AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 3 Discrete

Thursday 11 June 2020 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)
- You must ensure you have the other optional Question Paper/Answer Book for which you are entered (**either** Mechanics **or** Statistics). You will have 2 hours to complete **both** papers.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

For Examiner's Use	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

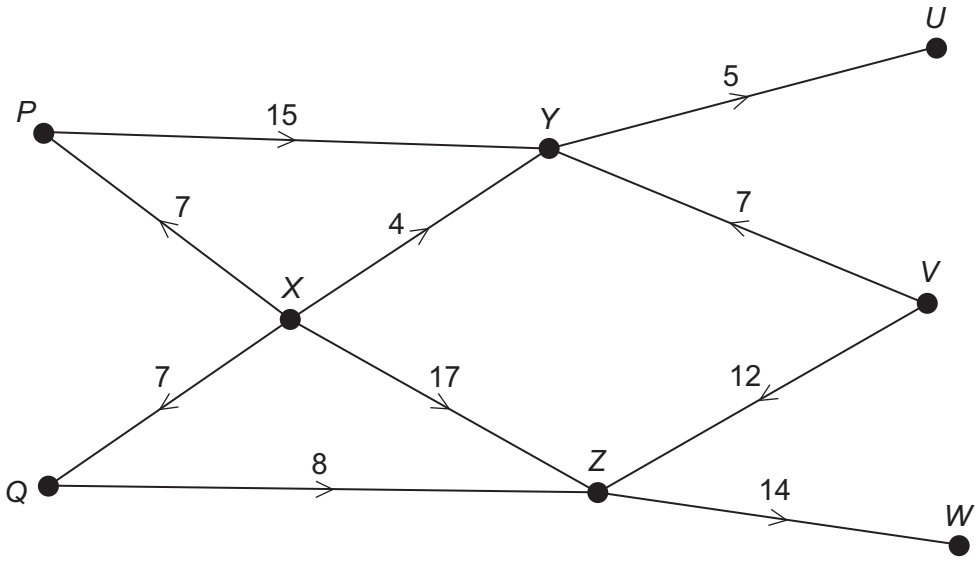
Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 50.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

Do not write outside the box

Answer **all** questions in the spaces provided.

1 The diagram below shows a network of pipes with their capacities.



A supersource and a supersink will be added to the network.

To which nodes should the supersource and supersink be connected?

Tick (✓) **one** box.

[1 mark]

Supersource	Supersink	
P, *Q*	*U*, *V*, *W*	☐
U, *V*, *W*	*P*, *Q*	☐
V, *X*	*U*, *W*	✓
U, *W*	*V*, *X*	☐

Do not write outside the box

2 Which of the following statements is true about the operation of matrix multiplication on the set of all 2×2 real matrices?

Tick (✓) **one** box.

[1 mark]

Matrix multiplication is associative and commutative. ☐

Matrix multiplication is associative but not commutative. ☑

Matrix multiplication is commutative but not associative. ☐

Matrix multiplication is not commutative and not associative. ☐

Turn over for the next question

Do not write outside the box

3 A company is installing an internal telephone network between the offices in a council building. Each office is required to be connected with telephone cables, either directly or indirectly, to every other office in the building.

The lengths of cable, in metres, needed to connect the offices are shown in the table below.

	Education	**Housing**	**Refuse Collection**	**Payroll**	**Social Care**	**Transport**
Education	–	27	13	35	16	24
Housing	27	–	29	30	22	24
Refuse Collection	13	29	–	26	23	17
Payroll	35	30	26	–	20	40
Social Care	16	22	23	20	–	21
Transport	24	24	17	40	21	–

The council wants the total length of cable that is used to be as small as possible.

The cost to the council to install one metre of cable is £8

3 (a) (i) Find the minimum total cost to the council to install the cable required for the internal telephone network.

[4 marks]

Minimum spanning tree (Prim's algorithm, starting from Education):

Education – Refuse Collection 13

Education – Social Care 16

Refuse Collection – Transport 17

Social Care – Payroll 20

Social Care – Housing 22

Total length = 13 + 16 + 17 + 20 + 22 = 88 m

Minimum total cost = 8 × 88

= £704

Do not write outside the box

3 (a) (ii) Suggest a reason why the total cost to the council for installing the internal telephone network is likely to be different from your answer to part **(a)(i)**.

[1 mark]

The cost is likely to be higher than £704, as the council may also need to buy telephones (and pay for labour to fit the cable) for the internal network.

3 (b) Before the company starts installing the cable, it is told that the Education office cannot be connected directly to the Transport office due to issues with the building.

Explain the possible impact of this on your answer to part **(a)(i)**.

[2 marks]

The Education office is not connected directly to the Transport office in the minimum spanning tree (Education – Transport, 24 m, is not used). So the minimum total length of cable required is still 88 m, and hence the cost of installing the cable does not change (£704).

Turn over for the next question

Do not write outside the box

4 Joe, a courier, is required to deliver parcels to six different locations, *A*, *B*, *C*, *D*, *E* and *F*.

Joe needs to start and finish his journey at the depot.

The distances, in miles, between the depot and the six different locations are shown in the table below.

	Depot	***A***	***B***	***C***	***D***	***E***	***F***
Depot	–	18	17	15	16	19	30
A	18	–	29	20	25	35	21
B	17	29	–	26	30	16	14
C	15	20	26	–	28	31	27
D	16	25	30	28	–	34	24
E	19	35	16	31	34	–	28
F	30	21	14	27	24	28	–

The minimum total distance that Joe can travel in order to make all six deliveries, starting and finishing at the depot, is L miles.

4 (a) Using the nearest neighbour algorithm starting from the depot, find an upper bound for L.

[2 marks]

Nearest neighbour from the depot:

Depot → C → A → F → B → E → D → Depot

15 + 20 + 21 + 14 + 16 + 34 + 16

= 136

Upper bound for L = 136 miles

Do not write outside the box

4 (b) By deleting the depot, find a lower bound for L.

[3 marks]

Minimum spanning tree of A, B, C, D, E, F (Kruskal's algorithm):

B – F 14, B – E 16, A – C 20, A – F 21, D – F 24

Total weight of MST = 14 + 16 + 20 + 21 + 24 = 95

Two least weight arcs joined to the depot:

Depot – C 15 and Depot – D 16, 15 + 16 = 31

Lower bound = 95 + 31

= 126

Lower bound for L = 126 miles

4 (c) Joe starts from the depot, delivers parcels to all six different locations and arrives back at the depot, covering 134 miles in the process.

Joe claims that this is the minimum total distance that is possible for the journey.

Comment on Joe's claim.

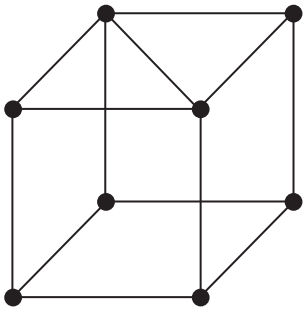
[1 mark]

126 ≤ L ≤ 136, and 134 lies between the bounds. Joe's claim may be correct, but without further work we can only say that 134 miles is an upper bound for L; it may be possible to improve on it.

Turn over for the next question

Do not write outside the box

5 The planar graph P is shown below.



5 (a) Determine the number of faces of P.

[2 marks]

P is a connected planar graph with

$v = 8$ vertices and $e = 13$ edges

Euler's formula: $v - e + f = 2$

$8 - 13 + f = 2$

$f = 7$

Therefore P has 7 faces.

5 (b) Akwasi claims that P is semi-Eulerian as it is connected and it has exactly two vertices with even degree.

Comment on the validity of Akwasi's claim.

[2 marks]

For a connected graph to be semi-Eulerian it must have exactly two vertices of odd degree (all other vertices even).

In P, two vertices have degree 4 and the other six vertices have degree 3, so P has six vertices of odd degree.

Therefore P is not semi-Eulerian and Akwasi's claim is incorrect.

Do not write outside the box

Turn over for the next question

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

6 The group $(G, \blacktriangle)$ has the elements e, r, r^2, q, qr and qr^2, where $r^2 = r \blacktriangle r$, $qr = q \blacktriangle r$, $qr^2 = q \blacktriangle r^2$ and e is the identity element of G.

The elements q and r have the following properties:

$$r \blacktriangle r \blacktriangle r = e$$

$$q \blacktriangle q = e$$

$$r^2 \blacktriangle q = q \blacktriangle r$$

6 (a) (i) State the order of G.

[1 mark]

6

6 (a) (ii) Prove that the inverse of qr is qr.

[3 marks]

Using $r^2 \blacktriangle q = q \blacktriangle r$, so $qr = r^2q$:

$qr \blacktriangle qr = (q \blacktriangle r) \blacktriangle (r^2 \blacktriangle q)$

$= q \blacktriangle (r \blacktriangle r \blacktriangle r) \blacktriangle q$ (associativity)

$= q \blacktriangle e \blacktriangle q$ ($r \blacktriangle r \blacktriangle r = e$)

$= q \blacktriangle q$ (e is the identity)

$= e$ ($q \blacktriangle q = e$)

Since $qr \blacktriangle qr = e$, the identity element of G,

the inverse of qr is qr.

Do not write outside the box

6 (b) Complete the Cayley table for elements of G.

[3 marks]

▲	e	r	r^2	q	qr	qr^2
e	e	r	r^2	q	qr	qr^2
r	r	r^2	e	qr^2	q	qr
r^2	r^2	e	r	qr	qr^2	q
q	q	qr	qr^2	e	r	r^2
qr	qr	qr^2	q	r^2	e	r
qr^2	qr^2	q	qr	r	r^2	e

6 (c) State the name of a group which is isomorphic to G.

[1 mark]

G is isomorphic to the symmetry group of an equilateral triangle (the dihedral group D_3).

Turn over ▶

Do not write outside the box

7 An engineering company makes brake kits and clutch kits to sell to motorsport teams.

The table below summarises the time taken and costs involved in making the two different types of kit.

Type of kit	Time taken to make a kit (hours)	Cost to engineering company per kit (£)	Profit to engineering company per kit (£)
Brake kit	5	500	2000
Clutch kit	3	200	1000

The workers at the engineering company have a combined 2500 hours available to make the kits every month.

The engineering company has £200 000 available to cover the costs of making the kits every month.

To meet the minimum demands of the motorsport teams, the engineering company must make at least 100 of each type of kit every month.

7 (a) Using a graphical method on the grid opposite, find the number of each type of kit that the engineering company should make every month, in order to maximise its total monthly profit.

Show clearly how you obtain your answer.

[8 marks]

Let x = number of brake kits made each month

Let y = number of clutch kits made each month

Time: $5x + 3y \leq 2500$

Cost: $500x + 200y \leq 200\,000 \Rightarrow 5x + 2y \leq 2000$

Demand: $x \geq 100$, $y \geq 100$

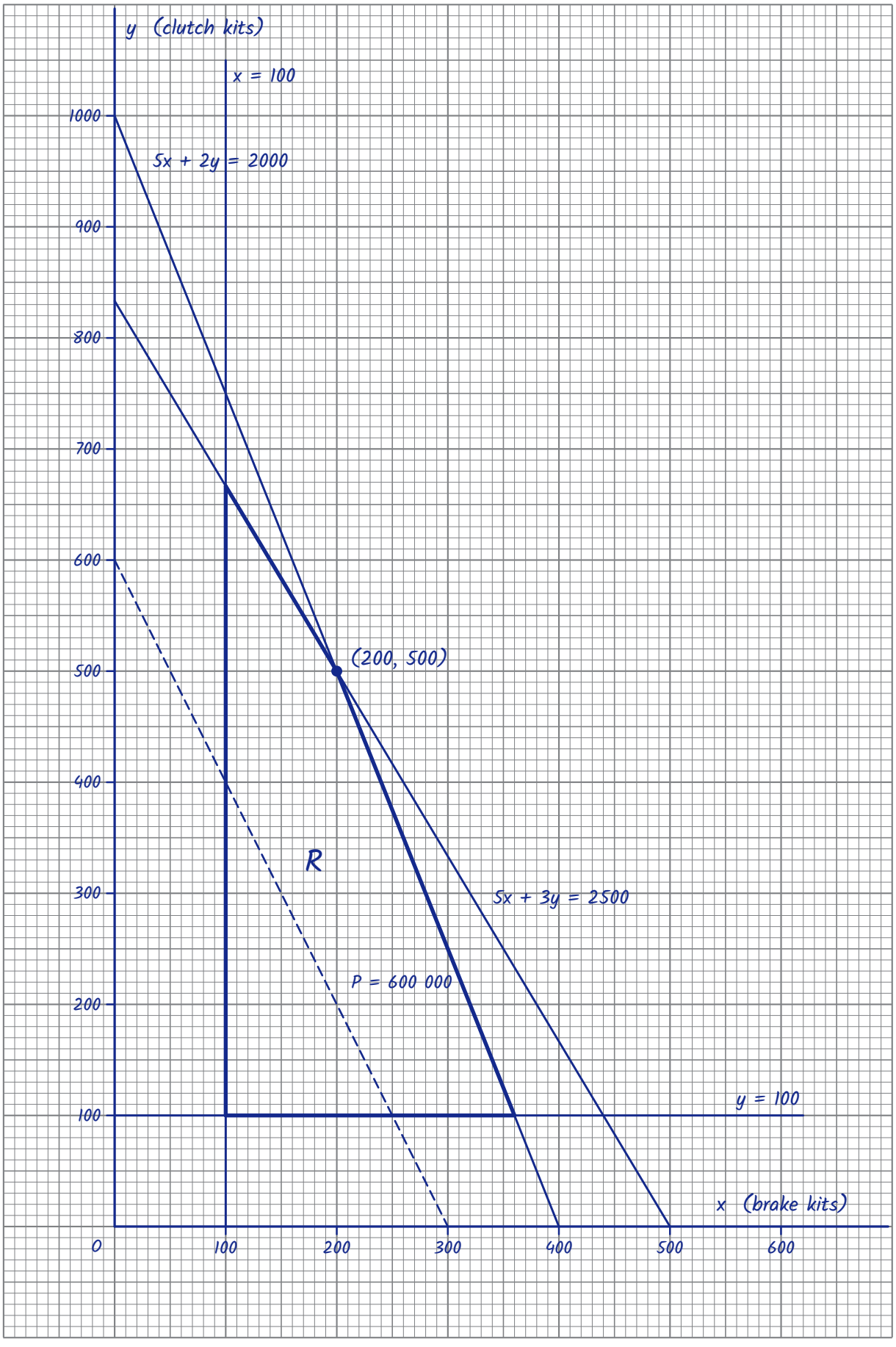
Maximise $P = 2000x + 1000y$

Graph drawn on the grid opposite: feasible region labelled R.

Objective line $2000x + 1000y = 600\,000$ $(2x + y = 600)$ drawn dashed.

(continued on the next page)

Do not write outside the box



Do not write outside the box

Q7 (a) continued

Vertices of the feasible region R:

$5x + 3y = 2500$ and $5x + 2y = 2000 \Rightarrow y = 500,\ x = 200$

(100, 100): $P = 200\,000 + 100\,000 = 300\,000$

(360, 100): $P = 720\,000 + 100\,000 = 820\,000$

(200, 500): $P = 400\,000 + 500\,000 = 900\,000$

(100, 666⅔): $P = 200\,000 + 666\,667 \approx 866\,667$

Sliding the objective line $2x + y = k$ away from the origin, the last point of R that it reaches is the vertex (200, 500).

Maximum $P = 2000(200) + 1000(500) = £900\,000$

The company should make 200 brake kits and 500 clutch kits every month (maximum monthly profit £900 000).

Do not write outside the box

7 (b) Give a reason why the engineering company may not be able to make the number of each kit that you found in part **(a)**.

[1 mark]

Fewer than 2500 working hours may actually be available, for example if some workers are ill or on holiday, so the company may not have time to make 200 brake kits and 500 clutch kits.

7 (c) During one particular month the engineering company removes the need to make at least 100 of each type of kit.

Explain whether or not this has an effect on your answer to part **(a)**.

[2 marks]

The optimal vertex (200, 500) does not lie on $x = 100$ or $y = 100$, so neither of the 'at least 100' constraints is active at the optimum. Removing them only adds the vertices $(0, 833\frac{1}{3})$, $P \approx 833\,333$ and (400, 0), $P = 800\,000$, which both give less than £900 000. So there is no effect: the company should still make 200 brake kits and 500 clutch kits, with maximum profit £900 000.

Turn over for the next question

Do not write outside the box

8 Daryl and Clare play a zero-sum game.

The game is represented by the following pay-off matrix for Daryl.

Daryl \ Clare	**Strategy**	**X**	**Y**	**Z**
	A	−1	3	4
	B	3	2	1

8 (a) (i) Show that the game does not have a stable solution.

[3 marks]

Row minima: A: −1, B: 1 max(row minima) = 1

Column maxima: X: 3, Y: 3, Z: 4 min(column maxima) = 3

max(row minima) = 1 ≠ 3 = min(column maxima)

so there is no saddle point and the game does not have a stable solution.

8 (a) (ii) The value of the game for Daryl is V.

Using your answer to part **(a)(i)**, state the interval in which V lies.

[1 mark]

$1 < V < 3$

Do not write outside the box

8 (b) Daryl plays strategy **A** with probability $\sin^2\theta$ and strategy **B** with probability $\cos^2\theta$, where $0° \leq \theta \leq 90°$

8 (b) (i) Show that the expected gain for Daryl when Clare plays strategy **X** is

$$3 - 4\sin^2\theta$$

[2 marks]

Expected gain (X) = $-1 \times \sin^2\theta + 3 \times \cos^2\theta$

$= -\sin^2\theta + 3(1 - \sin^2\theta)$ $\quad (\sin^2\theta + \cos^2\theta = 1)$

$= -\sin^2\theta + 3 - 3\sin^2\theta$

$= 3 - 4\sin^2\theta$

8 (b) (ii) Find an expression in terms of $\sin^2\theta$ for the expected gain for Daryl when Clare plays strategy **Y**.

[1 mark]

Expected gain (Y) = $3\sin^2\theta + 2\cos^2\theta$

$= 3\sin^2\theta + 2(1 - \sin^2\theta)$

$= 2 + \sin^2\theta$

8 (b) (iii) Find an expression in terms of $\sin^2\theta$ for the expected gain for Daryl when Clare plays strategy **Z**.

[1 mark]

Expected gain (Z) = $4\sin^2\theta + 1 \times \cos^2\theta$

$= 4\sin^2\theta + 1 - \sin^2\theta$

$= 1 + 3\sin^2\theta$

Question 8 continues on the next page

Do not write outside the box

8 (c) Daryl plays her optimal mixed strategy.

Using your answers to part **(b)**, find the value of θ, giving your answer to three significant figures.

[4 marks]

Let $p = \sin^2\theta$, $0 \leq p \leq 1$. Gains:

X: $3 - 4p$, Y: $2 + p$, Z: $1 + 3p$

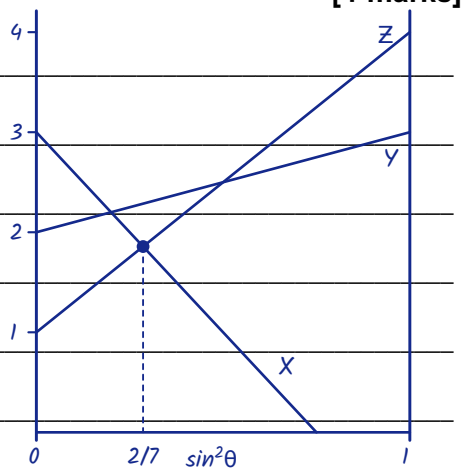
Daryl maximises her minimum gain:

the highest point of the lower

boundary is where X meets Z.

(X = Y at $p = 0.2$ gives 2.2, but there

Z = 1.6 is lower, so not optimal.)



$3 - 4\sin^2\theta = 1 + 3\sin^2\theta$

$2 = 7\sin^2\theta$

$\sin^2\theta = 2/7$ (check: gain = 13/7, Y = 16/7 > 13/7)

$\sin\theta = \sqrt{(2/7)} = 0.534522...$

$\theta = \sin^{-1}(0.534522...) = 32.311...°$

$\theta = 32.3°$

END OF QUESTIONS

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Copyright information

For confidentiality purposes, all acknowledgements of third-party copyright material are published in a separate booklet. This booklet is published after each live examination series and is available for free download from www.aqa.org.uk.

Permission to reproduce all copyright material has been applied for. In some cases, efforts to contact copyright-holders may have been unsuccessful and AQA will be happy to rectify any omissions of acknowledgements. If you have any queries please contact the Copyright Team.

Copyright © 2020 AQA and its licensors. All rights reserved.