AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 3 Discrete

Friday 7 June 2024 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.
- You must ensure you have the other optional Question Paper/Answer Book for which you are entered (**either** Mechanics or Statistics). You will have 2 hours to complete **both** papers.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 50.

For Examiner's Use	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

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Answer **all** questions in the spaces provided.

1 Which one of the following sets forms a group under the given binary operation?

Tick (✓) **one** box.

[1 mark]

Set	Binary Operation	
{1, 2, 3}	Addition modulo 4	
{1, 2, 3}	Multiplication modulo 4	
{0, 1, 2, 3}	Addition modulo 4	✓
{0, 1, 2, 3}	Multiplication modulo 4	

2 A student is trying to find the solution to the travelling salesperson problem for a network.

They correctly find two lower bounds for the solution: 15 and 19

They also correctly find two upper bounds for the solution: 48 and 51

Based on the above information only, which of the following pairs give the best lower bound and best upper bound for the solution of this problem?

Tick (✓) **one** box.

[1 mark]

Best Lower Bound	Best Upper Bound	
15	48	
15	51	
19	48	✓
19	51	

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3 The simple-connected graph G has the adjacency matrix

	A	B	C	D
A	0	1	1	1
B	1	0	1	0
C	1	1	0	1
D	1	0	1	0

Which one of the following statements about G is true?

Tick (✓) **one** box.

[1 mark]

G is a tree ☐

G is complete ☐

G is Eulerian ☐

G is planar ☑

Turn over for the next question

Do not write outside the box

4 Daniel and Jackson play a zero-sum game.

The game is represented by the following pay-off matrix for Daniel.

Jackson

Daniel	Strategy	W	X	Y	Z
	A	3	−2	1	4
	B	5	1	−4	1
	C	2	−1	1	2
	D	−3	0	2	−1

Neither player has any strategies which can be ignored due to dominance.

4 (a) Prove that the game does not have a stable solution.

Fully justify your answer.

[3 marks]

Row minima: A −2, B −4, C −1, D −3

max(row minima) = −1

Column maxima: W 5, X 1, Y 2, Z 4

min(column maxima) = 1

max(row minima) = −1 ≠ 1 = min(column maxima)

so there is no saddle point, therefore the game

does not have a stable solution.

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4 (b) Determine the play-safe strategy for each player.

[1 mark]

Play-safe strategy for Daniel C

Play-safe strategy for Jackson X

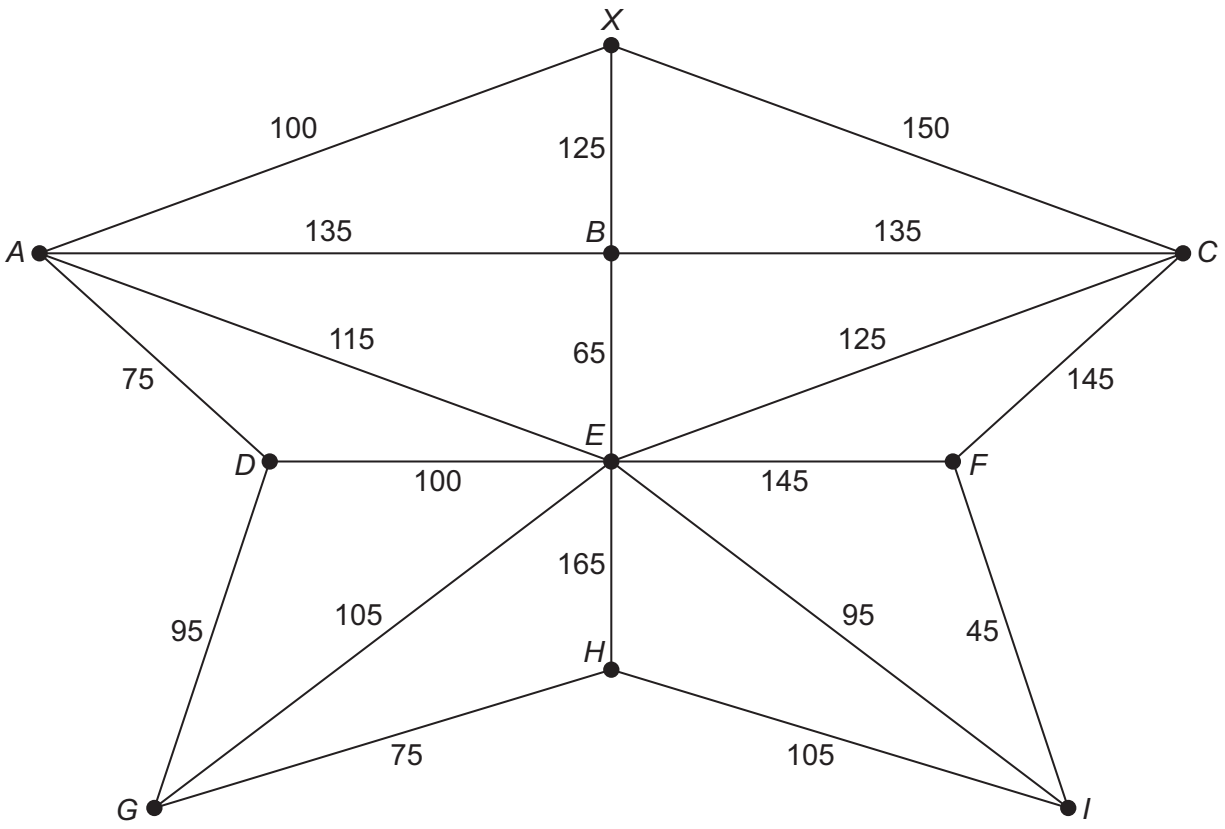
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5 The owners of a sports stadium want to install electric car charging points in each of the stadium's nine car parks.

An engineer creates a plan which requires installing electrical connections so that each car park is connected, directly or indirectly, to the stadium's main electricity power supply.

The engineer produces the network shown below, where the nodes represent the stadium's main electricity power supply *X* and the nine car parks *A*, *B*, …, *I*



Each arc represents a possible electrical connection which could be installed.

The weight on each arc represents the time, in hours, it would take to install the electrical connection. The electrical connections can only be installed one at a time.

To reduce disruption, the owners of the sports stadium want the required electrical connections to be installed in the minimum possible total time.

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5 (a) (i) Determine the electrical connections that should be installed.

[2 marks]

Minimum spanning tree, using Kruskal's algorithm:

FI 45 ✓ BE 65 ✓ AD 75 ✓ GH 75 ✓ DG 95 ✓ EI 95 ✓

DE 100 ✓ XA 100 ✓ HI 105 ✗ EG 105 ✗ AE 115 ✗

XB 125 ✗ CE 125 ✓ (9 arcs, all 10 nodes connected)

Install: XA, AD, DG, GH, DE, BE, EI, FI, CE

5 (a) (ii) Find the minimum possible total time needed to install the required electrical connections.

[1 mark]

100 + 75 + 95 + 75 + 100 + 65 + 95 + 45 + 125

= 775 hours

5 (b) Following the installation of the electrical connections, some of the car parks have an indirect connection to the stadium's main electricity power supply.

Give one limitation of this installation.

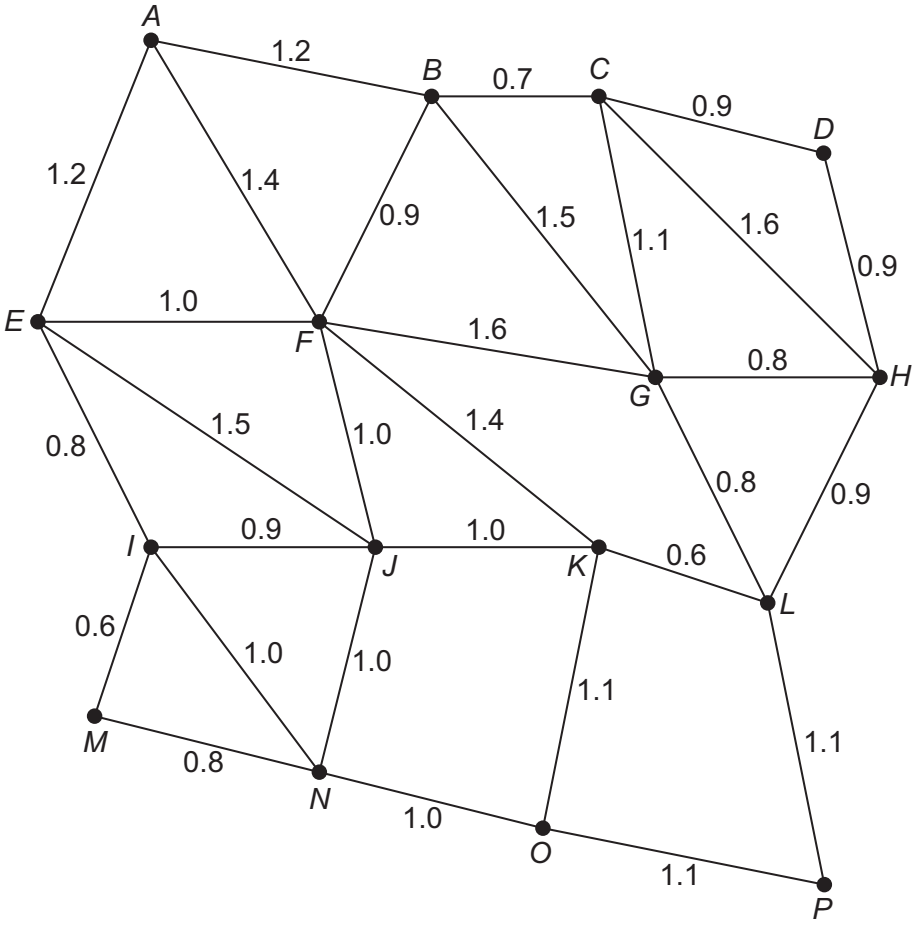
[1 mark]

If one electrical connection fails, it may result in more than one car park losing power and being unable to charge cars.

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6 A company delivers parcels to houses in a village, using a van.

The network below shows the roads in the village. Each node represents a road junction and the weight of each arc represents the length, in miles, of the road between the junctions.



The total length of all of the roads in the village is 31.4 miles.

On one particular day, the driver is due to make deliveries to at least one house on each road, so the van must travel along each road at least once. However, the driver has forgotten to add fuel to the van and it only has 4.5 litres of fuel to use to make its deliveries.

The van uses, on average, 1 litre of fuel to travel 7.8 miles along the roads of this village. Whilst making each delivery, the driver turns off the van's engine so it does not use any fuel.

Determine whether the van has enough fuel for the driver to make all of the deliveries to houses on each road of the village, starting and finishing at the same junction.

Fully justify your answer.

[6 marks]

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Every road must be travelled and the van must return to its start,

so this is a route inspection problem.

Odd degree nodes: A (3), G (5), J (5), O (3)

Shortest distances between odd nodes:

AG = 2.7 (ABG) JO = 2.0 (JNO)

AJ = 2.4 (AFJ) GO = 2.5 (GLKO)

AO = 3.9 (AFKO) GJ = 2.4 (GLKJ)

Pairings:

(AG)(JO) = 2.7 + 2.0 = 4.7 (minimum)

(AJ)(GO) = 2.4 + 2.5 = 4.9

(AO)(GJ) = 3.9 + 2.4 = 6.3

Repeat ABG and JNO, an extra 4.7 miles

Minimum total distance = 31.4 + 4.7 = 36.1 miles

Minimum fuel required = 36.1 ÷ 7.8 = 4.628... = 4.63 litres

(equivalently, 4.5 litres only covers 4.5 × 7.8 = 35.1 miles < 36.1)

4.63 > 4.5, so the van does not have enough fuel to make all of

its deliveries on every road and arrive back at the junction

it started from.

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7 (a) By considering associativity, show that the set of integers does not form a group under the binary operation of subtraction.

Fully justify your answer.

[2 marks]

Let $x = 3, y = 2, z = 1$: $(3 - 2) - 1 = 1 - 1 = 0$

$3 - (2 - 1) = 3 - 1 = 2$

As $(3 - 2) - 1 \neq 3 - (2 - 1)$, subtraction is not associative on the set of integers.

Associativity is required for all groups, therefore the set of integers does not form a group under subtraction.

7 (b) The group G is formed by the set

$$\{1, 7, 8, 11, 12, 18\}$$

under the operation of multiplication modulo 19

7 (b) (i) Complete the Cayley table for G

[3 marks]

$\times_{19}$	1	7	8	11	12	18
1	1	7	8	11	12	18
7	7	11	18	1	8	12
8	8	18	7	12	1	11
11	11	1	12	7	18	8
12	12	8	1	18	11	7
18	18	12	11	8	7	1

7 (b) (ii) State the inverse of 11 in G

[1 mark]

$11 \times 7 = 77 = 4 \times 19 + 1 \equiv 1 \pmod{19}$

Inverse of 11 is 7

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7 (c) (i) State, with a reason, the possible orders of the proper subgroups of G

[2 marks]

By Lagrange's theorem, the order of a subgroup must divide the order of the group. G has order 6, with divisors 1, 2, 3, 6

A proper subgroup cannot have order 6, so the possible orders of the proper subgroups of G are 1, 2 and 3

7 (c) (ii) Find all the proper subgroups of G

Give your answers in the form $\left(\langle g \rangle, \times_{19}\right)$ where $g \in G$

[3 marks]

Order 1: $\{1\}$

Order 2: $18^2 = 324 \equiv 1 \pmod{19}$, so $\{1, 18\}$

Order 3: $7^2 \equiv 11$, $7^3 \equiv 11 \times 7 \equiv 1$, so $\{1, 7, 11\}$ $(= \langle 11 \rangle$ also$)$

8 and 12 have order 6, so they generate G itself

Proper subgroups:

$(\langle 1 \rangle, \times_{19})$, $(\langle 18 \rangle, \times_{19})$, $(\langle 7 \rangle, \times_{19})$

7 (c) (iii) The group H is such that $G \cong H$

State a possible name for H

[1 mark]

Cyclic group of order 6 (C_6)

Turn over ▶

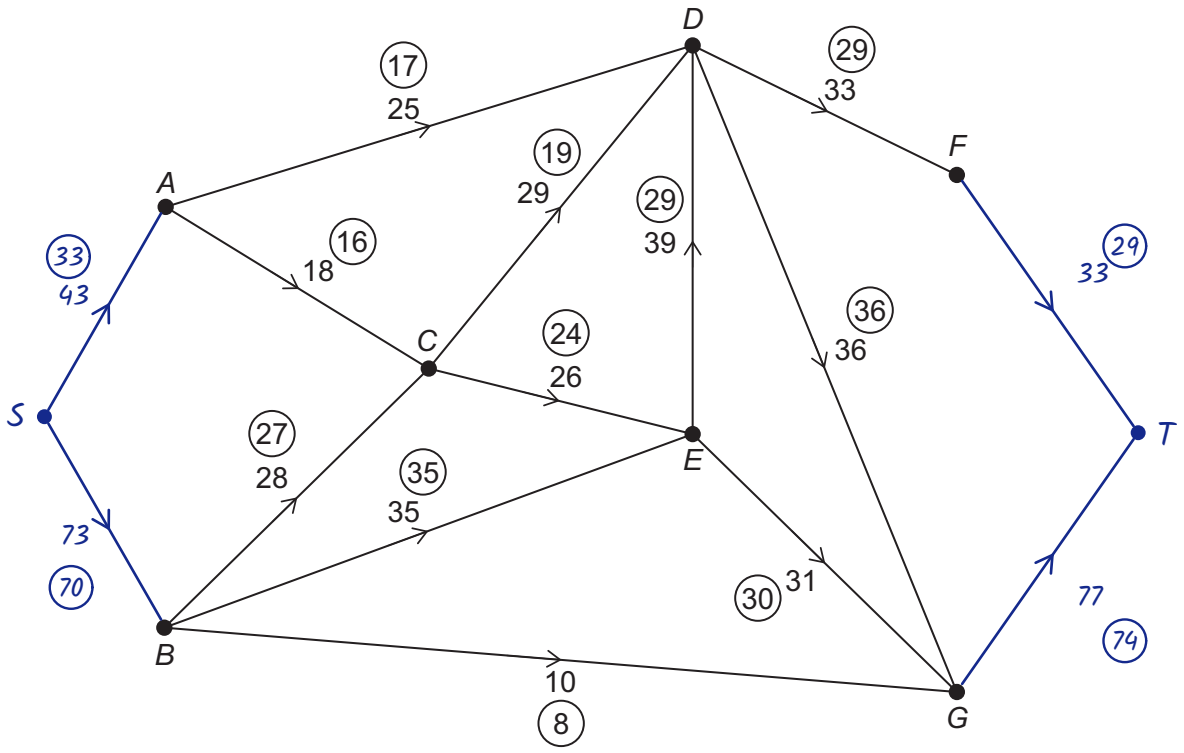
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8 **Figure 1** shows a network of water pipes.

The number on each arc represents the upper capacity for each pipe in litres per second.

The numbers in the circles represent an initial feasible flow of 103 litres per second.

Figure 1



8 (a) On **Figure 1** above, add a supersource S and a supersink T to the network.

[2 marks]

8 (b) Using flow augmentation, find the maximum flow through the network.

You must indicate any flow augmenting paths clearly in the table below.

You may use **Figure 2**, on the opposite page, in your solution.

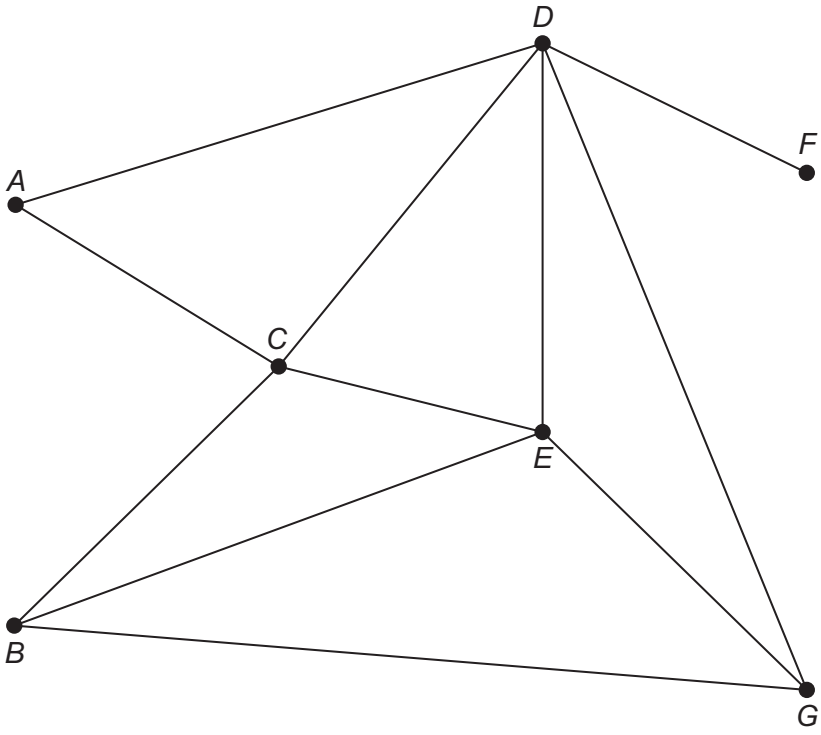
[4 marks]

Augmenting Path	Extra Flow
SADFT	4
SBGT	2
SBCEGT	1

Maximum Flow 110 litres per second

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Figure 2



8 (c) While the flow through the network is at its maximum value, the pipe *EG* develops a leak.

To repair the leak, an engineer turns off the flow of water through *EG*

The engineer claims that the maximum flow of water through the network will reduce by 31 litres per second.

Comment on the validity of the engineer's claim.

[2 marks]

Before the leak, EG is saturated (flow 31 = capacity 31).

BG, DG and DF are also all saturated, so none of the 31 litres per second through EG can be re-routed to the sinks (EG is part of the minimum cut DF + DG + EG + BG = 110).

Hence the maximum flow decreases by 31 litres per second (to 79), so the engineer's claim is correct.

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9 Janet and Samantha play a zero-sum game.

The game is represented by the following pay-off matrix for Janet.

		Samantha		
	Strategy	**S_1**	**S_2**	**S_3**
Janet	**J_1**	2	7	6
	J_2	5	5	1
	J_3	4	3	8
	J_4	1	6	4

9 (a) Explain why Janet should never play strategy **J_4**

[1 mark]

As $1 \leq 2$, $6 \leq 7$ and $4 \leq 6$, strategy J_1 dominates strategy J_4,

so Janet would always do at least as well by playing J_1

9 (b) Janet wants to maximise her winnings from the game.

She defines the following variables.

p_1 = the probability of Janet playing strategy **J_1**

p_2 = the probability of Janet playing strategy **J_2**

p_3 = the probability of Janet playing strategy **J_3**

v = the value of the game for Janet

Janet then formulates her situation as the following linear programming problem.

Maximise $P = v$

subject to
$$2p_1 + 5p_2 + 4p_3 \geq v$$
$$7p_1 + 5p_2 + 3p_3 \geq v$$
$$6p_1 + p_2 + 8p_3 \geq v$$

and
$$p_1 + p_2 + p_3 \leq 1$$
$$p_1, p_2, p_3 \geq 0$$

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9 (b) (i) Complete the initial Simplex tableau for Janet's situation in the grid below.

[2 marks]

P	v	p_1	p_2	p_3	r	s	t	u	**value**
1	−1	0	0	0	0	0	0	0	0
0	1	−2	−5	−4	1	0	0	0	0
0	1	−7	−5	−3	0	1	0	0	0
0	1	−6	−1	−8	0	0	1	0	0
0	0	1	1	1	0	0	0	1	1

9 (b) (ii) Hence, perform **one** iteration of the Simplex algorithm, showing your answer on the grid below.

[2 marks]

P	v	p_1	p_2	p_3	r	s	t	u	**value**
1	0	−2	−5	−4	1	0	0	0	0
0	1	−2	−5	−4	1	0	0	0	0
0	0	−5	0	1	−1	1	0	0	0
0	0	−4	4	−4	−1	0	1	0	0
0	0	1	1	1	0	0	0	1	1

Pivot on the v column (r row): R1 + R2, R3 − R2, R4 − R2

Question 9 continues on the next page

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9 (c) Further iterations of the Simplex algorithm are performed until an optimal solution is reached.

The grid below shows part of the final Simplex tableau.

p_1	p_2	**value**
1	0	$\frac{1}{12}$
0	1	$\frac{1}{2}$

Find the probability of Janet playing strategy $\mathbf{J_3}$ when she is playing to maximise her winnings from the game.

[1 mark]

$p_3 = 1 - p_1 - p_2 = 1 - \frac{1}{12} - \frac{1}{2} = \frac{5}{12}$

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Turn over for the next question

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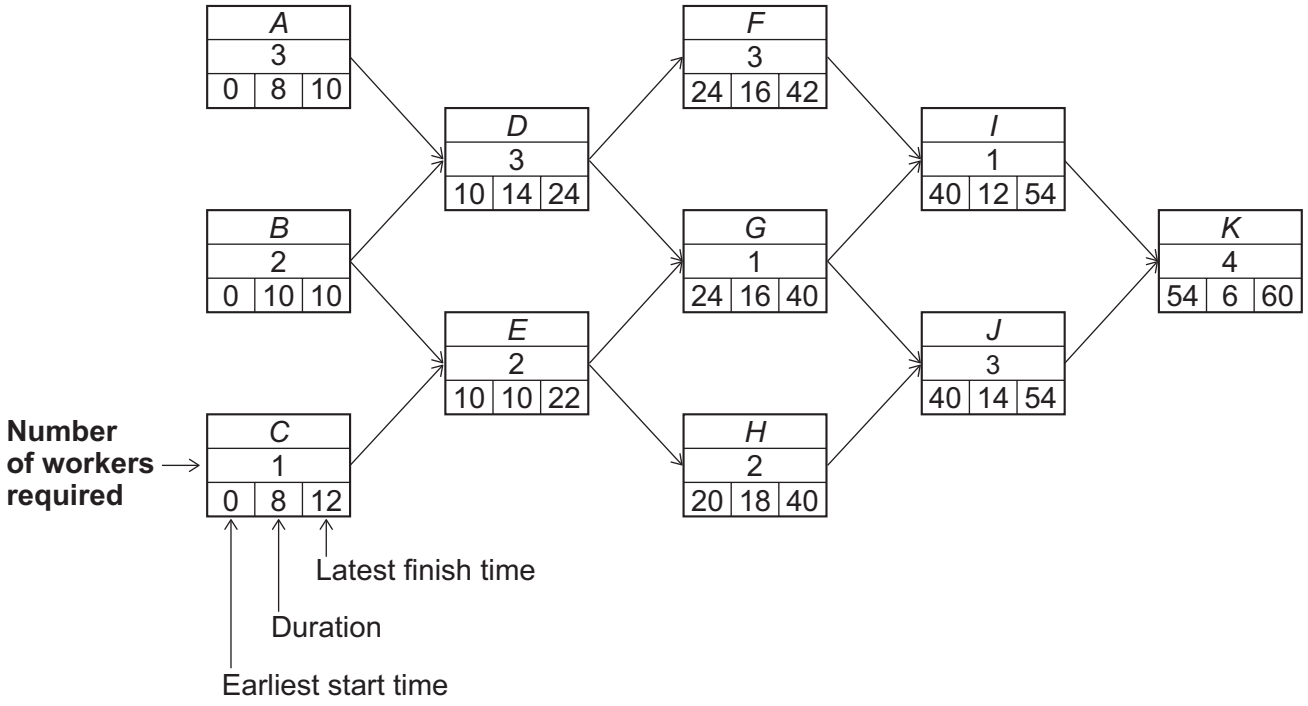
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10 A project is undertaken by Higton Engineering Ltd.

The project is broken down into 11 separate activities *A*, *B*, ..., *K*

Figure 3 below shows a completed activity network for the project, along with the earliest start time, duration, latest finish time and the number of workers required for each activity. All times and durations are given in days.

Figure 3



10 (a) Write down the critical path.

[1 mark]

BDGJK

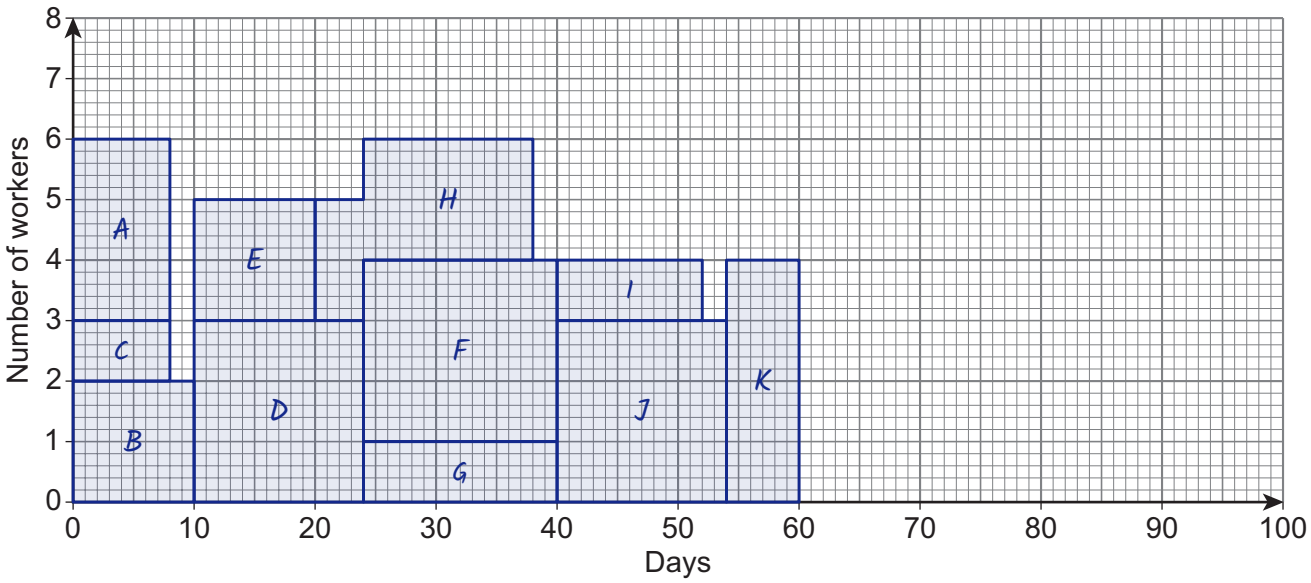
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10 (b) Using **Figure 4** below, draw a resource histogram for the project to show how the project can be completed in the minimum possible time.

Assume that each activity is to start as early as possible.

[3 marks]

Figure 4



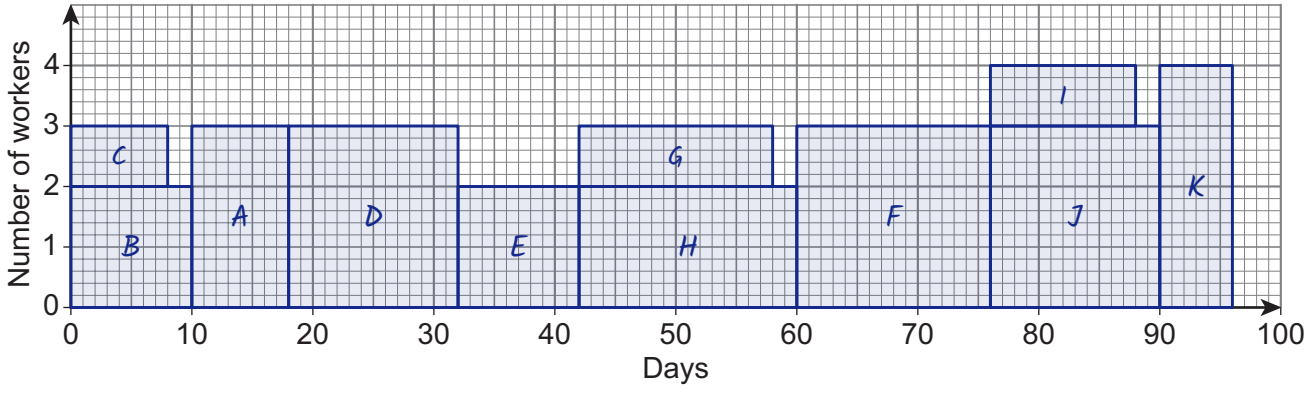
10 (c) Higton Engineering Ltd only has **four** workers available to work on the project.

Find the minimum completion time for the project.

Use **Figure 5** below in your answer.

[3 marks]

Figure 5



Minimum completion time 96 days

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