A-level
FURTHER MATHEMATICS
7367/2

Paper 2

Mark scheme

June 2020

Version: 1.1 Final Mark Scheme



Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

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Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
R	mark is for reasoning
A	mark is dependent on M marks and is for accuracy
B	mark is independent of M marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
'their'	indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles:

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

Q	Marking Instructions	AO	Marks	Typical Solution
1	Circles $\mathbf{a} \times (\mathbf{a} - \mathbf{b})$	2.2a	B1	$\mathbf{a} \times (\mathbf{a} - \mathbf{b})$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical Solution
2	Circles $\arg(-a - b\mathrm{i}) = \pi - \varphi$	2.2a	B1	$\arg(-a - b\mathrm{i}) = \pi - \varphi$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical Solution
3	Circles $\frac{5\sqrt{6}}{12}$	1.1b	B1	$\frac{5\sqrt{6}}{12}$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
4	Multiplies matrices **A** and **B** to form the product **AB** with at least one element of the product correct. Condone **BA**	1.1a	M1	$\mathbf{AB} = \begin{bmatrix} x^2-3x-4 & (x-3)(x+2) \\ (x-4)(x+2) & x^2+2 \end{bmatrix}$ $x = -2$ $\mathbf{AB} = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore k = 6$
	Forms the correct product **AB** (may be unsimplified)	1.1b	A1	
	Deduces from correct product that **AB**$= 6\mathbf{I}$, when $x = -2$	2.2a	R1	
	Total		**3**	

Q	Marking Instructions	AO	Marks	Typical Solution
5	Selects a correct approach which would lead to solving the inequality eg Multiplies the inequality by $(x-1)^2$ or Rearranges to an inequality with 0 on LHS or RHS or Replaces "≤" with "=" and multiplies by $(x-1)$	1.1a	M1	$\frac{(x-1)^2(2x+3)}{x-1} \leq (x-1)^2(x+5)$ $(x-1)(2x+3) \leq (x-1)^2(x+5)$ $(x-1)\{(x-1)(x+5)-(2x+3)\} \geq 0$ $(x-1)\{x^2+2x-8\} \geq 0$ $(x-1)(x+4)(x-2) \geq 0$ Considering cubic curve: $x \geq 2$ or between -4 and 1 But $x \neq 1$ So $x \geq 2$ or $-4 \leq x < 1$
	Manipulates their equation/inequality to allow the critical values to be found	1.1a	M1	
	Obtains critical values of -4, 1 and 2	1.1a	M1	
	Gives one correct region from $x \geq 2, -4 \leq x < 1$ Condone $-4 \leq x \leq 1$ Must have three critical values.	1.1b	A1	
	Obtains correct solution $x \geq 2, -4 \leq x < 1$	1.1b	A1	
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
6	Obtains or uses the sum of the integers from 1 to 999	1.1b	B1	$\sum_{r=1}^{999} r = \frac{999 \times 1000}{2} = 499500$ $\sum_{r=1}^{31} r^2 = \frac{31 \times 32 \times 63}{6} = 10416$ $\sum_{r=1}^{9} r^3 = \frac{9^2 \times 10^2}{4} = 2025$ Sixth powers: $1 + 64 + 729 = 794$ Required total: $499500 - 10416 - 2025 + 794$ $= 487853$
	Deduces that there are 31 square numbers or 9 cube numbers between 1 and 999, inclusive	2.2a	B1	
	Subtracts their sums of squares and cubes from the sum of integers.	1.1a	M1	
	Identifies at least one of the sixth powers (1, 64, 729) which are duplicated in the sums of squares and cubes	3.1a	M1	
	Obtains the correct sum of 487853	1.1b	A1	
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
7	Uses formula for volume of revolution Condone omission of π	1.1a	M1	$V = \pi \int_0^{0.8} y^2 \mathrm{d}x$ $\pi \times \frac{0.2}{3} \times \{2.46740 + 0.41409 + 4 \times 1.87536 + 4 \times 0.85988 + 2 \times 1.34393 +\}$ = 3.4579
	Identifies and uses required x-values as 0, 0.2, 0.4, 0.6, 0.8 (PI by their $y, y^2, \pi y^2$ values)	1.1a	M1	
	Correctly calculates values of y^2or πy^2	1.1b	A1	
	Substitutes their ordinates into Simpson's rule with consistent h for their number of ordinates. Condone use of y rather than y^2or πy^2	1.1a	M1	
	Obtains correct answer, 3.4579	1.1b	A1	
	Total		**5**	

x	0	0.2	0.4	0.6	0.8
y^2	2.46740	1.87536	1.34393	0.85988	0.41409

Q	Marking Instructions	AO	Marks	Typical solution
8(a)	Shows understanding that value of the determinant is unchanged when a column or row operation is used.	1.1a	M1	$a\begin{vmatrix} 2 & x+b & x^2+b^2 \\ -1 & a & -a^2 \\ 1 & b & b^2 \end{vmatrix}$
	Demonstrates understanding that a factor can be extracted from the determinant.	1.1a	M1	$a\begin{vmatrix} 0 & x-b & x^2-b^2 \\ -1 & a & -a^2 \\ 1 & b & b^2 \end{vmatrix}$
	Expands the determinant	1.1a	M1	
	Correctly extracts one factor	1.1b	A1	$a(x-b)\begin{vmatrix} 0 & 1 & x+b \\ -1 & a & -a^2 \\ 1 & b & b^2 \end{vmatrix}$
	Correctly finds two factors	1.1b	A1	
	Obtains the determinant in fully factorised form $-a(a+b)(a+x)(x-b)$ OE Correct answer seen: 6 marks	1.1b	A1	$a(x-b)(a+b)\begin{vmatrix} 0 & 1 & x+b \\ 0 & 1 & b-a \\ 1 & b & b^2 \end{vmatrix}$ $a(x-b)(a+b)\{(b-a)-(x+b)\}$ $-a(x-b)(a+b)(x+a)$
8(b)	Forms an expression for the volume scale factor $= \frac{300}{0.625}$	3.1a	M1	$a=5, b=3$ and $SF=\pm 480$
	Forms an equation for their volume scale factor = det **M**	1.1a	M1	$-5\times 8\times(x-3)(x+5)=\pm 480$
	Deduces all four correct values of x: -3, 1, $-1\pm 2\sqrt{7}$	2.2a	A1	$(x-3)(x+5)=\pm 12$ $x=1,\ x=-3$ $x=-1+2\sqrt{7}\,, x=-1-2\sqrt{7}$
	Total		**9**	

Q	Marking Instructions	AO	Marks	Typical Solution
9	Explains that $\mathbf{C}^2$ represents a rotation of $\frac{\pi}{6}$ or that **C** represents a rotation of $\frac{\pi}{12}$	2.4	E1	$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2-b^2 & -2ab \\ 2ab & a^2-b^2 \end{bmatrix}$ $2ab = \frac{1}{2}$ $a^2 - b^2 = \frac{\sqrt{3}}{2}$ $b = \frac{1}{4a}$ $a^2 - \frac{1}{16a^2} = \frac{\sqrt{3}}{2}$ $16a^4 - 8\sqrt{3}a^2 - 1 = 0$ $a^2 = \frac{\sqrt{3}+2}{4}$ $a = \frac{\sqrt{\sqrt{3}+2}}{2}$ since $a > 0$ $\mathbf{C^2}$ represents a rotation of $\frac{\pi}{6}$, therefore **C** represents a rotation of $\frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$ So $a = \cos\frac{\pi}{12}$ and $\cos\frac{\pi}{12} = \frac{\sqrt{\sqrt{3}+2}}{2}$
	Squares matrix **C** with at least two correct elements.	3.1a	M1	
	Forms two simultaneous equations in a and b using their squared **C** and the given $\mathbf{C}^2$	1.1a	M1	
	Forms two correct simultaneous equations.	1.1b	A1	
	Eliminates a or b and forms a quadratic equation in b^2 or a^2	1.1a	M1	
	Finds the correct value of $a^2 = \frac{\sqrt{3}+2}{4}$	1.1b	A1	
	Completes a rigorous argument to show that $\cos\frac{\pi}{12} = \frac{\sqrt{\sqrt{3}+2}}{2}$ by explaining that **C** represents a rotation of $\frac{\pi}{12}$	2.1	R1	
	Total		**7**	

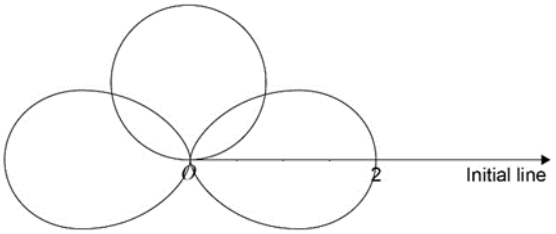
Q	Marking Instructions	AO	Marks	Typical Solution
10	Shows that $u_n = \frac{5^n-5}{5^n-1}$ is true for $n = 1$	1.1b	B1	Let $n = 1$; then the formula gives $u_1 = \frac{5^1-5}{5^1-1} = 0$ so the result is true for $n = 1$ Assume the result is true for $n = k$: Then $u_{k+1} = \frac{5}{6-\left(\frac{5^k-5}{5^k-1}\right)}$ $6 - \left(\frac{5^k-5}{5^k-1}\right) = \frac{6(5^k-1)-(5^k-5)}{5^k-1}$ $= \frac{6\times 5^k - 5^k - 6 + 5}{5^k-1}$ $= \frac{5\times 5^k - 1}{5^k-1} = \frac{5^{k+1}-1}{5^k-1}$ $\therefore u_{k+1} = 5 \times \frac{5^k-1}{5^{k+1}-1} = \frac{5^{k+1}-5}{5^{k+1}-1}$ and the result also holds for $n = k+1$ The formula for u_n is true for $n = 1$; if true for $n = k$, then it's also true for $n = k+1$ and hence by induction $u_n = \frac{5^n-5}{5^n-1}$ for $n \geq 1$
	States the assumption that $u_n = \frac{5^n-5}{5^n-1}$ is true for $n = k$	2.4	M1	
	Uses the recurrence relation and the assumption to express u_{k+1} in terms of k	3.1a	M1	
	Expresses u_{k+1} as a single fraction.	1.1a	M1	
	Completes rigorous working to deduce that $u_{k+1} = \frac{5^{k+1}-5}{5^{k+1}-1}$	2.2a	R1	
	Concludes a reasoned argument by stating that the formula for u_n is true for $n = 1$; that if true for $n = k$, then it's also true for $n = k+1$ and hence by induction $u_n = \frac{5^n-5}{5^n-1}$ for $n \geq 1$	2.1	R1	
	Total		**6**	

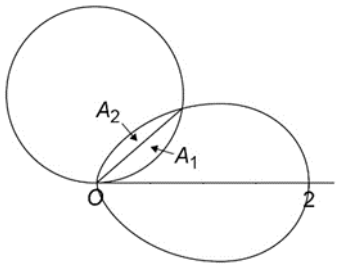
Q	Marking Instructions	AO	Marks	Typical Solution
11(a)	Finds the second or third simplified term of the series for $\frac{\sin x}{x}$ or finds the general term	1.1a	M1	$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{(-1)^r x^{2r}}{(2r+1)!} + \cdots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^r x^{2r}}{(2r)!}$
	Finds general term of series for $\frac{\sin x}{x}$ with x^{2r}	1.1a	M1	
	Subtracts general terms of $\frac{\sin x}{x}$ and $\cos x$	1.1a	M1	$\frac{1}{(2r+1)!} - \frac{1}{(2r)!} = \frac{1-(2r+1)}{(2r+1)!}$ $= \frac{-2r}{(2r+1)!}$
	Completes a rigorous argument to show the required result, including $\frac{1}{(2r+1)!} - \frac{1}{(2r)!} = \frac{1-(2r+1)}{(2r+1)!} = \frac{-2r}{(2r+1)!}$ **AG**	2.1	R1	$\therefore \frac{(-1)^r x^{2r}}{(2r+1)!} - \frac{(-1)^r x^{2r}}{(2r)!} = \frac{(-1)^r x^{2r}(-2r)}{(2r+1)!}$ $= (-1)^{r+1} \frac{2r}{(2r+1)!} x^{2r}$
11(b)	Selects a method to determine the value of the limit by finding the first non-zero term of one series.	3.1a	M1	First non-zero terms of series expansion of $\frac{\sin x}{x} - \cos x$ are $\frac{x^2}{3}$ and $-\frac{x^4}{30}$ First non-zero terms of series expansion of $1 - \cos x$ are $\frac{x^2}{2}$ and $-\frac{x^4}{24}$ $\lim_{x\to 0}\left[\frac{\frac{x^2}{3} - \frac{x^4}{30} + \cdots}{\frac{x^2}{2} - \frac{x^4}{24} + \cdots}\right] = \lim_{x\to 0}\left[\frac{\frac{1}{3} - \frac{x^2}{30} + \cdots}{\frac{1}{2} - \frac{x^2}{24} + \cdots}\right] = \frac{1/3}{1/2}$ $= \frac{2}{3}$
	Explains or shows that each series has terms in higher powers of x	2.4	E1	
	Deduces that the required limit can be determined by dividing numerator and denominator by the lowest power of x or by using l'Hôpital's rule.	2.2a	M1	
	Completes a rigorous argument to show the required result $\lim_{x\to 0}\left[\frac{\frac{\sin x}{x} - \cos x}{1 - \cos x}\right] = \frac{2}{3}$	2.1	R1	
	Total		**8**	

Q	Marking Instructions	AO	Marks	Typical solution
12(a)	Selects a method to find the required result by integrating by parts.	3.1a	M1	$I = \left[\frac{1}{2}\mathrm{e}^{2t}\sin t\right]_a^b - \frac{1}{2}\int_a^b e^{2t}\cos t\,dt$
	Obtains correct result of integration by parts.	1.1b	A1	
	Uses integration by parts a second time, consistent with their choice of u and v' in their first integration by parts.	1.1a	M1	$I = \left[\frac{1}{2}\mathrm{e}^{2t}\sin t\right]_a^b - \frac{1}{2}\left\{\left[\frac{1}{2}\mathrm{e}^{2t}\cos t\right]_a^b + \frac{1}{2}\int_a^b \mathrm{e}^{2t}\sin t\,dt\right\}$
	Obtains correct result of second integration by parts FT their first integration by parts.	1.1b	A1	
	Deduces that the second integration by parts gives an equation in I which can be solved	2.2a	M1	$I = \left[\frac{1}{2}\mathrm{e}^{2t}\sin t - \frac{1}{4}\mathrm{e}^{2t}\cos t\right]_a^b - \frac{1}{4}I$
	Completes rigorous argument to show that the result of the second integration by parts gives $I = \left[\frac{2}{5}e^{2t}\sin t - \frac{1}{5}e^{2t}\cos t\right]_a^b$	2.1	R1	$I = \left[\frac{2}{5}\mathrm{e}^{2t}\sin t - \frac{1}{5}\mathrm{e}^{2t}\cos t\right]_a^b$

12(b)	Selects a method to solve the differential equation by finding an integrating factor.	3.1a	M1	$\frac{\mathrm{d}v}{\mathrm{d}t}+v=5\mathrm{e}^{t}\sin t$ IF: $\mathrm{e}^{\int 1\mathrm{d}t}=e^{t}$ $\frac{\mathrm{d}}{\mathrm{d}t}\left(v\mathrm{e}^{t}\right)=5\mathrm{e}^{2t}\sin t$ $v\mathrm{e}^{t}=2\mathrm{e}^{2t}\sin t-\mathrm{e}^{2t}\cos t+c$ When $t=0$, $v=0$ $\therefore c=1$ $v\mathrm{e}^{t}=2\mathrm{e}^{2t}\sin t-\mathrm{e}^{2t}\cos t+1$ $v\mathrm{e}^{2\pi}=2\mathrm{e}^{4\pi}\sin 2\pi-\mathrm{e}^{4\pi}\cos 2\pi+1$ $=-\mathrm{e}^{4\pi}+1$ $v=-\mathrm{e}^{2\pi}+\mathrm{e}^{-2\pi}$ Speed $=\mathrm{e}^{2\pi}-\mathrm{e}^{-2\pi}$
	Multiplies the differential equation by their integrating factor	1.1a	M1	
	Integrates LHS correctly to obtain ve^{t}	1.1b	A1	
	Integrates their RHS correctly with or without limits FT their q and r from (a)	1.1b	B1F	
	Uses the initial conditions when evaluating a definite integral or determining the constant of integration.	3.4	M1	
	States the correct exact value of the speed, which must be positive FT their q and r from (a) provided $v < 0$ when $t = 2\pi$	3.2a	A1	
	Total		**12**	

Q	Marking Instructions	AO	Marks	Typical Solution
13(a)	Evaluates Charlotte's method by substituting her particular integral and its derivatives into the differential equation.	2.3	M1	Continuing Charlotte's method gives $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-2y=4\lambda\mathrm{e}^{-2x}-2\lambda\mathrm{e}^{-2x}-2\lambda\mathrm{e}^{-2x}=0$
	Explains why Charlotte's method fails to find a particular integral for the differential equation.	2.3	E1	This would make $10\mathrm{e}^{-2x}$ equal to zero, which is impossible, so the method fails.
13(b)	Evaluates Charlotte's method by explaining that she should first have found the complementary function or the auxiliary equation.	2.3	E1	Charlotte needs to find the complementary function first. $m^2+m-2=0$ $m=1$ or $m=-2$ CF: $y=A\mathrm{e}^{x}+B\mathrm{e}^{-2x}$ The RHS has a similar form to the CF and so we need to introduce a factor of x into the particular integral. PI: $y=\lambda x\mathrm{e}^{-2x}$ $y'=\lambda\mathrm{e}^{-2x}(-2x+1)$ $y''=\lambda\mathrm{e}^{-2x}(4x-4)$ $\lambda\mathrm{e}^{-2x}(4x-4-2x+1-2x)=10\mathrm{e}^{-2x}$ $\lambda=-\frac{10}{3}$ General solution: $y=A\mathrm{e}^{x}+B\mathrm{e}^{-2x}-\frac{10}{3}x\mathrm{e}^{-2x}$
	Obtains the auxiliary equation and its solutions $u=-2,1$	1.1b	B1	
	Writes down the complementary function. FT their solutions of their auxiliary equation.	1.1b	B1F	
	Selects a method to solve the differential equation by stating the correct PI $y=\lambda xe^{-2x}$	3.1a	B1	
	Differentiates their PI twice (must be different from Charlotte's PI)	1.1a	M1	
	Obtains correct 1st and 2nd derivative of the correct PI	1.1b	A1	
	Substitutes their PI and its derivatives into the differential equation.	1.1a	M1	
	Correctly reasons that the general solution of the differential equation is $y=Ae^{x}+Be^{-2x}-\frac{10}{3}xe^{-2x}$	2.1	R1	
	Total		**10**	

Q	Marking Instructions	AO	Marks	Typical Solution
14(a)	Draws another quarter of the curve in the correct location.	1.1a	M1	
	Draws the complete curve.	1.1b	A1	

14(b)	Selects a method to find the area of the shaded region by splitting it into two regions with a line from the pole to the point of intersection of C_1 and C_2 PI by correct limits	3.1a	B1	
	Forms an equation for the intersection of C_1 and C_2	1.1a	M1	Point of intersection: $1+\cos 2\theta = 2\sin\theta$ $2\cos^2\theta = 2\sin\theta$ $2-2\sin^2\theta = 2\sin\theta$ $\sin^2\theta + \sin\theta - 1 = 0$ $\sin\theta = \frac{-1\pm\sqrt{5}}{2}$
	Solves the equation to find the value of $\sin\theta = \frac{-1+\sqrt{5}}{2}$ at the point of intersection.	2.2a	A1	
	Uses an integral of the form $\frac{1}{2}\int r^2\,d\theta$ to find an area enclosed by a polar curve.	3.1a	B1	θ is acute so $\theta = \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right) = \alpha$ as defined in the question.
	Selects a method to integrate $\sin^2\theta$ or $\cos^2 2\theta$ by using a double angle formula	3.1a	M1	$A_1 = \frac{1}{2}\int_0^{\alpha}(2\sin\theta)^2\mathrm{d}\theta$ $A_1 = \int_0^{\alpha}(1-\cos 2\theta)\mathrm{d}\theta$
	Correctly integrates $\int(2\sin\theta)^2\,d\theta$	1.1b	A1	$A_1 = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\alpha}$ $A_1 = \alpha - \frac{1}{2}\sin 2\alpha$
	Correctly integrates $\int(1+\cos 2\theta)^2\,d\theta$	1.1b	A1	$A_2 = \frac{1}{2}\int_{\alpha}^{\frac{\pi}{2}}(1+\cos 2\theta)^2\mathrm{d}\theta$ $A_2 = \frac{1}{2}\int_{\alpha}^{\frac{\pi}{2}}(1+2\cos 2\theta+\cos^2 2\theta)\mathrm{d}\theta$
	Substitutes limits into one of their integrals to determine an area.	1.1a	M1	$A_2 = \frac{1}{2}\int_{\alpha}^{\frac{\pi}{2}}\left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)\mathrm{d}\theta$ $A_2 = \left[\frac{3}{4}\theta+\frac{1}{2}\sin 2\theta+\frac{1}{16}\sin 4\theta\right]_{\alpha}^{\frac{\pi}{2}}$ $A_2 = \frac{3\pi}{8} - \left(\frac{3}{4}\alpha+\frac{1}{2}\sin 2\,\alpha+\frac{1}{16}\sin 4\alpha\right)$
	Completes a rigorous argument to show that the area of the shaded region is $\frac{3}{8}\pi+\frac{1}{4}\alpha-\sin 2\alpha-\frac{1}{16}\sin 4\alpha$	2.1	R1	Area enclosed $= A_1+A_2$ $= \frac{3\pi}{8}+\frac{1}{4}\alpha-\sin 2\alpha-\frac{1}{16}\sin 4\alpha$
	Total		**11**	

Q	Marking Instructions	AO	Marks	Typical Solution
15(a)	Obtains two vectors in the plane Π	1.1a	B1	$\overrightarrow{AB} = \begin{bmatrix} 0 \\ -6 \\ -8 \end{bmatrix}$ $\overrightarrow{BC} = \begin{bmatrix} -4 \\ 7.2 \\ 9.6 \end{bmatrix}$ Normal vector $\mathbf{n} = \begin{vmatrix} \mathbf{i} & 0 & -5 \\ \mathbf{j} & 3 & 9 \\ \mathbf{k} & 4 & 12 \end{vmatrix} = \begin{bmatrix} 0 \\ -20 \\ 15 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ Equation of plane Π: $-4y + 3z = 16$
	Selects a method to find a vector normal to the plane Π either by taking the vector product of their two vectors in Π or by taking the scalar product of a general vector with their two vectors in Π	3.1a	M1	
	Obtains a correct Cartesian equation of the plane Π	1.1b	A1	
15(b)(i)	Selects a method to show that L_1 lies in Π either by substituting a general point on L_1 into their equation of Π , or by substituting a point on L_1 in their equation of Π and using a scalar product or a vector product to show that the direction vector of L_1 is perpendicular to their normal to Π`	3.1a	M1	For any point on L_1, $y = -0.4 + 3\mu$ and $z = 4.8 + 4\mu$ $\therefore -4y + 3z = -4(-0.4 + 3\mu) + 3(4.8 + 4\mu)$ $= 1.6 - 12\mu + 14.4 + 12\mu = 16$ So the point lies in the plane Π and therefore L_1 lies in the plane Π.
	Completes a rigorous argument to show that L_1 lies in Π	2.1	R1	

15(b)(ii)	Selects a method to show that every point on L_1 is equidistant from B and C by finding the general vector from a point on L_1 to B or C or by finding the midpoint (5, -0.4, 4.8) of BC	3.1a	M1	Midpoint of BC is (5, -0.4, 4.8), which lies on L_1 Consider the direction vectors of L_1 and BC: $\begin{bmatrix}15\\3\\4\end{bmatrix}\cdot\begin{bmatrix}-5\\9\\12\end{bmatrix}=-75+27+48=0$ $\therefore$ L_1 is perpendicular to BC Since the midpoint of BC also lies on L_1 then L_1 is the perpendicular bisector of BC and hence every point on $L_1$is equidistant from B and C
	Finds the vector $\overrightarrow{BC}$ or forms an expression for the length of a general vector from a point on L_1 to B or C	1.1a	M1	
	Takes the scalar product of $\overrightarrow{BC}$ and the direction vector of L_1 or forms an equation for the distances from a point on L_1 to B and C being equal.	1.1a	M1	
	Completes a reasoned argument to show that every point on L_1 is equidistant from B and C	2.1	R1	

15(c)	Deduces that the direction vector of L_2 is perpendicular to AB or to their normal to Π	2.2a	M1	L_2 is the perpendicular bisector of AB in the plane Π. Midpoint of AB is M_{AB} (7, -1, 4) Direction vector $\mathbf{s}$ for L_2 is perpendicular to both AB and $\mathbf{n}$. $\begin{vmatrix} \mathbf{i} & 0 & 0 \\ \mathbf{j} & 3 & -4 \\ \mathbf{k} & 4 & 3 \end{vmatrix} = \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix}$ so let $\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $L_2 : \mathbf{r} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
	Deduces that L_2 passes through the midpoint of AB	2.2a	M1	
	Selects a method to find the direction vector of L_2 by taking the vector product of $\overrightarrow{AB}$ and their normal to Π	3.1a	M1	
	Completes a reasoned mathematical argument to obtain a correct equation of L_2	2.1	R1	
15(d)	Deduces that D is at the intersection of $L_1$and L_2	2.2a	B1	D is the point of intersection of L_1 and L_2 . $5 + 15\mu = \lambda + 7$ $0.4 + 3\mu = -1$ $4.8 + 4\mu = 4$ $\mu = -0.2$ $\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ $D(2, -1, 4)$
	Equates equations of L_1 and their L_2	1.1a	M1	
	Obtains the correct coordinates of D FT their equation of L_2 Condone position vector of D	1.1b	A1	
	Total		**16**	