$, $y = 2\sqrt{6}$

Split at $x = 2\sqrt{2}$: A_1 under H $(0 \le x \le 2\sqrt{2})$, A_2 under C $(2\sqrt{2} \le x \le 4\sqrt{2})$

$A_1 = \int_0^{2\sqrt{2}} \sqrt{(x^2 + 16)}\, dx$

Let $x = 4\sinh u$: $dx = 4\cosh u\, du$, $\sqrt{(16\sinh^2 u + 16)} = 4\cosh u$

$x = 0 \Rightarrow u = 0$; $x = 2\sqrt{2} \Rightarrow \sinh u = \frac{\sqrt{2}}{2}$; let $\alpha = \sinh^{-1}\frac{\sqrt{2}}{2}$

$A_1 = \int_0^{\alpha} 16\cosh^2 u\, du = 8\int_0^{\alpha} (\cosh 2u + 1)\, du$

$= [4\sinh 2u + 8u]_0^{\alpha} = 4\sinh 2\alpha + 8\alpha$

$\sinh\alpha = \frac{\sqrt{2}}{2}$, $\cosh\alpha = \sqrt{(1 + \frac{1}{2})} = \frac{\sqrt{6}}{2}$

$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha = 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{6}}{2} = \frac{\sqrt{12}}{2} = \sqrt{3}$

$\alpha = \ln(\frac{\sqrt{2}}{2} + \sqrt{(\frac{1}{2} + 1)}) = \ln(\frac{\sqrt{2} + \sqrt{6}}{2})$

$A_1 = 4\sqrt{3} + 8\ln(\frac{\sqrt{2} + \sqrt{6}}{2})$

Do not write outside the box

For A_2: $\tan\theta = \frac{2\sqrt{6}}{2\sqrt{2}} = \sqrt{3}$, so OP makes an angle $\frac{\pi}{3}$ with the x-axis

Sector of C (radius $\sqrt{32}$) from $\theta = 0$ to $\frac{\pi}{3}$: area $= \frac{1}{2} \times 32 \times \frac{\pi}{3} = \frac{16\pi}{3}$

Triangle O, $(2\sqrt{2}, 0)$, P: area $= \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{6} = 2\sqrt{12} = 4\sqrt{3}$

A_2 = sector − triangle $= \frac{16\pi}{3} - 4\sqrt{3}$

Shaded area $= A_1 + A_2 = 4\sqrt{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right) + \frac{16\pi}{3} - 4\sqrt{3}$

$= \frac{16\pi}{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$ as required

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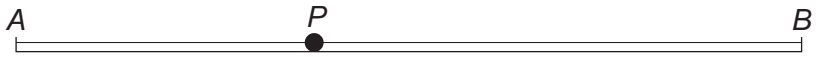
15 **In this question use $g = 9.8\,\text{m s}^{-2}$**

A particle P of mass m is attached to two light elastic strings, AP and BP.

The other ends of the strings, A and B, are attached to fixed points which are 4 metres apart on a rough horizontal surface at the bottom of a container.

The coefficient of friction between P and the surface is 0.68

- When the extension of string AP is e_A metres, the tension in AP is $24me_A$
- When the extension of string BP is e_B metres, the tension in BP is $10me_B$
- The natural length of string AP is 1 metre
- The natural length of string BP is 1.3 metres



15 (a) Show that when $AP = 1.5$ metres, the tension in AP is equal to the tension in BP.

[1 mark]

$AP = 1.5$: $e_A = 0.5$, $T_{AP} = 24m \times 0.5 = 12m$

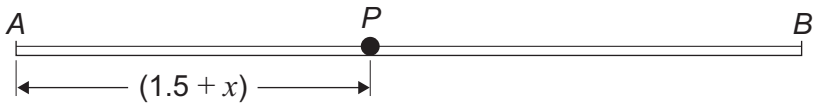
$BP = 4 - 1.5 = 2.5$: $e_B = 2.5 - 1.3 = 1.2$, $T_{BP} = 10m \times 1.2 = 12m$

So the tension in AP equals the tension in BP (both 12m N).

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15 (b) P is held at the point between A and B where $AP = 1.9$ metres, and then released from rest.

At time t seconds after P is released, $AP = (1.5 + x)$ metres.



Show that when P is moving towards A,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 34x = 6.664$$

[3 marks]

$AP = 1.5 + x$: $e_A = 0.5 + x$, $T_{AP} = 24m(0.5 + x)$ towards A

$BP = 2.5 - x$: $e_B = 1.2 - x$, $T_{BP} = 10m(1.2 - x)$ towards B

Moving towards A, friction acts towards B: $F = 0.68 \times 9.8m = 6.664m$

Newton's second law (positive towards B):

$m\frac{d^2x}{dt^2} = 10m(1.2 - x) - 24m(0.5 + x) + 6.664m$

$\frac{d^2x}{dt^2} = 12 - 10x - 12 - 24x + 6.664$

$\frac{d^2x}{dt^2} + 34x = 6.664$ as required

Question 15 continues on the next page

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15 (c) The container is then filled with oil, and P is again released from rest at the point between A and B where $AP = 1.9$ metres.

At time t seconds after P is released, the oil causes a resistive force of magnitude $10mv$ newtons to act on the particle, where $v\,\text{m s}^{-1}$ is the speed of the particle.

Find x in terms of t when P is moving towards A.

[9 marks]

Resistance $10mv$ acts towards B, and $v = -\dot{x}$ as P moves towards A:

$m\ddot{x} = 10m(1.2 - x) - 24m(0.5 + x) + 6.664m - 10m\dot{x}$

$\ddot{x} + 10\dot{x} + 34x = 6.664$

Auxiliary equation: $\lambda^2 + 10\lambda + 34 = 0$

$\lambda = \dfrac{-10 \pm \sqrt{(100 - 136)}}{2} = -5 \pm 3i$

CF: $x = e^{-5t}(A \cos 3t + B \sin 3t)$

PI: $x = k \Rightarrow 34k = 6.664 \Rightarrow k = 0.196$

General solution: $x = e^{-5t}(A \cos 3t + B \sin 3t) + 0.196$

$t = 0$, $AP = 1.9 \Rightarrow x = 0.4$: $0.4 = A + 0.196 \Rightarrow A = 0.204$

$\dot{x} = -5e^{-5t}(A \cos 3t + B \sin 3t) + e^{-5t}(-3A \sin 3t + 3B \cos 3t)$

From rest: $t = 0$, $\dot{x} = 0 \Rightarrow 0 = -5A + 3B \Rightarrow B = \dfrac{5 \times 0.204}{3} = 0.34$

$x = e^{-5t}(0.204 \cos 3t + 0.34 \sin 3t) + 0.196$

END OF QUESTIONS

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