AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 2

Thursday 4 June 2020 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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13	
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15	
TOTAL	

Do not write outside the box

Answer **all** questions in the spaces provided.

1 Three of the four expressions below are equivalent to each other.

Which of the four expressions is **not** equivalent to any of the others?

Circle your answer.

[1 mark]

$\mathbf{a} \times (\mathbf{a} + \mathbf{b})$ $(\mathbf{a} + \mathbf{b}) \times \mathbf{b}$ $(\mathbf{a} - \mathbf{b}) \times \mathbf{b}$ $\mathbf{a} \times (\mathbf{a} - \mathbf{b})$

2 Given that $\arg(a + b\mathrm{i}) = \varphi$, where a and b are positive real numbers and $0 < \varphi < \frac{\pi}{2}$, three of the following four statements are correct.

Which statement is **not** correct?

Tick (✓) **one** box.

[1 mark]

$\arg(-a - b\mathrm{i}) = \pi - \varphi$ ✓

$\arg(a - b\mathrm{i}) = -\varphi$

$\arg(b + a\mathrm{i}) = \frac{\pi}{2} - \varphi$

$\arg(b - a\mathrm{i}) = \varphi - \frac{\pi}{2}$

Do not write outside the box

3 Find the gradient of the tangent to the curve

$$y = \sin^{-1} x$$

at the point where $x = \frac{1}{5}$

Circle your answer.

[1 mark]

$\frac{5\sqrt{6}}{12}$ $\quad$ $\frac{2\sqrt{6}}{5}$ $\quad$ $\frac{4\sqrt{3}}{25}$ $\quad$ $\frac{25}{24}$

Turn over for the next question

Do not write outside the box

4 The matrices **A** and **B** are defined as follows:

$$\mathbf{A} = \begin{bmatrix} x+1 & 2 \\ x+2 & -3 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} x-4 & x-2 \\ 0 & -2 \end{bmatrix}$$

Show that there is a value of x for which $\mathbf{AB} = k\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix and k is an integer to be found.

[3 marks]

$AB = \begin{bmatrix} x+1 & 2 \\ x+2 & -3 \end{bmatrix}\begin{bmatrix} x-4 & x-2 \\ 0 & -2 \end{bmatrix}$

$AB = \begin{bmatrix} (x+1)(x-4) & (x+1)(x-2)-4 \\ (x+2)(x-4) & (x+2)(x-2)+6 \end{bmatrix}$

$= \begin{bmatrix} x^2-3x-4 & (x-3)(x+2) \\ (x-4)(x+2) & x^2+2 \end{bmatrix}$

For $AB = kI$ the off-diagonal elements must be zero:

$(x-3)(x+2) = 0$ and $(x-4)(x+2) = 0 \Rightarrow x = -2$

When $x = -2$: $x^2-3x-4 = 4+6-4 = 6$, $x^2+2 = 4+2 = 6$

$AB = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 6I$

So $AB = kI$ when $x = -2$, with $k = 6$

Do not write outside the box

5 Solve the inequality

$$\frac{2x+3}{x-1} \leq x+5$$

[5 marks]

Multiply both sides by $(x-1)^2 > 0$ $(x \neq 1)$:

$(x-1)(2x+3) \leq (x-1)^2(x+5)$

$0 \leq (x-1)^2(x+5) - (x-1)(2x+3)$

$0 \leq (x-1)\{(x-1)(x+5) - (2x+3)\}$

$0 \leq (x-1)(x^2+4x-5-2x-3)$

$0 \leq (x-1)(x^2+2x-8)$

$0 \leq (x-1)(x+4)(x-2)$

Critical values: $x = -4$, $x = 1$, $x = 2$

The cubic $(x+4)(x-1)(x-2)$ has positive leading coefficient,

so it is ≥ 0 for $-4 \leq x \leq 1$ or $x \geq 2$

But $x \neq 1$ since the original expression is undefined there.

Solution: $-4 \leq x < 1$ or $x \geq 2$

Do not write outside the box

6 Find the sum of all the integers from 1 to 999 inclusive that are not square or cube numbers.

[5 marks]

Sum of integers 1 to 999: $\Sigma r\ (r = 1 \text{ to } 999) = \frac{999 \times 1000}{2} = 499500$

Squares ≤ 999: 1^2, ..., 31^2 ($31^2 = 961$, $32^2 = 1024$), so 31 squares

Cubes ≤ 999: 1^3, ..., 9^3 ($9^3 = 729$, $10^3 = 1000$), so 9 cubes

$\Sigma r^2\ (r = 1 \text{ to } 31) = \frac{31 \times 32 \times 63}{6} = 10416$

$\Sigma r^3\ (r = 1 \text{ to } 9) = \frac{9^2 \times 10^2}{4} = 2025$

Numbers that are both squares and cubes are sixth powers: 1, 64, 729.

These have been subtracted twice, so add them back once:

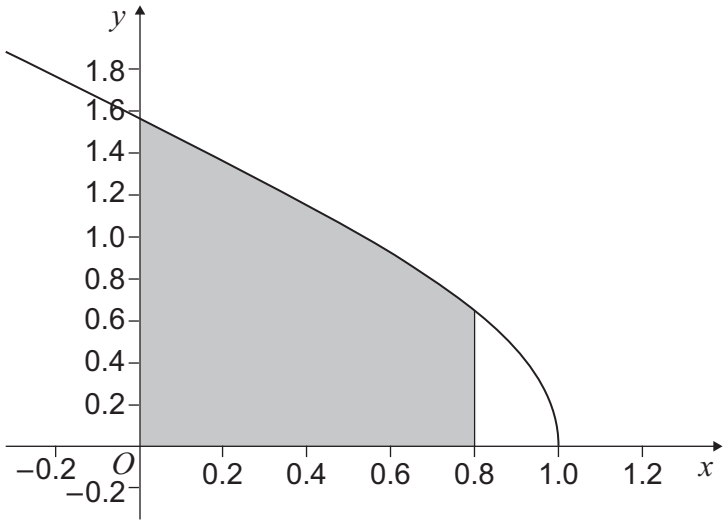
$1 + 64 + 729 = 794$

Required sum = $499500 - 10416 - 2025 + 794$

$= 487853$

Do not write outside the box

7 The diagram shows part of the graph of $y = \cos^{-1} x$



The finite region enclosed by the graph of $y = \cos^{-1} x$, the y-axis, the x-axis and the line $x = 0.8$ is rotated by 2π radians about the x-axis.

Use Simpson's rule with five ordinates to estimate the volume of the solid formed. Give your answer to four decimal places.

[5 marks]

$V = \pi \int_0^{0.8} y^2 \, dx, \quad y = \cos^{-1}x, \quad h = 0.2$

x:	0	0.2	0.4	0.6	0.8
y^2:	2.46740	1.87536	1.34393	0.85988	0.41409

$V \approx \pi \times \frac{0.2}{3} \times \{2.46740 + 0.41409 + 4(1.87536 + 0.85988) + 2(1.34393)\}$

$= \pi \times \frac{0.2}{3} \times 16.51030...$

$= 3.457910...$

$V = 3.4579$ (4 d.p.)

Turn over ▶

Do not write outside the box

8 (a) Factorise $\begin{vmatrix} 2a+b+x & x+b & x^2+b^2 \\ 0 & a & -a^2 \\ a+b & b & b^2 \end{vmatrix}$ as fully as possible.

[6 marks]

$C1 \rightarrow C1 - C2$:

$$\Delta = \begin{vmatrix} 2a & x+b & x^2+b^2 \\ -a & a & -a^2 \\ a & b & b^2 \end{vmatrix} = a \begin{vmatrix} 2 & x+b & x^2+b^2 \\ -1 & a & -a^2 \\ 1 & b & b^2 \end{vmatrix}$$

$R1 \rightarrow R1 - 2R3$:

$$\Delta = a \begin{vmatrix} 0 & x-b & x^2-b^2 \\ -1 & a & -a^2 \\ 1 & b & b^2 \end{vmatrix} = a(x-b) \begin{vmatrix} 0 & 1 & x+b \\ -1 & a & -a^2 \\ 1 & b & b^2 \end{vmatrix}$$

$R2 \rightarrow R2 + R3$:

$$\Delta = a(x-b) \begin{vmatrix} 0 & 1 & x+b \\ 0 & a+b & b^2-a^2 \\ 1 & b & b^2 \end{vmatrix} = a(x-b)(a+b) \begin{vmatrix} 0 & 1 & x+b \\ 0 & 1 & b-a \\ 1 & b & b^2 \end{vmatrix}$$

Expand down column 1:

$\Delta = a(x-b)(a+b) \times 1 \times [1 \times (b-a) - 1 \times (x+b)]$

$= a(x-b)(a+b)(-a-x)$

$\Delta = -a(a+b)(a+x)(x-b)$

Do not write outside the box

8 (b) The matrix **M** is defined by

$$\mathbf{M} = \begin{bmatrix} 13+x & x+3 & x^2+9 \\ 0 & 5 & -25 \\ 8 & 3 & 9 \end{bmatrix}$$

Under the transformation represented by **M**, a solid of volume 0.625 m^3 becomes a solid of volume 300 m^3

Use your answer to part **(a)** to find the possible values of x.

[3 marks]

Compare M with the determinant in (a): $a = 5$, $b = 3$

Volume scale factor = $|\det M| = \frac{300}{0.625} = 480$, so $\det M = \pm 480$

$\det M = -5(5 + 3)(5 + x)(x - 3) = -40(x + 5)(x - 3)$

$-40(x + 5)(x - 3) = \pm 480 \Rightarrow (x + 5)(x - 3) = \pm 12$

$x^2 + 2x - 15 = -12 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow (x + 3)(x - 1) = 0$

$x^2 + 2x - 15 = 12 \Rightarrow x^2 + 2x - 27 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{112}}{2} = -1 \pm 2\sqrt{7}$

$x = 1, \quad x = -3, \quad x = -1 + 2\sqrt{7}, \quad x = -1 - 2\sqrt{7}$

Do not write outside the box

9 The matrix $\mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are positive real numbers,

and $\mathbf{C}^2 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Use $\mathbf{C}$ to show that $\cos\frac{\pi}{12}$ can be written in the form $\frac{\sqrt{\sqrt{m}+n}}{2}$, where m and n are integers.

[7 marks]

$C^2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2-b^2 & -2ab \\ 2ab & a^2-b^2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

So $a^2 - b^2 = \frac{\sqrt{3}}{2}$ and $2ab = \frac{1}{2} \Rightarrow b = \frac{1}{4a}$

$a^2 - \frac{1}{16a^2} = \frac{\sqrt{3}}{2} \Rightarrow 16a^4 - 8\sqrt{3}a^2 - 1 = 0$

$a^2 = \frac{8\sqrt{3} \pm \sqrt{(192+64)}}{32} = \frac{8\sqrt{3} \pm 16}{32} = \frac{\sqrt{3} \pm 2}{4}$

$a^2 > 0$ so $a^2 = \frac{\sqrt{3}+2}{4}$, and $a > 0$ so $a = \frac{\sqrt{(\sqrt{3}+2)}}{2}$

$C^2 = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$, a rotation of $\frac{\pi}{6}$ about O

So det $C^2 = 1$, i.e. $(a^2+b^2)^2 = 1$, giving $a^2 + b^2 = 1$. Hence C is a rotation about O through θ, where $\cos\theta = a > 0$ and $\sin\theta = b > 0$, so θ is acute.

Applying C twice gives a rotation through 2θ, so $2\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{12}$ (as $0 < 2\theta < \pi$)

Hence $\cos\frac{\pi}{12} = a = \frac{\sqrt{(\sqrt{3}+2)}}{2}$, which is of the form $\frac{\sqrt{(\sqrt{m}+n)}}{2}$

with $m = 3$, $n = 2$

Do not write outside the box

Turn over for the next question

Do not write outside the box

10 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 0 \qquad u_{n+1} = \frac{5}{6 - u_n}$$

Prove by induction that, for all integers $n \geq 1$,

$$u_n = \frac{5^n - 5}{5^n - 1}$$

[6 marks]

$n = 1$: $\frac{5^1 - 5}{5^1 - 1} = \frac{0}{4} = 0 = u_1$, so the result is true for $n = 1$

Assume true for $n = k$: $u_k = \frac{5^k - 5}{5^k - 1}$

Then $u_{k+1} = \frac{5}{6 - u_k} = 5 \div \left(6 - \frac{5^k - 5}{5^k - 1}\right)$

$6 - \frac{5^k - 5}{5^k - 1} = \frac{6(5^k - 1) - (5^k - 5)}{5^k - 1} = \frac{6 \times 5^k - 6 - 5^k + 5}{5^k - 1}$

$= \frac{5 \times 5^k - 1}{5^k - 1} = \frac{5^{k+1} - 1}{5^k - 1}$

$u_{k+1} = 5 \times \frac{5^k - 1}{5^{k+1} - 1} = \frac{5^{k+1} - 5}{5^{k+1} - 1}$

which is the formula with $n = k + 1$, so the result is true for $n = k + 1$

The formula is true for $n = 1$, and if it is true for $n = k$ then it is also true for $n = k + 1$. Hence, by induction,

$u_n = \frac{5^n - 5}{5^n - 1}$ for all integers $n \geq 1$

Do not write outside the box

Turn over for the next question

Do not write outside the box

11 (a) Starting from the series given in the formulae booklet, show that the general term of the Maclaurin series for

$$\frac{\sin x}{x} - \cos x$$

is

$$(-1)^{r+1}\frac{2r}{(2r+1)!}x^{2r}$$

[4 marks]

From the booklet: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots$

so $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots + \frac{(-1)^r x^{2r}}{(2r+1)!} + \ldots$

and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots$

General term of $\frac{\sin x}{x} - \cos x$:

$\frac{(-1)^r x^{2r}}{(2r+1)!} - \frac{(-1)^r x^{2r}}{(2r)!} = (-1)^r x^{2r}\left(\frac{1}{(2r+1)!} - \frac{1}{(2r)!}\right)$

$\frac{1}{(2r+1)!} - \frac{1}{(2r)!} = \frac{1-(2r+1)}{(2r+1)!} = \frac{-2r}{(2r+1)!}$

So the general term is $(-1)^r x^{2r} \times \frac{-2r}{(2r+1)!}$

$= (-1)^{r+1}\frac{2r}{(2r+1)!}x^{2r}$

Do not write outside the box

11 (b) Show that

$$\lim_{x \to 0} \left[\frac{\frac{\sin x}{x} - \cos x}{1 - \cos x} \right] = \frac{2}{3}$$

[4 marks]

Using (a): $\frac{\sin x}{x} - \cos x = \frac{2}{3!}x^2 - \frac{4}{5!}x^4 + \ldots = \frac{x^2}{3} - \frac{x^4}{30} + \ldots$

$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots = \frac{x^2}{2} - \frac{x^4}{24} + \ldots$

Both series start with an x^2 term and all other terms have higher powers of x, so divide numerator and denominator by x^2:

$\frac{\sin x/x - \cos x}{1 - \cos x} = \frac{x^2/3 - x^4/30 + \ldots}{x^2/2 - x^4/24 + \ldots} = \frac{1/3 - x^2/30 + \ldots}{1/2 - x^2/24 + \ldots}$

As $x \to 0$, all terms in x^2 and higher powers $\to 0$, so

$\lim (x \to 0) \left[\frac{\sin x/x - \cos x}{1 - \cos x} \right] = \frac{1/3}{1/2} = \frac{2}{3}$

Do not write outside the box

12 (a) Given that $I = \int_a^b e^{2t} \sin t \, dt$, show that

$$I = \left[q e^{2t} \sin t + r e^{2t} \cos t \right]_a^b$$

where q and r are rational numbers to be found.

[6 marks]

By parts with $u = \sin t$, $dv/dt = e^{2t}$:

$I = \left[\frac{1}{2} e^{2t} \sin t\right]_a^b - \frac{1}{2} \int_a^b e^{2t} \cos t \, dt$

By parts again with $u = \cos t$, $dv/dt = e^{2t}$:

$\int_a^b e^{2t} \cos t \, dt = \left[\frac{1}{2} e^{2t} \cos t\right]_a^b + \frac{1}{2} \int_a^b e^{2t} \sin t \, dt$

$= \left[\frac{1}{2} e^{2t} \cos t\right]_a^b + \frac{1}{2} I$

So $I = \left[\frac{1}{2} e^{2t} \sin t\right]_a^b - \frac{1}{2}\left\{\left[\frac{1}{2} e^{2t} \cos t\right]_a^b + \frac{1}{2} I\right\}$

$I = \left[\frac{1}{2} e^{2t} \sin t - \frac{1}{4} e^{2t} \cos t\right]_a^b - \frac{1}{4} I$

$\frac{5}{4} I = \left[\frac{1}{2} e^{2t} \sin t - \frac{1}{4} e^{2t} \cos t\right]_a^b$

$I = \left[\frac{2}{5} e^{2t} \sin t - \frac{1}{5} e^{2t} \cos t\right]_a^b$

$q = \frac{2}{5}, \quad r = -\frac{1}{5}$

Do not write outside the box

12 (b) A small object is initially at rest. The subsequent motion of the object is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + v = 5\mathrm{e}^t \sin t$$

where v is the velocity at time t.

Find the speed of the object when $t = 2\pi$, giving your answer in exact form.

[6 marks]

Integrating factor: $e^{\int 1\,dt} = e^t$

$e^t \frac{dv}{dt} + e^t v = 5e^{2t} \sin t \quad \Rightarrow \quad \frac{d}{dt}(ve^t) = 5e^{2t} \sin t$

Using (a): $ve^t = 5\left(\frac{2}{5}e^{2t} \sin t - \frac{1}{5}e^{2t} \cos t\right) + c$

$= 2e^{2t} \sin t - e^{2t} \cos t + c$

Initially at rest: $t = 0$, $v = 0$: $\quad 0 = 0 - 1 + c \quad \Rightarrow \quad c = 1$

$ve^t = 2e^{2t} \sin t - e^{2t} \cos t + 1$

$t = 2\pi$: $\quad ve^{2\pi} = 2e^{4\pi} \sin 2\pi - e^{4\pi} \cos 2\pi + 1 = -e^{4\pi} + 1$

$v = -e^{2\pi} + e^{-2\pi} \quad (< 0)$

Speed $= |v| = e^{2\pi} - e^{-2\pi}$

Turn over ▶

Do not write outside the box

13 Charlotte is trying to solve this mathematical problem:

> Find the general solution of the differential equation
>
> $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 10\mathrm{e}^{-2x}$$

Charlotte's solution starts as follows:

> Particular integral: $y = \lambda \mathrm{e}^{-2x}$
>
> so
>
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\lambda \mathrm{e}^{-2x}$$
>
> and
>
> $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\lambda \mathrm{e}^{-2x}$$

13 (a) Show that Charlotte's method will fail to find a particular integral for the differential equation.

[2 marks]

Substituting Charlotte's PI into the left-hand side:

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4\lambda e^{-2x} - 2\lambda e^{-2x} - 2\lambda e^{-2x} = 0$

This would need $0 = 10e^{-2x}$, which is impossible for any λ, so her method fails (e^{-2x} is part of the complementary function).

Do not write outside the box

13 (b) Explain how Charlotte should have started her solution differently and find the general solution of the differential equation.

[8 marks]

Charlotte should first have found the complementary function from the auxiliary equation:

$m^2 + m - 2 = 0 \Rightarrow (m + 2)(m - 1) = 0 \Rightarrow m = 1$ or $m = -2$

CF: $y = Ae^x + Be^{-2x}$

The right-hand side $10e^{-2x}$ is of the same form as a term in the CF, so she should have started with the PI $y = \lambda xe^{-2x}$.

$y = \lambda xe^{-2x}$

$\frac{dy}{dx} = \lambda e^{-2x} - 2\lambda xe^{-2x} = \lambda e^{-2x}(1 - 2x)$

$\frac{d^2y}{dx^2} = -2\lambda e^{-2x} - 2\lambda e^{-2x} + 4\lambda xe^{-2x} = \lambda e^{-2x}(4x - 4)$

$\lambda e^{-2x}(4x - 4 + 1 - 2x - 2x) = 10e^{-2x}$

$-3\lambda = 10 \Rightarrow \lambda = -\frac{10}{3}$

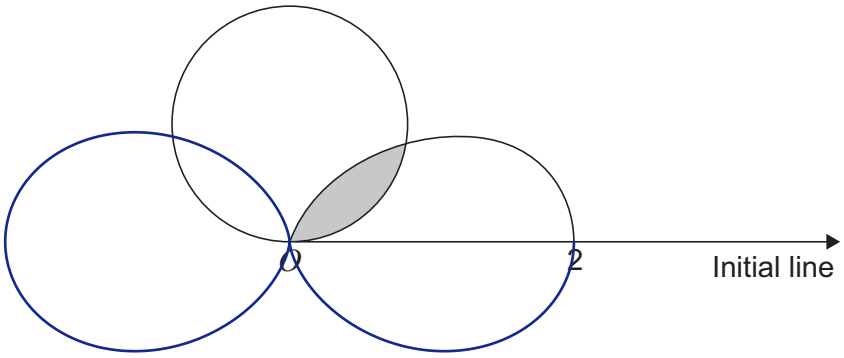
PI: $y = -\frac{10}{3}xe^{-2x}$

General solution: $y = Ae^x + Be^{-2x} - \frac{10}{3}xe^{-2x}$

Do not write outside the box

14 The diagram shows the polar curve C_1 with equation $r = 2\sin\theta$

The diagram also shows part of the polar curve C_2 with equation $r = 1 + \cos 2\theta$



14 (a) On the diagram above, complete the sketch of C_2

[2 marks]

14 (b) Show that the area of the region shaded in the diagram is equal to

$$k\pi + m\alpha - \sin 2\alpha + q\sin 4\alpha$$

where $\alpha = \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$, and k, m and q are rational numbers.

[9 marks]

Intersection: $2\sin\theta = 1 + \cos 2\theta = 2 - 2\sin^2\theta$

$\sin^2\theta + \sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{-1 \pm \sqrt{5}}{2}$

θ is acute so $\sin\theta = \frac{\sqrt{5}-1}{2}$, $\theta = \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right) = \alpha$

Split the region by the line from O to the point of intersection: A_1 is bounded by C_1 for $0 \leq \theta \leq \alpha$, A_2 by C_2 for $\alpha \leq \theta \leq \pi/2$.

Do not write outside the box

Q14(b) continued

$A_1 = \frac{1}{2}\int_0^{\alpha} (2\sin\theta)^2\, d\theta = \int_0^{\alpha} 2\sin^2\theta\, d\theta = \int_0^{\alpha} (1 - \cos 2\theta)\, d\theta$

$A_1 = [\theta - \frac{1}{2}\sin 2\theta]_0^{\alpha} = \alpha - \frac{1}{2}\sin 2\alpha$

$A_2 = \frac{1}{2}\int_{\alpha}^{\pi/2} (1 + \cos 2\theta)^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$

$= \frac{1}{2}\int_{\alpha}^{\pi/2} (1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta))\, d\theta$

$= \frac{1}{2}\int_{\alpha}^{\pi/2} (\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta)\, d\theta$

$= [\frac{3}{4}\theta + \frac{1}{2}\sin 2\theta + \frac{1}{16}\sin 4\theta]_{\alpha}^{\pi/2}$

$= \frac{3\pi}{8} + 0 + 0 - (\frac{3}{4}\alpha + \frac{1}{2}\sin 2\alpha + \frac{1}{16}\sin 4\alpha)$

Shaded area $= A_1 + A_2$

$= \alpha - \frac{1}{2}\sin 2\alpha + \frac{3\pi}{8} - \frac{3}{4}\alpha - \frac{1}{2}\sin 2\alpha - \frac{1}{16}\sin 4\alpha$

$= \frac{3}{8}\pi + \frac{1}{4}\alpha - \sin 2\alpha - \frac{1}{16}\sin 4\alpha$

which is of the form $k\pi + m\alpha - \sin 2\alpha + q\sin 4\alpha$ with

$k = \frac{3}{8}, \quad m = \frac{1}{4}, \quad q = -\frac{1}{16}$

Do not write outside the box

15 The points $A(7, 2, 8)$, $B(7, -4, 0)$ and $C(3, 3.2, 9.6)$ all lie in the plane Π.

15 (a) Find a Cartesian equation of the plane Π.

[3 marks]

$\overrightarrow{AB} = \begin{bmatrix} 0 \\ -6 \\ -8 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} -4 \\ 1.2 \\ 1.6 \end{bmatrix}$

$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} (-6)(1.6) - (-8)(1.2) \\ (-8)(-4) - 0(1.6) \\ 0(1.2) - (-6)(-4) \end{bmatrix} = \begin{bmatrix} 0 \\ 32 \\ -24 \end{bmatrix} = -8\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$

Use $\mathbf{n} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$: $\mathbf{r} \cdot \mathbf{n} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = 0 - 8 + 24 = 16$

Cartesian equation of Π: $-4y + 3z = 16$

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15 (b) The line L_1 has equation $\mathbf{r} = \begin{bmatrix} 5 \\ -0.4 \\ 4.8 \end{bmatrix} + \mu \begin{bmatrix} 15 \\ 3 \\ 4 \end{bmatrix}$

15 (b) (i) Show that L_1 lies in the plane Π.

[2 marks]

Any point on L_1: $(5 + 15\mu, -0.4 + 3\mu, 4.8 + 4\mu)$

$-4y + 3z = -4(-0.4 + 3\mu) + 3(4.8 + 4\mu) = 1.6 - 12\mu + 14.4 + 12\mu = 16$

This holds for all μ, so every point of L_1 lies in Π, i.e. L_1 lies in the plane Π.

15 (b) (ii) Show that every point on L_1 is equidistant from B and C.

[4 marks]

Midpoint of BC: $M = \left(\frac{7+3}{2}, \frac{-4+3.2}{2}, \frac{0+9.6}{2}\right) = (5, -0.4, 4.8)$

This is the point on L_1 with $\mu = 0$, so M lies on L_1.

$$BC = \begin{bmatrix} 3-7 \\ 3.2+4 \\ 9.6-0 \end{bmatrix} = \begin{bmatrix} -4 \\ 7.2 \\ 9.6 \end{bmatrix} = 0.8 \begin{bmatrix} -5 \\ 9 \\ 12 \end{bmatrix}$$

$\begin{bmatrix} 15 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 9 \\ 12 \end{bmatrix} = -75 + 27 + 48 = 0$, so L_1 is perpendicular to BC

For X on L_1: $MX = \mu d$, $d = (15, 3, 4)$, $d \cdot MB = d \cdot MC = 0$, so

$XB^2 = |MB - \mu d|^2 = MB^2 + \mu^2|d|^2 = MC^2 + \mu^2|d|^2 = XC^2$ $(MB = MC)$

So L_1 passes through the midpoint of BC and is perpendicular to BC: it is the perpendicular bisector of BC, and hence every point on L_1 is equidistant from B and C.

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15 (c) The line L_2 lies in the plane Π, and every point on L_2 is equidistant from A and B.

Find an equation of the line L_2

[4 marks]

L_2 is the perpendicular bisector of AB in Π, so it passes through the midpoint of AB and is perpendicular to both AB and n.

Midpoint of AB: $\left(\frac{7+7}{2}, \frac{2-4}{2}, \frac{8+0}{2}\right) = (7, -1, 4)$

$$AB \times n = \begin{bmatrix} 0 \\ -6 \\ -8 \end{bmatrix} \times \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} (-6)(3) - (-8)(-4) \\ (-8)(0) - 0(3) \\ 0(-4) - (-6)(0) \end{bmatrix} = \begin{bmatrix} -50 \\ 0 \\ 0 \end{bmatrix}$$

Direction of L_2: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

L_2: $\mathbf{r} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

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15 (d) The points A, B and C all lie on a circle G.
The point D is the centre of circle G.

Find the coordinates of D.

[3 marks]

D is equidistant from A, B and C and lies in Π, so D lies on both L_1 and L_2: D is their point of intersection.

$5 + 15\mu = 7 + \lambda$, $\quad -0.4 + 3\mu = -1$, $\quad 4.8 + 4\mu = 4$

From the y equation: $3\mu = -0.6 \Rightarrow \mu = -0.2$; z: $4.8 + 4(-0.2) = 4$ ✓

$x = 5 + 15(-0.2) = 2 \quad (\lambda = -5)$

Check: $DA^2 = 5^2 + 3^2 + 4^2 = 50$, $DB^2 = 5^2 + 3^2 + 4^2 = 50$,

$DC^2 = 1^2 + 4.2^2 + 5.6^2 = 50$ ✓

$D = (2, -1, 4)$

END OF QUESTIONS

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