AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 3 Discrete

Friday 13 June 2025 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.
- You must ensure you have the other optional Question Paper/Answer Book for which you are entered (**either** Mechanics or Statistics). You will have 2 hours to complete **both** papers.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 50.

For Examiner's Use	
Question	Mark
1	
2	
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8	
9	
10	
TOTAL	

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

Answer **all** questions in the spaces provided.

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1 The group A is of order 9

The group B is a proper, non-trivial subgroup of A

Find the order of B

Circle your answer.

[1 mark]

1 (3) 9 18

Do not write outside the box

2 The simplex algorithm is used to solve a linear programming problem, which results in the following correct simplex tableau.

P	x	y	r	s	**value**
1	$\frac{2}{3}$	0	$\frac{5}{3}$	0	25
0	$\frac{5}{3}$	0	$-\frac{1}{3}$	1	7
0	$\frac{1}{3}$	1	$\frac{1}{3}$	0	5

Which one of the following gives the correct optimal values of P, x and y?

Tick (✓) **one** box.

[1 mark]

$P = 1,\ x = \frac{2}{3},\ y = 0$ ☐

$P = 1,\ x = \frac{5}{3},\ y = 1$ ☐

$P = 25,\ x = 0,\ y = 5$ ☑

$P = 25,\ x = 7,\ y = 5$ ☐

Turn over for the next question

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3 The group $(X, *)$ is an abelian group of order 4

The elements of X are

$$e,\ a,\ b \text{ and } c$$

where e is the identity element of $(X, *)$

The following is a partially completed Cayley table for $(X, *)$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b		e	a
c	c			e

Which one of the following fully completed Cayley tables for $(X, *)$ is correct?

Tick (✓) **one** box.

[1 mark]

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	a	e	a
c	c	b	c	e

☐

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	a	e	a
c	c	c	b	e

☐

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	a	b	e

☐

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

☑

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4 A water company is planning to install a new sewer system between 5 districts of a city.

The lengths, in miles, of possible sewer pipes between districts is shown in the table.

	Blacon	Hoole	Lache	Newton	Upton
Blacon	–	3.3	4.0	2.9	2.7
Hoole	3.3	–	3.6	1.3	1.7
Lache	4.0	3.6	–	3.4	4.3
Newton	2.9	1.3	3.4	–	1.4
Upton	2.7	1.7	4.3	1.4	–

The water company wants to create a network connecting, directly or indirectly, each of the 5 districts using the minimum possible total length of sewer pipes.

4 (a) Find the minimum possible total length of sewer pipes required.

[3 marks]

Prim's algorithm, starting from Blacon (B):

B to U (Blacon to Upton) 2.7, U to N (Upton to Newton) 1.4

N to H (Newton to Hoole) 1.3, N to L (Newton to Lache) 3.4

Minimum spanning tree: BU, UN, NH, NL

Total length = 2.7 + 1.4 + 1.3 + 3.4

Minimum total length = 8.8 miles

4 (b) As part of their planning, the water company carries out a geological survey of the city.

As a result, the water company cannot connect Hoole directly to Upton in the plans for the new sewer system.

Explain the effect, if any, this has on your answer to part **(a)**.

[2 marks]

The arc Hoole to Upton (1.7) is not in the minimum spanning tree,

so not being able to connect Hoole directly to Upton has no effect:

the minimum total length of sewer pipes is still 8.8 miles.

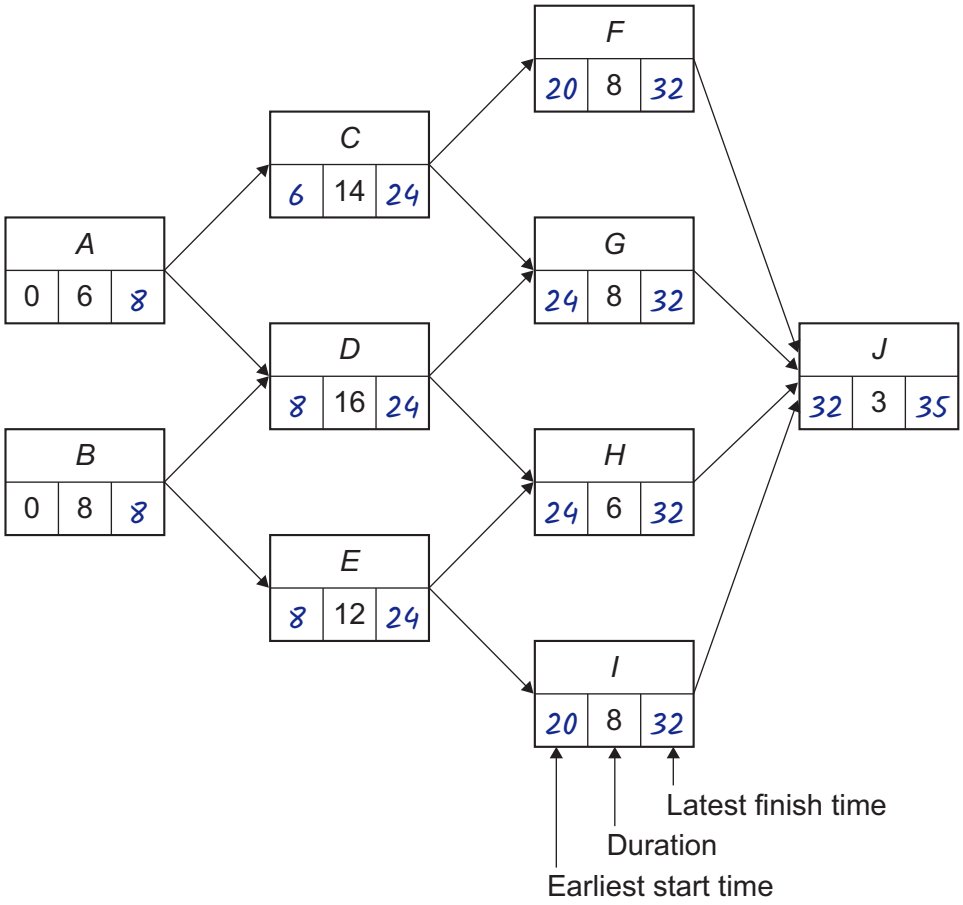
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5 A project is being planned by Lepton Electricity Installations.

The project is broken down into 10 separate activities *A*, *B*, …, *J* and the activity network for the project is shown in **Figure 1**

The duration of each activity is given in days.

Figure 1



5 (a) (i) Find the earliest start time and the latest finish time for each activity and write these values on the activity network in **Figure 1**

[2 marks]

5 (a) (ii) Determine the float of activity *F*

[1 mark]

Float of F = LF − duration − ES = 32 − 8 − 20

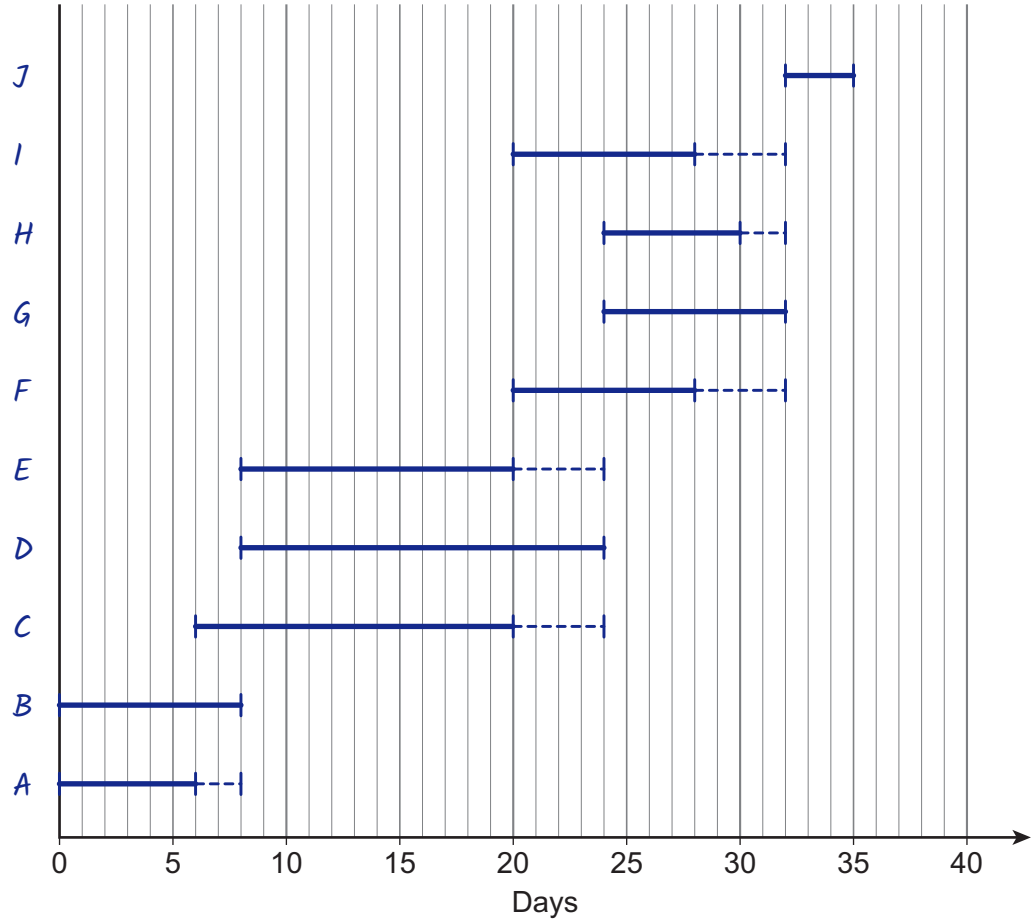
= 4 days

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5 (b) On **Figure 2** draw a cascade diagram (Gantt chart) for the project, assuming that each activity starts as early as possible.

[3 marks]

Figure 2

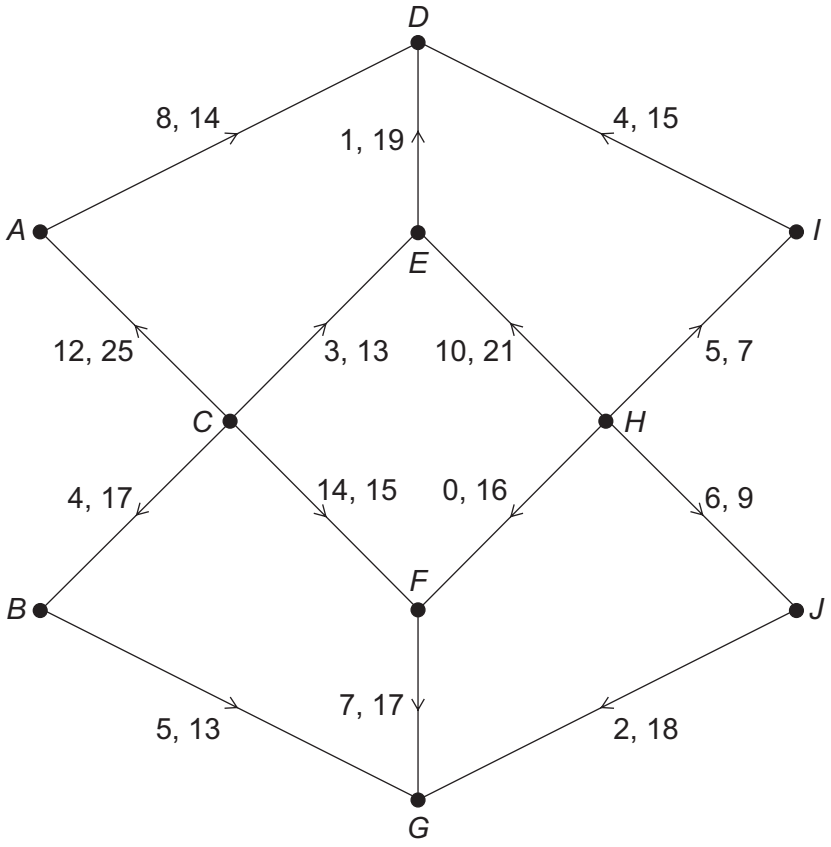


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6 The network below represents a system of pipes that transport chemicals at a factory.

The numbers on each arc represent the lower and upper capacity of each pipe in litres per second.



6 (a) The management team at the factory plan to connect a supersource and a supersink to the network.

Determine which node(s) should connect to the supersource and which node(s) should connect to the supersink.

[2 marks]

Node(s) connected to supersource C and H

Node(s) connected to supersink D and G

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6 (b) Find the value of the cut $\{A, B, C, E, F, H\}\ \{D, G, I, J\}$

[1 mark]

Arcs from {A, B, C, E, F, H} to {D, G, I, J} (upper capacities):

AD 14 + ED 19 + HI 7 + HJ 9 + BG 13 + FG 17 (no arcs go back)

Value of cut = 7 + 19 + 14 + 13 + 17 + 9 = 79 litres per second

6 (c) (i) Andrew, a manager at the factory, finds a flow of 79 litres per second through the network. He claims this is the maximum possible flow through the network.

Explain whether this claim is correct.

Fully justify your answer.

[2 marks]

The value of the cut {A, B, C, E, F, H} {D, G, I, J} is 79 l/s,

which is the same as the value of Andrew's flow of 79 l/s.

By the maximum flow-minimum cut theorem, the maximum flow is

equal to the value of the minimum cut, and no flow can exceed 79,

so this cut is a minimum cut and the maximum flow is 79 l/s.

Hence Andrew's claim is correct.

6 (c) (ii) Andrew also claims that introducing a supersource and a supersink to the network will increase the maximum flow through the network.

Explain why this claim is incorrect.

[1 mark]

Adding a supersource and a supersink does not change any capacities

in the network, so the value of the minimum cut is unchanged (79),

and therefore the maximum flow through the network does not change.

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7 A large bakery makes and sells two different types of loaf: malt and rye.

Each day, the bakery makes x malt loaves and y rye loaves.

The bakery wants to maximise the total profit, in £s, it generates each day from these two types of loaf.

The bakery formulates their situation as a linear programming problem by considering:

the total profit, £P, generated on all the malt loaves and rye loaves made each day

the total time available to make the loaves each day

the ingredients available to make the loaves each day

a minimum number of each type of loaf made each day.

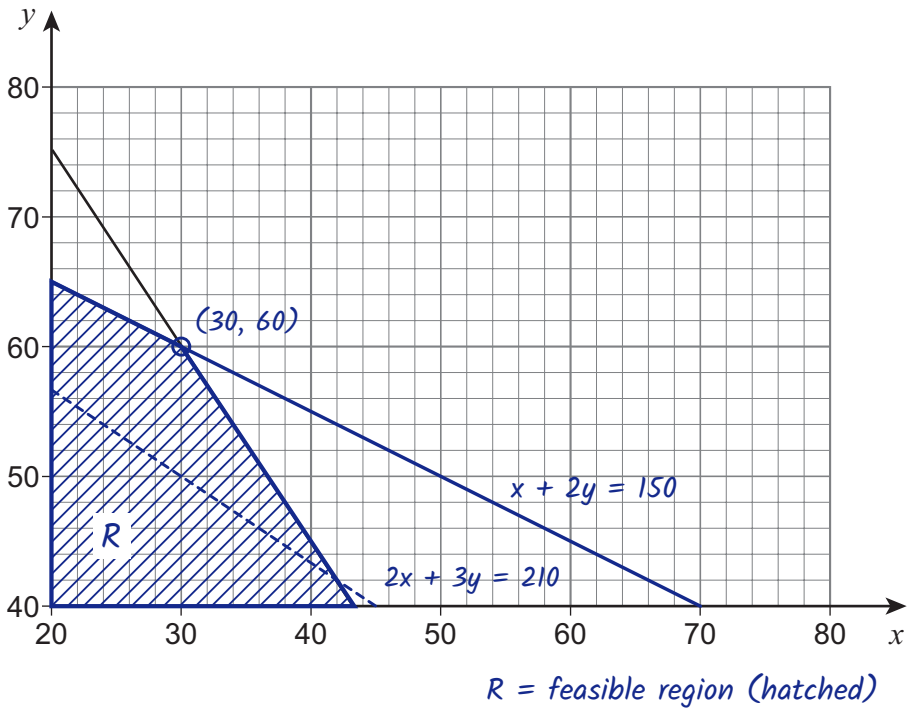
The bakery's linear programming problem is shown below.

Maximise $P = 2x + 3y$

subject to
$$x + 2y \leq 150$$
$$3x + 2y \leq 210$$
$$x \geq 20$$
$$y \geq 40$$

and x, y are integers

7 (a) The graph shows the line $3x + 2y = 210$



Do not write outside the box

7 (a) (i) On the graph, draw the line $x + 2y = 150$

[1 mark]

7 (a) (ii) On the graph, identify the feasible region of the linear programming problem.

[1 mark]

7 (b) (i) Hence, find the number of malt loaves and the number of rye loaves the bakery should make each day.

[2 marks]

Objective line $2x + 3y = k$ (gradient $-2/3$) slid away from the origin;

last vertex of R it touches is (30, 60). Check vertices: $P = 2x + 3y$

(20, 40): 160 (20, 65): 235 (30, 60): 240 (43.3, 40): 206.7

Optimal point (30, 60), which has integer values.

Number of malt loaves 30

Number of rye loaves 60

7 (b) (ii) Find the maximum total profit, in £s, the bakery will generate each day.

[1 mark]

$P = 2 \times 30 + 3 \times 60 = 60 + 180$

Maximum total profit = £240

7 (b) (iii) Give a reason why your answer to part **(b)(ii)** may not be accurate.

[1 mark]

It assumes that all the loaves made each day are sold. If some loaves are not sold (e.g. wasted), the actual profit will be less than £240.

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8 The connected planar graph G has n^2 vertices, $2n$ edges and $3n - 4$ faces, where n is an integer.

8 (a) (i) Show that $n = 2$

[3 marks]

Euler's formula for a connected planar graph: $v - e + f = 2$

$n^2 - 2n + (3n - 4) = 2$

$n^2 + n - 4 - 2 = 0$

$n^2 + n - 6 = 0$

$(n - 2)(n + 3) = 0$

$n = 2$ or $n = -3$

$n = -3$ is impossible since the number of edges $2n = -6 < 0$

(and $3n - 4 = -13$ faces), so reject $n = -3$

$\therefore n = 2$

8 (a) (ii) Hence, write down the number of vertices, edges and faces for G

[1 mark]

Number of vertices $2^2 = 4$

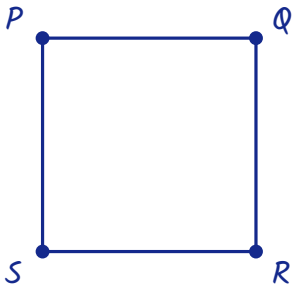
Number of edges $2 \times 2 = 4$

Number of faces $3 \times 2 - 4 = 2$

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8 (b) (i) In the case when G is Eulerian, draw G

[1 mark]



All vertices have degree 2 (even)

8 (b) (ii) Alexie claims that in the case when G is Eulerian, G is also a bipartite graph.

Comment on the validity of Alexie's claim.

[2 marks]

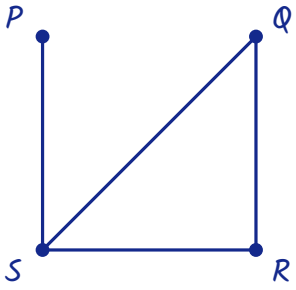
Split the vertices into the two sets {P, R} and {Q, S}.

Every edge (PQ, QR, RS, SP) joins a vertex in {P, R} to a vertex in {Q, S}, so G is bipartite: G is isomorphic to $K_{2,2}$

Hence Alexie's claim is valid.

8 (c) (i) In the case when G is semi-Eulerian, draw G

[1 mark]



Exactly two odd vertices:
S (degree 3) and P (degree 1)

8 (c) (ii) Zackary claims that in the case when G is semi-Eulerian, G is also a tree.

Comment on the validity of Zackary's claim.

[2 marks]

G contains a cycle (QRS), but a tree cannot contain a cycle.

(Also a tree with 4 vertices has 3 edges, but G has 4 edges, and a tree has only 1 face, but G has 2 faces.) So G is not a tree.

Hence Zackary's claim is not valid.

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9 (a) John and Elizabeth play a zero-sum game that does **not** have a stable solution.

This game is represented by the following pay-off matrix for John.

		Elizabeth		
	Strategy	**X**	**Y**	**Z**
John	**A**	3	−3	−1
	B	−1	4	3
	C	−2	1	1

9 (a) (i) State, with a reason, which strategy John should never play.

[1 mark]

Strategy C, since it is dominated by strategy B:

$-1 > -2$, $4 > 1$ and $3 > 1$

so John should never play strategy C.

9 (a) (ii) Find the optimal mixed strategy for John.

Fully justify your answer.

[6 marks]

Remove C. Let John play A with probability p and B with probability $1 - p$.

Expected gains for John:

If Elizabeth plays X: $3p - 1(1 - p) = 4p - 1$

If Elizabeth plays Y: $-3p + 4(1 - p) = 4 - 7p$

If Elizabeth plays Z: $-p + 3(1 - p) = 3 - 4p$

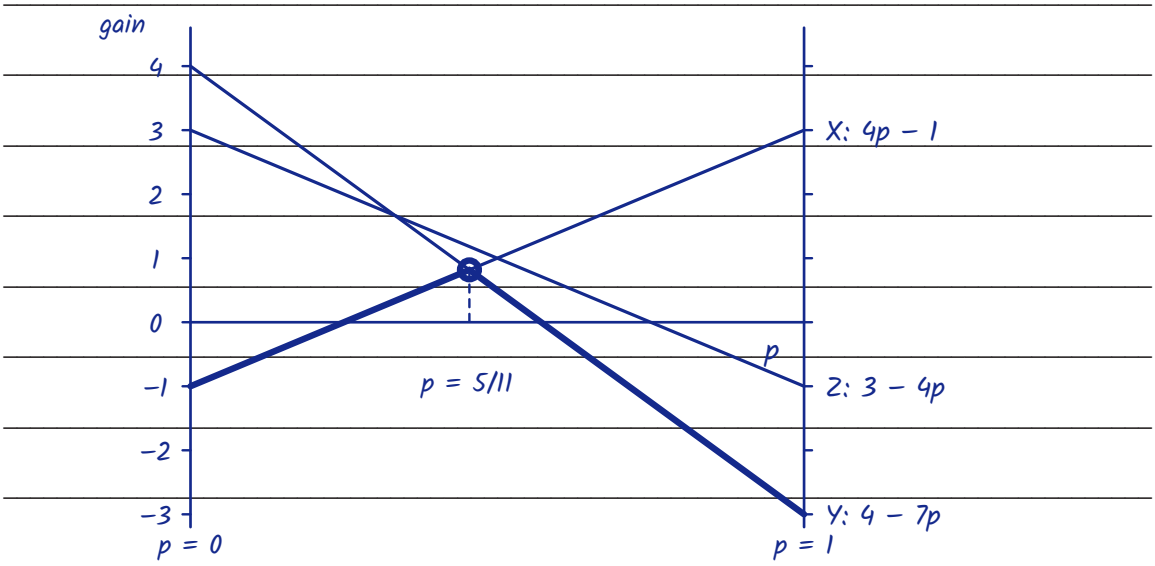
Graph of expected gains against p (on the next page).

Elizabeth will choose the strategy giving John the least gain, so John wants the highest point on the lowest boundary.

From the graph, this is where X and Y intersect:

$4p - 1 = 4 - 7p$

Do not write outside the box



11p = 5

p = 5/11 value of game = 4(5/11) − 1 = 9/11

Check Z at p = 5/11: 3 − 4(5/11) = 13/11 > 9/11, so Z is not the lowest line there and the optimal point is on X and Y.

Optimal mixed strategy for John:

play A with probability 5/11, B with probability 6/11 and C with probability 0.

Question 9 continues on the next page

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9 (b) John and Elizabeth play a different zero-sum game that does **not** have a stable solution.

This game is represented by the following pay-off matrix for Elizabeth.

		John		
	Strategy	**J_1**	**J_2**	**J_3**
Elizabeth	**E_1**	6	2	4
	E_2	3	4	3
	E_3	7	1	8

To determine her optimal mixed strategy for this game, Elizabeth begins by defining the following variables.

v = the value of the game for Elizabeth

p_1 = the probability of Elizabeth playing strategy **E_1**

p_2 = the probability of Elizabeth playing strategy **E_2**

p_3 = the probability of Elizabeth playing strategy **E_3**

Elizabeth then **starts** to formulate her situation as a linear programming problem by writing

Maximise $P = v$

subject to $p_1 + p_2 + p_3 \leq 1$

$p_1, p_2, p_3 \geq 0$

Write down the **three** remaining inequalities needed for the linear programming problem for Elizabeth's situation to be **fully** formulated.

[2 marks]

Inequality 1 $6p_1 + 3p_2 + 7p_3 \geq v$

Inequality 2 $2p_1 + 4p_2 + p_3 \geq v$

Inequality 3 $4p_1 + 3p_2 + 8p_3 \geq v$

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10 The set S contains the elements (a, b) where $a, b \in \mathbb{R}$ **except** $(0, 0)$

The group G is formed by S under the binary operation $\lozenge$ where

$$(a, b) \lozenge (c, d) = (a \times d + b \times c, b \times d - a \times c)$$

10 (a) Prove that $(0, 1)$ is the identity element of G

[2 marks]

$(a, b) \lozenge (0, 1) = (a \times 1 + b \times 0,\ b \times 1 - a \times 0)$

$= (a, b)$

$(0, 1) \lozenge (a, b) = (0 \times b + 1 \times a,\ 1 \times b - 0 \times a)$

$= (a, b)$

So $(a, b) \lozenge (0, 1) = (a, b) = (0, 1) \lozenge (a, b)$ for every (a, b) in G,

and $(0, 1)$ is in S (it is not $(0, 0)$),

$\therefore (0, 1)$ is the identity element of G.

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10 (b) It is given that $(a, b) \in G$

Find, in terms of a and b, the inverse of (a, b) in G

[3 marks]

Let the inverse of (a, b) be (c, d), so $(a, b) \diamond (c, d) = (0, 1)$

$(ad + bc,\ bd - ac) = (0, 1)$

$ad + bc = 0 \quad (1)$ $\qquad bd - ac = 1 \quad (2)$

$a \times (1)$: $a^2d + abc = 0$ $\qquad b \times (2)$: $b^2d - abc = b$

Add: $(a^2 + b^2)d = b \Rightarrow d = b/(a^2 + b^2)$

$(a^2 + b^2 \neq 0$ since $(a, b) \neq (0, 0))$

$b \times (1)$: $abd + b^2c = 0$ $\qquad a \times (2)$: $abd - a^2c = a$

Subtract: $(a^2 + b^2)c = -a \Rightarrow c = -a/(a^2 + b^2)$

Inverse of (a, b) is $\left(-\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2} \right)$

END OF QUESTIONS

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