AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 1

Thursday 22 May 2025 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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TOTAL	

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Answer **all** questions in the spaces provided.

1 The equation $x^2 + 4x + c = 0$ has non-real roots.

Find the range of possible values of c

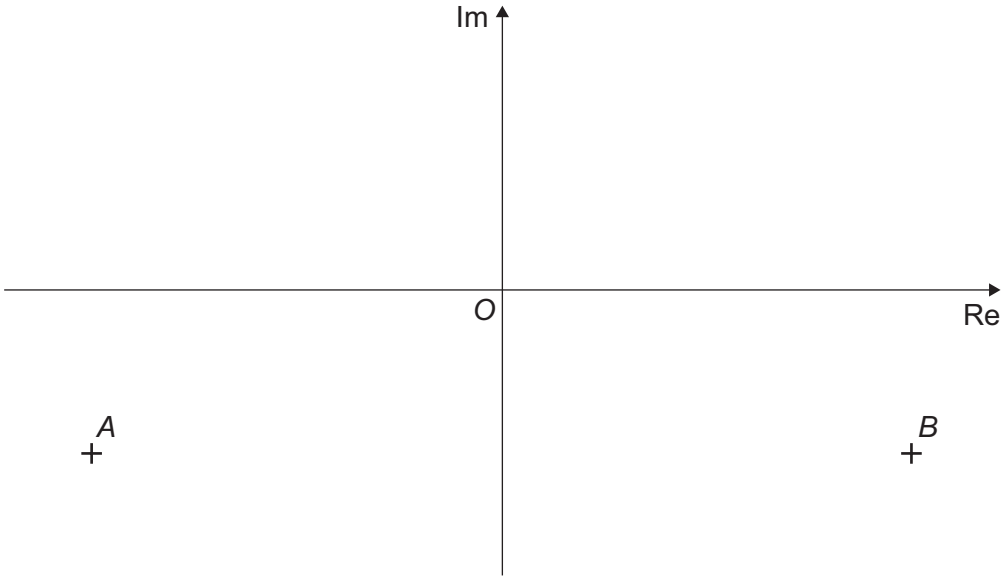
Circle your answer.

[1 mark]

$c < 4$ $\quad$ $c \leq 4$ $\quad$ $c \geq 4$ $\quad$ (circled) $c > 4$

2 The point A on the Argand diagram represents the complex number z

The point B is the reflection in the imaginary axis of the point A



Which complex number does the point B represent?

Circle your answer.

[1 mark]

$-z$ $\quad$ z^* $\quad$ (circled) $-z^*$ $\quad$ $\frac{1}{z}$

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3 Which **one** of the following expressions **cannot** be evaluated by l'Hôpital's rule?

Circle your answer.

[1 mark]

$\lim_{x\to 0}\left(\frac{\sqrt{x}}{\cos x}\right)$ $\lim_{x\to 0}\left(\frac{x^2}{\sin x}\right)$ $\lim_{x\to 0}\left(\frac{\tanh x}{2x}\right)$ $\lim_{x\to 0}\left(\frac{e^x-1}{\ln(1+x)}\right)$

4 Which **one** of the following statements is always correct?

Tick (✓) **one** box.

[1 mark]

$\sinh^2 x = \frac{1}{2}(1-\cosh 2x)$ ☐

$\operatorname{sech}^2 x + \tanh^2 x = 1$ ☑

$\operatorname{cosech}^2 x + \coth^2 x = 1$ ☐

$\cosh^2 x = \frac{1}{2}(\cosh 2x - 1)$ ☐

Turn over for the next question

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5 The complex number z_1 has argument $\tan^{-1}\left(\frac{1}{5}\right)$

The complex number $z_2 = -2 - 4i$

Find $\arg\left(\frac{z_1}{z_2}\right)$

Give your answer as a number in the range $-\pi < \alpha < \pi$, to **two** decimal places.

[3 marks]

$\arg z_1 = \tan^{-1}\frac{1}{5} = 0.19739...$

$z_2 = -2 - 4i$ lies in the third quadrant, so

$\arg z_2 = -\pi + \tan^{-1}\frac{4}{2} = -\pi + 1.10714... = -2.03444...$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

$= 0.19739... - (-2.03444...)$

$= 2.23183...$ (which is in the range $-\pi < \alpha < \pi$)

$\arg\left(\frac{z_1}{z_2}\right) = 2.23$ (2 d.p.)

Do not write outside the box

6 Use Simpson's rule with 5 ordinates to find an approximation to

$$\int_0^2 \frac{1}{\sqrt{1+x^4}}\,\mathrm{d}x$$

Give your answer to **three** decimal places.

[3 marks]

$h = (2 - 0) \div 4 = 0.5$

x	0	0.5	1	1.5	2
y	1	0.970142...	0.707106...	0.406138...	0.242535...

$\int_0^2 \frac{1}{\sqrt{(1+x^4)}}\,dx \approx \frac{0.5}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$

$= \frac{1}{6}[1 + 0.242535... + 4(0.970142... + 0.406138...)$

$+ 2(0.707106...)]$

$= \frac{1}{6} \times 8.161873...$

$= 1.360312...$

≈ 1.360 (3 d.p.)

Turn over for the next question

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7 The cubic equation

$$5z^3 + 4z^2 - z + 3 = 0$$

has roots α, β and γ

Find an equation, with integer coefficients, that has roots $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$

[4 marks]

Let $w = 2z - 1$, so $z = \frac{w+1}{2}$

$5(\frac{w+1}{2})^3 + 4(\frac{w+1}{2})^2 - \frac{w+1}{2} + 3 = 0$

$\frac{5(w+1)^3}{8} + (w+1)^2 - \frac{w+1}{2} + 3 = 0$

$\times 8$: $5(w+1)^3 + 8(w+1)^2 - 4(w+1) + 24 = 0$

$5(w^3 + 3w^2 + 3w + 1) + 8(w^2 + 2w + 1) - 4w - 4 + 24 = 0$

$5w^3 + 15w^2 + 15w + 5 + 8w^2 + 16w + 8 - 4w + 20 = 0$

$5w^3 + 23w^2 + 27w + 33 = 0$

Equation with roots $2\alpha - 1$, $2\beta - 1$, $2\gamma - 1$:

$5w^3 + 23w^2 + 27w + 33 = 0$

Do not write outside the box

8 The curve C_1 has equation

$$\frac{x^2}{4}+\frac{y^2}{25}=1$$

The curve C_1 is translated by the vector $\begin{bmatrix}1\\3\end{bmatrix}$ to give the curve C_2

Find the coordinates of the points where the curve C_2 intersects the x-axis.

[3 marks]

C_2: $\frac{(x-1)^2}{4}+\frac{(y-3)^2}{25}=1$

Meets the x-axis when $y = 0$:

$\frac{(x-1)^2}{4}+\frac{(0-3)^2}{25}=1$

$\frac{(x-1)^2}{4}=1-\frac{9}{25}=\frac{16}{25}$

$(x-1)^2=\frac{64}{25}$

$x-1=\pm\frac{8}{5}$

$x=\frac{13}{5}$ or $x=-\frac{3}{5}$

Points: $(\frac{13}{5}, 0)$ and $(-\frac{3}{5}, 0)$

Turn over for the next question

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ANSWER IN THE SPACES PROVIDED

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9 The vectors **a** and **b** are such that

$|\mathbf{a} \times \mathbf{b}| = 12$

$\mathbf{a} \bullet \mathbf{b} = -\sqrt{3}$

$|\mathbf{a}| = 7$

Find the value of $|\mathbf{b}|$

[4 marks]

Let θ be the angle between a and b.

$a \cdot b = |a||b|\cos\theta$: $\quad -\sqrt{3} = 7|b|\cos\theta$

$|a \times b| = |a||b|\sin\theta$: $\quad 12 = 7|b|\sin\theta$

$\cos\theta = \frac{-\sqrt{3}}{7|b|}$, $\quad \sin\theta = \frac{12}{7|b|}$

Using $\cos^2\theta + \sin^2\theta = 1$:

$\frac{3}{49|b|^2} + \frac{144}{49|b|^2} = 1$

$49|b|^2 = 147$

$|b|^2 = 3$

$|b| = \sqrt{3}$ $\quad$ (since $|b| > 0$)

Turn over for the next question

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10 Astrid is solving this mathematics problem:

> The series S_n is defined by
>
> $$S_n = 2 + 4 + 6 + ... + 2n \quad (n \in \mathbb{Z}, n \geq 1)$$
>
> Prove by induction that
>
> $$S_n = n(n+1)$$

Astrid's solution is as follows:

> Assume the result is true for $n = k$
>
> Then
>
> $S_k = k(k+1)$
>
> $S_{k+1} = S_k + 2(k+1)$
>
> $S_{k+1} = k(k+1) + 2(k+1)$
>
> $S_{k+1} = (k+2)(k+1)$
>
> $S_{k+1} = (k+1)((k+1)+1)$
>
> So the result is also true for $n = k+1$
>
> The result is true for $n = 1$
>
> It is true for $n = k$, and also true for $n = k + 1$
>
> Hence, by induction $S_n = n(n+1)$ for all integers $n \geq 1$

10 (a) (i) Chloe says that Astrid missed out an essential part of the proof, which could have been written at the start.

Explain what Astrid missed out.

[1 mark]

Astrid did not show that the result is true for n = 1 (the base case) before the inductive step.

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10 (a) (ii) Write down the working that Astrid missed out.

[1 mark]

When $n = 1$: $S_1 = 2$ (the first term)

and $n(n + 1) = 1 \times (1 + 1) = 2$, so the result is true for $n = 1$

10 (b) (i) One statement in the last three lines of Astrid's solution is written incorrectly.

Which statement is written incorrectly?

Tick (✓) **one** box.

[1 mark]

The result is true for $n = 1$	☐
It is true for $n = k$, and also true for $n = k + 1$	✓
Hence, by induction $S_n = n(n+1)$ for all integers $n \geq 1$	☐

10 (b) (ii) Write out a correct statement which should replace the incorrect statement identified in part **(b)(i)**

[1 mark]

If the result is true for $n = k$, then it is also true for

$n = k + 1$

Turn over for the next question

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11 The function f is defined by

$$f(x) = \frac{1}{1+e^x} \qquad (x \in \mathbb{R})$$

11 (a) Show that

$$f''(x) = \frac{-e^x + e^{2x}}{(1+e^x)^3}$$

[3 marks]

$f(x) = (1 + e^x)^{-1}$

$f'(x) = -(1 + e^x)^{-2} \times e^x = \frac{-e^x}{(1 + e^x)^2}$

Product rule on $f'(x) = -e^x(1 + e^x)^{-2}$:

$f''(x) = -e^x(1 + e^x)^{-2} + (-e^x)(-2)(1 + e^x)^{-3}(e^x)$

$= -e^x(1 + e^x)^{-2} + 2e^{2x}(1 + e^x)^{-3}$

$= \frac{-e^x(1 + e^x) + 2e^{2x}}{(1 + e^x)^3}$

$= \frac{-e^x - e^{2x} + 2e^{2x}}{(1 + e^x)^3}$

$f''(x) = \frac{-e^x + e^{2x}}{(1 + e^x)^3}$ as required

Do not write outside the box

11 (b) Hence, find the Maclaurin expansion of $f(x)$ up to and including the term in x^3

[4 marks]

$f(0) = \frac{1}{1+1} = \frac{1}{2}$

$f'(0) = \frac{-1}{(1+1)^2} = -\frac{1}{4}$

$f''(0) = \frac{-1+1}{(1+1)^3} = 0$

$f'(x) = (e^{2x} - e^x)(1+e^x)^{-3}$

$f'''(x) = (2e^{2x} - e^x)(1+e^x)^{-3} - 3e^x(e^{2x} - e^x)(1+e^x)^{-4}$

$f'''(0) = \frac{2-1}{2^3} - \frac{3(1)(0)}{2^4} = \frac{1}{8}$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$f(x) = \frac{1}{2} - \frac{1}{4}x + 0x^2 + \frac{1}{8} \times \frac{x^3}{6} + \ldots$

$f(x) \approx \frac{1}{2} - \frac{1}{4}x + \frac{1}{48}x^3$

Turn over ▶

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12 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{M} = \frac{1}{20}\begin{bmatrix} 19 & 3 \\ 3 & 11 \end{bmatrix}$$

[5 marks]

$\det(M - \lambda I) = 0$: $\left(\frac{19}{20} - \lambda\right)\left(\frac{11}{20} - \lambda\right) - \frac{9}{400} = 0$

$\lambda^2 - \frac{3}{2}\lambda + \frac{209}{400} - \frac{9}{400} = 0$

$2\lambda^2 - 3\lambda + 1 = 0$

$(2\lambda - 1)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = 1 \text{ or } \lambda = 0.5$

$\lambda = 1$: $\frac{1}{20}\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -x + 3y = 0$

$x = 3y$, eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = 0.5$: $\frac{1}{20}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 3x + y = 0$

$y = -3x$, eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Eigenvalue 1, eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$; eigenvalue 0.5, eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

12 (b) State, with a reason, the Cartesian equation of the line of invariant points of the matrix **M**

[2 marks]

Invariant points satisfy $Mv = v$, i.e. eigenvalue $\lambda = 1$,

so the line of invariant points is in the direction $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Line of invariant points: $y = \frac{1}{3}x$

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12 (c) Find matrices **U**, **D** and $\mathbf{U}^{-1}$, such that **D** is diagonal and $\mathbf{M} = \mathbf{UDU}^{-1}$

[3 marks]

$U = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$

$\det U = 3 \times 3 - (-1)(1) = 10$

$U^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$ $\quad$ so $M = UDU^{-1}$

12(d) Hence, find the matrix **L** such that $\mathbf{M}^n \to \mathbf{L}$ as $n \to \infty$

[3 marks]

$M^n = UD^nU^{-1}$

$M^n = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 0.5^n \end{bmatrix}\frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

As $n \to \infty$, $1^n = 1$ and $0.5^n \to 0$, so $D^n \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$L = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

$L = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$

Turn over for the next question

Do not write outside the box

13 The function f is defined by

$$f(z) = 4z^3 + rz^2 + 92z + s$$

where r and s are real numbers.

One of the roots of the equation $f(z) = 0$ is $-4 + 3i$

13 (a) Find the other two roots of the equation $f(z) = 0$

[4 marks]

The coefficients are real, so the complex roots occur in a conjugate pair: another root is $-4 - 3i$

Let the third root be γ.

Sum of pairwise products: $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{92}{4} = 23$

$(-4 + 3i)(-4 - 3i) + \gamma(-4 + 3i) + \gamma(-4 - 3i) = 23$

$25 - 8\gamma = 23$

$\gamma = \frac{1}{4}$

Other two roots: $-4 - 3i$ and $\frac{1}{4}$

Do not write outside the box

13 (b) Find the value of r and the value of s

[3 marks]

Sum of roots $= -\frac{r}{4}$:

$(-4 + 3i) + (-4 - 3i) + \frac{1}{4} = -\frac{r}{4}$

$-\frac{31}{4} = -\frac{r}{4} \quad \Rightarrow \quad r = 31$

Product of roots $= -\frac{s}{4}$:

$(-4 + 3i)(-4 - 3i) \times \frac{1}{4} = -\frac{s}{4}$

$\frac{25}{4} = -\frac{s}{4} \quad \Rightarrow \quad s = -25$

$r = 31, \quad s = -25$

Turn over for the next question

Do not write outside the box

14 (a) Using the exponential definitions of sinh x and cosh x, prove that

$$\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

[3 marks]

Let $y = \coth^{-1}(x)$, so $x = \coth y = \frac{\cosh y}{\sinh y}$

$x = \frac{\frac{1}{2}(e^y + e^{-y})}{\frac{1}{2}(e^y - e^{-y})} = \frac{e^y + e^{-y}}{e^y - e^{-y}} = \frac{e^{2y} + 1}{e^{2y} - 1}$

$x(e^{2y} - 1) = e^{2y} + 1$

$xe^{2y} - x = e^{2y} + 1$

$e^{2y}(x - 1) = x + 1$

$e^{2y} = \frac{x + 1}{x - 1}$

$2y = \ln\left(\frac{x + 1}{x - 1}\right)$

$y = \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right)$

$\therefore \quad \coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right)$ as required

Do not write outside the box

14 (b) Hence, solve the equation

$$\coth^{-1}(x) = -\ln 5$$

Give your answer in an exact form.

[3 marks]

$\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) = -\ln 5$

$\ln\left(\frac{x+1}{x-1}\right) = -2\ln 5 = \ln\left(\frac{1}{25}\right)$

$\frac{x+1}{x-1} = \frac{1}{25}$

$25(x+1) = x-1$

$25x + 25 = x - 1$

$24x = -26$

$x = -\frac{13}{12}$

Turn over for the next question

Do not write outside the box

15 Three planes have equations

$$\begin{aligned} x + 2y - z &= 9 \\ x - 3y + 3z &= t \\ 3x + y + z &= 4t \end{aligned}$$

where t is a constant.

The planes meet along a line of intersection.

15 (a) Find the value of t

[4 marks]

$x + 2y - z = 9$ (1)

$x - 3y + 3z = t$ (2)

$3x + y + z = 4t$ (3)

(2) − (1): $-5y + 4z = t - 9$

(3) − 3 × (1): $-5y + 4z = 4t - 27$

The planes meet in a line, so these two equations must be the same (consistent): $t - 9 = 4t - 27$

$3t = 18$

$t = 6$

Do not write outside the box

15 (b) Find a vector equation of the line of intersection.

Fully justify your answer.

[5 marks]

With $t = 6$: $x + 2y - z = 9$, $x - 3y + 3z = 6$, $3x + y + z = 24$

Let $x = 0$: $2y - z = 9$ and $-3y + 3z = 6 \Rightarrow -y + z = 2$

Adding: $y = 11$, so $z = 13$

Check in plane (3): $3(0) + 11 + 13 = 24$ ✓

so A(0, 11, 13) lies on all three planes.

Direction: $n_1 \times n_2 = (i + 2j - k) \times (i - 3j + 3k)$

$= (6 - 3)i - (3 + 1)j + (-3 - 2)k = 3i - 4j - 5k$

Check with n_3: $(3i + j + k)\cdot(3i - 4j - 5k) = 9 - 4 - 5 = 0$

So $3i - 4j - 5k$ is perpendicular to all three normals, i.e. parallel

to all three planes, and the line through A lies in all three planes.

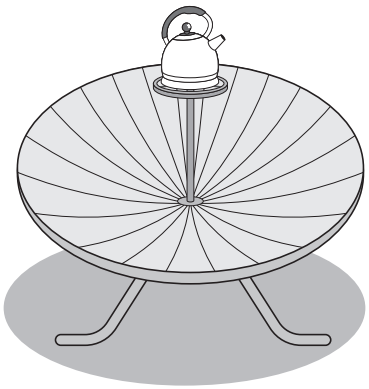
Line of intersection:

$r = 11j + 13k + \lambda(3i - 4j - 5k)$

Turn over for the next question

Do not write outside the box

16 The picture shows a solar cooker. Part of the solar cooker is a parabolic dish.



The shape of the internal surface of the dish is formed by rotating the part of the parabola $y^2 = 0.8x$ between $x = 0$ and $x = 0.25$ through 2π radians about the x-axis, where x and y are measured in metres.

Use integration to show that the internal surface area of the dish, to **three** decimal places, is 0.796 square metres.

Fully justify your answer.

[7 marks]

$y^2 = 0.8x \Rightarrow 2y\frac{dy}{dx} = 0.8 \Rightarrow \frac{dy}{dx} = \frac{0.4}{y}$

$(\frac{dy}{dx})^2 = \frac{0.16}{y^2} = \frac{0.16}{0.8x} = \frac{0.2}{x}$

$S = 2\pi \int_0^{0.25} y\sqrt{(1 + (\frac{dy}{dx})^2)}\, dx$

$= 2\pi \int_0^{0.25} \sqrt{(0.8x)}\sqrt{(1 + \frac{0.2}{x})}\, dx$

$= 2\pi \int_0^{0.25} \sqrt{(0.8x)}\sqrt{(\frac{x + 0.2}{x})}\, dx$

$= 2\pi\sqrt{0.8} \int_0^{0.25} (x + 0.2)^{1/2}\, dx$

$= 2\pi\sqrt{0.8}\, [\frac{2}{3}(x + 0.2)^{3/2}]_0^{0.25}$

$= \frac{4\pi\sqrt{0.8}}{3}(0.45^{3/2} - 0.2^{3/2})$

$= \frac{4\pi\sqrt{0.8}}{3}(0.301869... - 0.089442...)$

$= \frac{19\pi}{75} = 0.795870...$

Internal surface area = 0.796 m^2 (3 d.p.) as required

Do not write outside the box

Turn over for the next question

Do not write outside the box

17 **In this question use $g = 10$ m s^{-2}**

A particle P of mass 0.6 kg is attached to one end of each of two light elastic strings, AP and BP

The other ends of the strings, A and B, are attached to fixed points which are 7 metres apart, with A vertically above B

The natural length of the string AP is 2 metres.

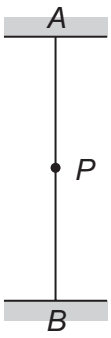
When the extension of the string AP is e metres, the tension in the string AP is $5e$ newtons.

The natural length of the string BP is 3 metres.

When the extension of the string BP is e metres, the tension in the string BP is $3e$ newtons.

The whole system is in a large tub of oil.

The diagram shows the particle P, the strings and the points A and B



The particle P is held at the point between A and B which is 0.5 metres vertically below its equilibrium position.

The particle is then released from rest.

During the subsequent motion the oil causes a resistive force of magnitude $\frac{4}{\sqrt{5}}v$ newtons to act on the particle, where v m s^{-1} is the speed of the particle.

At time t seconds after P is released, its displacement towards B from its equilibrium position is x metres.

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17 (a) Show that during the subsequent motion the particle satisfies the differential equation

$$0.6\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{4}{\sqrt{5}}\frac{\mathrm{d}x}{\mathrm{d}t}+8x=0$$

Fully justify your answer.

[5 marks]

Equilibrium: extensions e_A, e_B with $e_A + e_B = 7 - 2 - 3 = 2$

Resolving vertically: $5e_A = 3e_B + 0.6 \times 10$

$5e_A = 3(2 - e_A) + 6 \quad \Rightarrow \quad 8e_A = 12$

$e_A = 1.5$ m, $\quad e_B = 0.5$ m

At displacement x (towards B) from equilibrium:

tension in AP $= 5(1.5 + x)$ up, tension in BP $= 3(0.5 - x)$ down

resistance $\frac{4}{\sqrt{5}}v$ opposes the motion, i.e. $-\frac{4}{\sqrt{5}}\frac{dx}{dt}$ in the x direction

N2L, taking downwards (towards B) as positive:

$0.6\frac{d^2x}{dt^2} = 0.6 \times 10 + 3(0.5 - x) - 5(1.5 + x) - \frac{4}{\sqrt{5}}\frac{dx}{dt}$

$0.6\frac{d^2x}{dt^2} = 6 + 1.5 - 3x - 7.5 - 5x - \frac{4}{\sqrt{5}}\frac{dx}{dt}$

$0.6\frac{d^2x}{dt^2} = -8x - \frac{4}{\sqrt{5}}\frac{dx}{dt}$

$0.6\frac{d^2x}{dt^2} + \frac{4}{\sqrt{5}}\frac{dx}{dt} + 8x = 0$ as required

(BP stays taut as $x \leq 0.5$ throughout the damped motion.)

Question 17 continues on the next page

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17 (b) Find x in terms of t, giving your answer in exact form.

[6 marks]

Auxiliary equation: $0.6\lambda^2 + \frac{4}{\sqrt{5}}\lambda + 8 = 0$

$\lambda = \frac{-4/\sqrt{5} \pm \sqrt{(16/5 - 4 \times 0.6 \times 8)}}{2 \times 0.6} = \frac{-4/\sqrt{5} \pm \sqrt{(-16)}}{1.2}$

$\lambda = \frac{-4/\sqrt{5} \pm 4i}{1.2} = -\frac{2\sqrt{5}}{3} \pm \frac{10}{3}i$

$x = e^{-2\sqrt{5}t/3}\left(A \cos \frac{10t}{3} + B \sin \frac{10t}{3}\right)$

$t = 0,\ x = 0.5$: $0.5 = e^0(A + 0) \Rightarrow A = 0.5$

Product rule:

$\frac{dx}{dt} = -\frac{2\sqrt{5}}{3}e^{-2\sqrt{5}t/3}\left(A \cos \frac{10t}{3} + B \sin \frac{10t}{3}\right)$

$+ e^{-2\sqrt{5}t/3}\left(-\frac{10}{3}A \sin \frac{10t}{3} + \frac{10}{3}B \cos \frac{10t}{3}\right)$

Released from rest: $t = 0,\ \frac{dx}{dt} = 0$

$0 = -\frac{2\sqrt{5}}{3}A + \frac{10}{3}B$

$B = \frac{2\sqrt{5}}{10}A = \frac{\sqrt{5}}{5} \times 0.5 = \frac{\sqrt{5}}{10}$

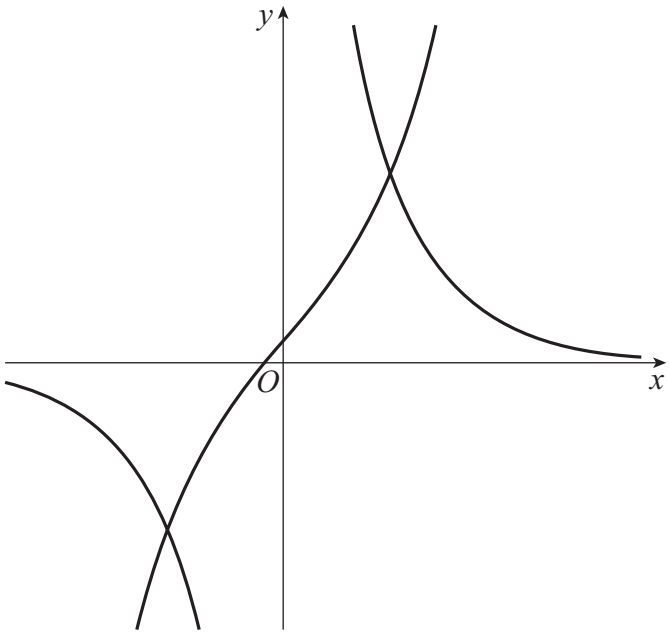
$x = e^{-2\sqrt{5}t/3}\left(\frac{1}{2} \cos \frac{10t}{3} + \frac{\sqrt{5}}{10} \sin \frac{10t}{3}\right)$

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18 The diagram shows part of the graph of $y = 15\,\text{cosech}\,x$ and part of the graph of $y = 4\sinh x + \frac{1}{2}$



18 (a) Solve the inequality

$$15\,\text{cosech}\,x < 4\sinh x + \frac{1}{2}$$

Give your answer in logarithmic form.

[4 marks]

Let $s = \sinh x$. Curves meet: $\frac{15}{s} = 4s + \frac{1}{2} \Rightarrow 8s^2 + s - 30 = 0$

$(8s - 15)(s + 2) = 0 \Rightarrow \sinh x = \frac{15}{8}$ or $\sinh x = -2$

$x = \text{arsinh}\frac{15}{8} = \ln\left(\frac{15}{8} + \sqrt{\left(\frac{225}{64} + 1\right)}\right) = \ln\left(\frac{15}{8} + \frac{17}{8}\right) = \ln 4$

$x = \text{arsinh}(-2) = \ln(-2 + \sqrt{(4 + 1)}) = \ln(\sqrt{5} - 2)$

$x > 0$ $(s > 0)$: $15 < 4s^2 + \frac{s}{2} \Leftrightarrow 8s^2 + s - 30 > 0 \Leftrightarrow s > \frac{15}{8}$

$x < 0$ $(s < 0)$: $15 > 4s^2 + \frac{s}{2} \Leftrightarrow 8s^2 + s - 30 < 0 \Leftrightarrow -2 < s < 0$

$\therefore \quad \ln(\sqrt{5} - 2) < x < 0 \quad$ or $\quad x > \ln 4$

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18 (b) Given that

$$f(x) = \ln\left(\tanh\left(\frac{1}{2}x\right)\right) \qquad (x > 0)$$

Show that

$$f'(x) = \text{cosech}\, x$$

[4 marks]

$f(x) = \ln(\tanh \frac{x}{2})$

$f'(x) = \frac{½\text{sech}^2(½x)}{\tanh(½x)}$ (chain rule)

$= \frac{1}{2\cosh^2(½x)} \times \frac{\cosh(½x)}{\sinh(½x)}$

$= \frac{1}{2\sinh(½x)\cosh(½x)}$

Using $\sinh 2A = 2\sinh A \cosh A$ with $A = ½x$:

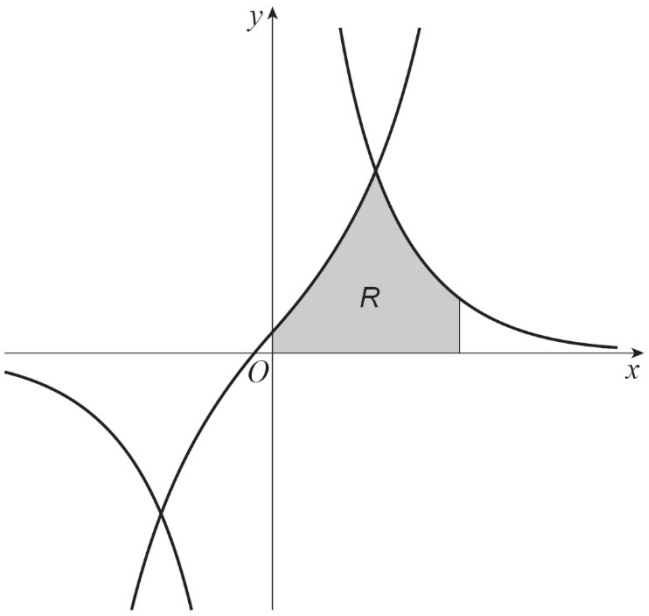
$= \frac{1}{\sinh x}$

$f'(x) = \text{cosech}\, x$ as required

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18 (c) The shaded region R is enclosed by the positive x-axis, the positive y-axis, the graph of $y = 4\sinh x + \frac{1}{2}$, the graph of $y = 15\,\text{cosech}\, x$ and the line $x = \ln 9$



Find the area of R

Give your answer in the form $\frac{p}{q} + \ln r + s \ln\left(\frac{t}{3}\right)$ where p, q, r, s and t are integers.

[7 marks]

For $x > 0$ the curves meet at $x = \ln 4$ (from part (a)).

$R = \int_0^{\ln 4}(4\sinh x + \frac{1}{2})\,dx + \int_{\ln 4}^{\ln 9} 15\,\text{cosech}\, x\,dx$

$\int_0^{\ln 4}(4\sinh x + \frac{1}{2})\,dx = [4\cosh x + \frac{1}{2}x]_0^{\ln 4}$

$\cosh(\ln 4) = \frac{1}{2}(4 + \frac{1}{4}) = \frac{17}{8}$

$= 4 \times \frac{17}{8} + \frac{1}{2}\ln 4 - (4 + 0) = \frac{17}{2} + \ln 2 - 4 = \frac{9}{2} + \ln 2$

$\int_{\ln 4}^{\ln 9} 15\,\text{cosech}\, x\,dx = 15[\ln \tanh(\frac{1}{2}x)]_{\ln 4}^{\ln 9}$ (using (b))

$\tanh(\ln a) = \frac{a - a^{-1}}{a + a^{-1}} = \frac{a^2 - 1}{a^2 + 1}$ so $\tanh(\frac{1}{2}\ln 9) = \tanh(\ln 3) = \frac{8}{10} = \frac{4}{5}$

and $\tanh(\frac{1}{2}\ln 4) = \tanh(\ln 2) = \frac{3}{5}$

$= 15(\ln\frac{4}{5} - \ln\frac{3}{5}) = 15\ln(\frac{4}{3})$

Area of $R = \frac{9}{2} + \ln 2 + 15\ln(\frac{4}{3})$

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