AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 2

Monday 3 June 2024 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
TOTAL	

Do not write outside the box

Answer **all** questions in the spaces provided.

1 It is given that

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ \lambda \\ -6 \end{bmatrix} = 0$$

where λ is a constant.

Find the value of λ

Circle your answer.

[1 mark]

–28 –8 (8) 28

2 The movement of a particle is described by the simple harmonic equation

$$\ddot{x} = -25x$$

where x metres is the displacement of the particle at time t seconds, and $\ddot{x}$ m s^{-2} is the acceleration of the particle.

The maximum displacement of the particle is 9 metres.

Find the maximum speed of the particle.

Circle your answer.

[1 mark]

15 m s^{-1} (45 m s^{-1}) 75 m s^{-1} 135 m s^{-1}

Do not write outside the box

3 The function g is defined by

$$g(x) = \text{sech } x \qquad (x \in \mathbb{R})$$

Which one of the following is the range of g?

Tick (✓) **one** box.

[1 mark]

$-\infty < g(x) \leq -1$ ☐

$-1 \leq g(x) < 0$ ☐

$0 < g(x) \leq 1$ ☑

$1 \leq g(x) \leq \infty$ ☐

4 The function f is a quartic function with real coefficients.

The complex number $5i$ is a root of the equation $f(x) = 0$

Which **one** of the following **must** be a factor of $f(x)$?

Circle your answer.

[1 mark]

$(x^2 - 25)$ $(x^2 - 5)$ $(x^2 + 5)$ $(x^2 + 25)$

Do not write outside the box

5 The first four terms of the series S can be written as

$$S = (1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + ...$$

5 (a) Write an expression, using $\sum$ notation, for the sum of the first n terms of S

[1 mark]

$\sum_{r=1}^{n} r(r+1)$

5 (b) Show that the sum of the first n terms of S is equal to

$$\frac{1}{3}n(n+1)(n+2)$$

[2 marks]

$\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)[(2n+1)+3]$

$= \frac{1}{6}n(n+1)(2n+4)$

$= \frac{1}{3}n(n+1)(n+2)$ as required

Do not write outside the box

6 The cubic equation

$$x^3 + 5x^2 - 4x + 2 = 0$$

has roots α, β and γ

Find a cubic equation, with integer coefficients, whose roots are 3α, 3β and 3γ

[3 marks]

Let $y = 3x$, so $x = \frac{y}{3}$

$\left(\frac{y}{3}\right)^3 + 5\left(\frac{y}{3}\right)^2 - 4\left(\frac{y}{3}\right) + 2 = 0$

$\frac{y^3}{27} + \frac{5y^2}{9} - \frac{4y}{3} + 2 = 0$

Multiply by 27: $y^3 + 15y^2 - 36y + 54 = 0$

Required cubic: $x^3 + 15x^2 - 36x + 54 = 0$

Turn over for the next question

Do not write outside the box

7 The matrices **A** and **B** are defined as follows.

$$\mathbf{A} = \begin{bmatrix} p-2 & p-1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2p-1 \\ 0 & 4-p \end{bmatrix}$$

Find the values of p such that **A** and **B** are commutative under matrix multiplication.

Fully justify your answer.

[4 marks]

$AB = \begin{bmatrix} p-2 & p-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2p-1 \\ 0 & 4-p \end{bmatrix}$

$= \begin{bmatrix} p-2 & (p-2)(2p-1)+(p-1)(4-p) \\ 0 & 4-p \end{bmatrix}$

$= \begin{bmatrix} p-2 & p^2-2 \\ 0 & 4-p \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 2p-1 \\ 0 & 4-p \end{bmatrix}\begin{bmatrix} p-2 & p-1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} p-2 & p-1+2p-1 \\ 0 & 4-p \end{bmatrix}$

$= \begin{bmatrix} p-2 & 3p-2 \\ 0 & 4-p \end{bmatrix}$

Since A and B are commutative under matrix multiplication, $AB = BA$.

The other three elements are equal for all p, so compare the top right elements: $p^2 - 2 = 3p - 2$

$p^2 - 3p = 0$

$p(p-3) = 0$

$p = 0$ or $p = 3$

Do not write outside the box

8 The vectors **a**, **b**, and **c** are such that $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{a} \times \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

Work out $(\mathbf{a} - 4\mathbf{b} + 3\mathbf{c}) \times (2\mathbf{a})$

[4 marks]

$(a - 4b + 3c) \times (2a) = 2(a \times a) - 8(b \times a) + 6(c \times a)$

$a \times a = 0, \quad b \times a = -(a \times b), \quad c \times a = -(a \times c)$

$(a - 4b + 3c) \times (2a) = 0 + 8(a \times b) - 6(a \times c)$

$= 8\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 16 \\ 8 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 18 \end{bmatrix}$

$= \begin{bmatrix} 16 \\ 8 \\ -18 \end{bmatrix}$

Do not write outside the box

9 A curve passes through the point (–2, 4.73) and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - x^2}{2x + 3y}$$

Use Euler's step by step method once, and then the midpoint formula

$$y_{r+1} = y_{r-1} + 2h\mathrm{f}(x_r, y_r), \quad x_{r+1} = x_r + h$$

once, each with a step length of 0.02, to estimate the value of y when $x = -1.96$

Give your answer to five significant figures.

[4 marks]

$x_0 = -2, \quad y_0 = 4.73, \quad h = 0.02, \quad f(x, y) = \frac{y^2 - x^2}{2x + 3y}$

Euler: $y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = 4.73 + 0.02 \times \frac{4.73^2 - (-2)^2}{2(-2) + 3(4.73)}$

$= 4.73 + 0.02 \times 1.803032\ldots = 4.766060648$

$x_1 = -1.98$

Midpoint formula: $y_2 = y_0 + 2h\, f(x_1, y_1)$

$y_2 = 4.73 + 0.04 \times \frac{4.766060648^2 - (-1.98)^2}{2(-1.98) + 3(4.766060648)}$

$= 4.73 + 0.04 \times 1.818011\ldots = 4.802720\ldots$

$x_2 = -1.96$

$y \approx 4.8027$ (5 s.f.)

Do not write outside the box

Turn over for the next question

Do not write outside the box

10 The matrix **C** is defined by

$$\mathbf{C} = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$$

Prove that the transformation represented by **C** has no invariant lines of the form $y = kx$

[4 marks]

An invariant line $y = kx$: points (x, kx) map to (x', y') with $y' = kx'$

$$\begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ kx \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$x' = 3x + 2kx, \qquad y' = -4x + 5kx$

$y' = kx'$: $\quad -4x + 5kx = k(3x + 2kx)$

$-4 + 5k = 3k + 2k^2$

$2k^2 - 2k + 4 = 0$

Discriminant $= (-2)^2 - 4(2)(4) = 4 - 32 = -28 < 0$

so the quadratic has no real roots: there is no real value of k. Hence the transformation represented by C has no invariant lines of the form $y = kx$.

Do not write outside the box

11 Latifa and Sam are studying polynomial equations of degree greater than 2, with real coefficients and no repeated roots.

Latifa says that if such an equation has exactly one real root, it must be of degree 3

Sam says that this is not correct.

State, giving reasons, whether Latifa or Sam is right.

[3 marks]

Sam is right. A polynomial with real coefficients has its complex (non-real) roots in conjugate pairs, so an equation of odd degree n can have exactly one real root and n − 1 non-real roots.

Counter example: $z^5 - 1 = 0$ has degree 5 and no repeated roots. Its roots are $z = 1$ and the four non-real roots $e^{\pm 2\pi i/5}$, $e^{\pm 4\pi i/5}$, so it has exactly one real root but is not of degree 3.

So Latifa is wrong and Sam is right.

Turn over for the next question

Do not write outside the box

12 The transformation S is represented by the matrix $\mathbf{M} = \begin{bmatrix} 1 & -6 \\ 2 & 7 \end{bmatrix}$

The transformation T is a reflection in the line $y = x\sqrt{3}$ and is represented by the matrix **N**

The point $P(x, y)$ is transformed first by S, then by T

The result of these transformations is the point $Q(3, 8)$

Find the coordinates of P

Give your answers to three decimal places.

[5 marks]

T is a reflection in $y = x\sqrt{3} = x \tan 60°$, so $\theta = 60°$

$N = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 120° & \sin 120° \\ \sin 120° & -\cos 120° \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$

P is transformed first by S then by T: $NM\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 6y \\ 2x + 7y \end{bmatrix}$

$NM\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(x - 6y) + \frac{1}{2}\sqrt{3}(2x + 7y) \\ \frac{1}{2}\sqrt{3}(x - 6y) + \frac{1}{2}(2x + 7y) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$(\sqrt{3} - \frac{1}{2})x + (3 + \frac{7\sqrt{3}}{2})y = 3$

$(\frac{\sqrt{3}}{2} + 1)x + (\frac{7}{2} - 3\sqrt{3})y = 8$

$1.232050808x + 9.062177826y = 3$

$1.866025404x - 1.696152423y = 8$

Solving simultaneously:

$x = 4.083467...,\quad y = -0.224122...$

P is the point $(4.083, -0.224)$

Do not write outside the box

Turn over for the next question

Do not write outside the box

13 (a) Use the method of differences to show that

$$\sum_{r=2}^{n} \frac{1}{(r-1)r(r+1)} = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}$$

[5 marks]

$\frac{1}{(r-1)r(r+1)} \equiv \frac{A}{r-1} + \frac{B}{r} + \frac{C}{r+1}$

$1 \equiv Ar(r+1) + B(r-1)(r+1) + C(r-1)r$

$r = 0$: $1 = -B \Rightarrow B = -1$

$r = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$

$r = -1$: $1 = 2C \Rightarrow C = \frac{1}{2}$

$\sum_{r=2}^{n} \frac{1}{(r-1)r(r+1)} = \sum_{r=2}^{n} \left(\frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)} \right)$

$r = 2$: $\frac{1}{2(1)} - \frac{1}{2} + \frac{1}{2(3)}$

$r = 3$: $\frac{1}{2(2)} - \frac{1}{3} + \frac{1}{2(4)}$

$r = 4$: $\frac{1}{2(3)} - \frac{1}{4} + \frac{1}{2(5)}$

$\vdots$

$r = n-2$: $\frac{1}{2(n-3)} - \frac{1}{n-2} + \frac{1}{2(n-1)}$

$r = n-1$: $\frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n}$

$r = n$: $\frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}$

For $3 \leq r \leq n-1$ each term $-\frac{1}{r}$ cancels with a $\frac{1}{2r}$ in the first column and a $\frac{1}{2r}$ in the third column.

The terms left are

$\sum_{r=2}^{n} \frac{1}{(r-1)r(r+1)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} + \frac{1}{2n} - \frac{1}{n} + \frac{1}{2(n+1)}$

$= \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}$ as required

Do not write outside the box

13 (b) Find the smallest integer n such that

$$\sum_{r=2}^{n} \frac{1}{(r-1)r(r+1)} > 0.24999$$

[3 marks]

$\sum_{r=2}^{n} \frac{1}{(r-1)r(r+1)} = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)} > 0.24999$

$0.00001 > \frac{1}{2n} - \frac{1}{2(n+1)} = \frac{(n+1)-n}{2n(n+1)} = \frac{1}{2n(n+1)}$

$2n(n+1) > 100\,000$

$2n^2 + 2n - 100\,000 > 0$

$2n^2 + 2n - 100\,000 = 0 \Rightarrow n = 223.10...$ or $n = -224.10...$

$n > 223.10...$ (n positive)

Smallest integer $n = 224$

Turn over ▶

Do not write outside the box

14 The matrix **M** is defined as

$$\mathbf{M} = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 2k+3 \\ 2 & 1 & 5 \end{bmatrix}$$

where k is a constant.

14 (a) Given that **M** is a non-singular matrix, find $\mathbf{M}^{-1}$ in terms of k

[5 marks]

$\det M = 5(15 - (2k + 3)) - 2(30 - 2(2k + 3)) + 1(6 - 6)$

$= 5(12 - 2k) - 2(24 - 4k) + 0 = 12 - 2k$

Matrix of minors:

$$\begin{bmatrix} 12 - 2k & 24 - 4k & 0 \\ 9 & 23 & 1 \\ 4k + 3 & 10k + 9 & 3 \end{bmatrix}$$

Matrix of cofactors:

$$\begin{bmatrix} 12 - 2k & 4k - 24 & 0 \\ -9 & 23 & -1 \\ 4k + 3 & -10k - 9 & 3 \end{bmatrix}$$

Transpose and divide by det M:

$$M^{-1} = \frac{1}{12 - 2k} \begin{bmatrix} 12 - 2k & -9 & 4k + 3 \\ 4k - 24 & 23 & -10k - 9 \\ 0 & -1 & 3 \end{bmatrix}$$

Do not write outside the box

14 (b) State any restrictions on the value of k

[1 mark]

M is singular when $12 - 2k = 0$, so $k \neq 6$

14 (c) Using your answer to part **(a)**, show that the solution to the set of simultaneous equations below is independent of the value of k

$$\begin{aligned} 5x + 2y + z &= 1 \\ 6x + 3y + (2k+3)z &= 4k+3 \\ 2x + y + 5z &= 9 \end{aligned}$$

[4 marks]

The equations are $M\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4k+3 \\ 9 \end{bmatrix}$ so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1}\begin{bmatrix} 1 \\ 4k+3 \\ 9 \end{bmatrix}$ $(k \neq 6)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12-2k}\begin{bmatrix} 12-2k & -9 & 4k+3 \\ 4k-24 & 23 & -10k-9 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4k+3 \\ 9 \end{bmatrix}$$

$$= \frac{1}{12-2k}\begin{bmatrix} 12-2k-36k-27+36k+27 \\ 4k-24+92k+69-90k-81 \\ -4k-3+27 \end{bmatrix}$$

$$= \frac{1}{12-2k}\begin{bmatrix} 12-2k \\ 6k-36 \\ 24-4k \end{bmatrix} = \frac{1}{12-2k}\begin{bmatrix} 12-2k \\ -3(12-2k) \\ 2(12-2k) \end{bmatrix}$$

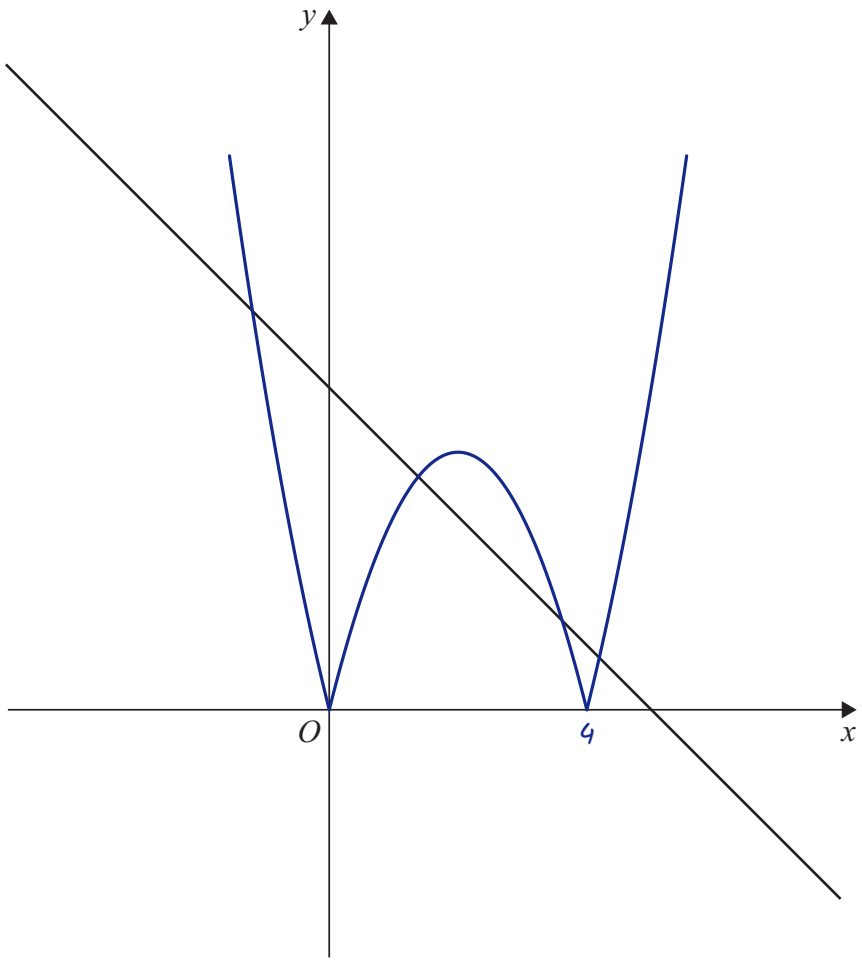
$$= \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$x = 1$, $y = -3$, $z = 2$, which does not depend on k, so the solution is independent of the value of k.

Turn over ▶

Do not write outside the box

15 The diagram shows the line $y = 5 - x$



15 (a) On the diagram above, sketch the graph of $y = |x^2 - 4x|$, including all parts of the graph where it intersects the line $y = 5 - x$

(You do not need to show the coordinates of the points of intersection.)

[3 marks]

Do not write outside the box

15 (b) Find the solution of the inequality

$$|x^2 - 4x| > 5 - x$$

Give your answer in an exact form.

[4 marks]

Points of intersection:

$x^2 - 4x = 5 - x \Rightarrow x^2 - 3x - 5 = 0 \Rightarrow x = \frac{3 \pm \sqrt{29}}{2}$

$-(x^2 - 4x) = 5 - x \Rightarrow x^2 - 5x + 5 = 0 \Rightarrow x = \frac{5 \pm \sqrt{5}}{2}$

In order: $\frac{3 - \sqrt{29}}{2} < 0 < \frac{5 - \sqrt{5}}{2} < \frac{5 + \sqrt{5}}{2} < 4 < \frac{3 + \sqrt{29}}{2}$

From the graph, $y = |x^2 - 4x|$ is above $y = 5 - x$ to the left of the first point, between the second and third points, and to the right of the fourth point.

$x < \frac{3 - \sqrt{29}}{2}$ or $\frac{5 - \sqrt{5}}{2} < x < \frac{5 + \sqrt{5}}{2}$ or $x > \frac{3 + \sqrt{29}}{2}$

Do not write outside the box

16 The function f is defined by

$$f(x) = \frac{ax + 5}{x + b}$$

where a and b are constants.

The graph of $y = f(x)$ has asymptotes $x = -2$ and $y = 3$

16 (a) Write down the value of a and the value of b

[2 marks]

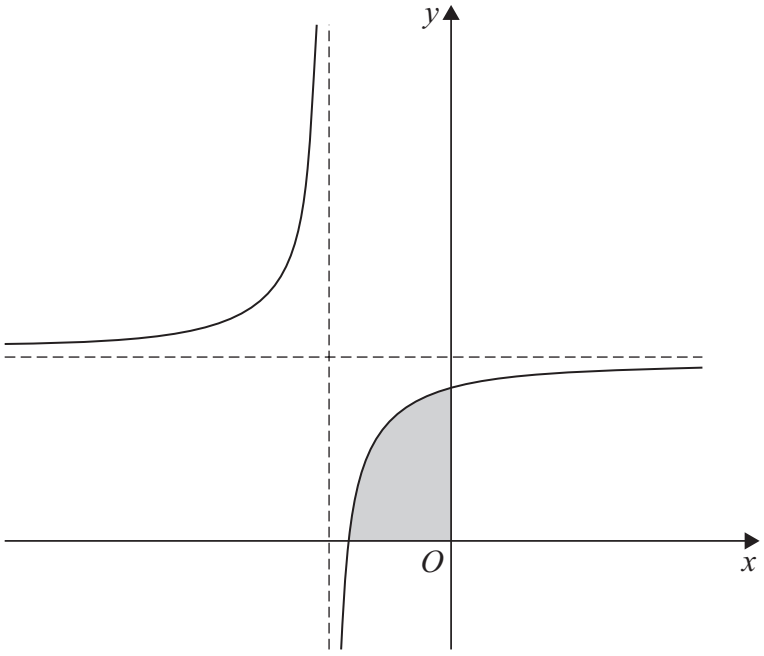
Vertical asymptote $x = -2$, so $x + b = 0$ at $x = -2$: $b = 2$

Horizontal asymptote $y = a$, so $a = 3$

$a = 3$, $b = 2$

16 (b) The diagram shows the graph of $y = f(x)$ and its asymptotes.

The shaded region *R* is enclosed by the graph of $y = f(x)$, the x-axis and the y-axis.



16 (b) (i) The shaded region *R* is rotated through 360° about the x-axis to form a solid.

Find the volume of this solid.

Give your answer to three significant figures.

[3 marks]

$f(x) = \frac{3x + 5}{x + 2} = 0$ when $x = -\frac{5}{3}$

$V = \pi \int_{-5/3}^{0} \left(\frac{3x + 5}{x + 2}\right)^2 dx$, $\frac{3x + 5}{x + 2} = 3 - \frac{1}{x + 2}$

Do not write outside the box

$V = \pi\int_{-5/3}^{0} \left(9 - \frac{6}{x+2} + \frac{1}{(x+2)^2}\right) dx$

$= \pi\left[9x - 6\ln(x+2) - \frac{1}{x+2}\right]_{-5/3}^{0}$

$= \pi\left[\left(-6\ln 2 - \frac{1}{2}\right) - \left(-15 - 6\ln\frac{1}{3} - 3\right)\right]$

$= \pi(17.5 - 6\ln 6) = 21.2040...$

$V = 21.2$ (3 s.f.)

16 (b) (ii) The shaded region R is rotated through 360° about the ***y*-axis** to form a solid.

Find the volume of this solid.

Give your answer to three significant figures.

[4 marks]

y-intercept: $y = f(0) = \frac{5}{2}$

$y = \frac{3x+5}{x+2} \Rightarrow xy + 2y = 3x + 5 \Rightarrow x(y-3) = 5 - 2y$

$x = \frac{5-2y}{y-3} = -2 - \frac{1}{y-3}$

$V = \pi\int_{0}^{5/2} x^2\,dy = \pi\int_{0}^{5/2} \left(\frac{5-2y}{y-3}\right)^2 dy$

$= \pi\int_{0}^{5/2} \left(4 + \frac{4}{y-3} + \frac{1}{(y-3)^2}\right) dy$

$= \pi\left[4y + 4\ln|y-3| - \frac{1}{y-3}\right]_{0}^{5/2}$

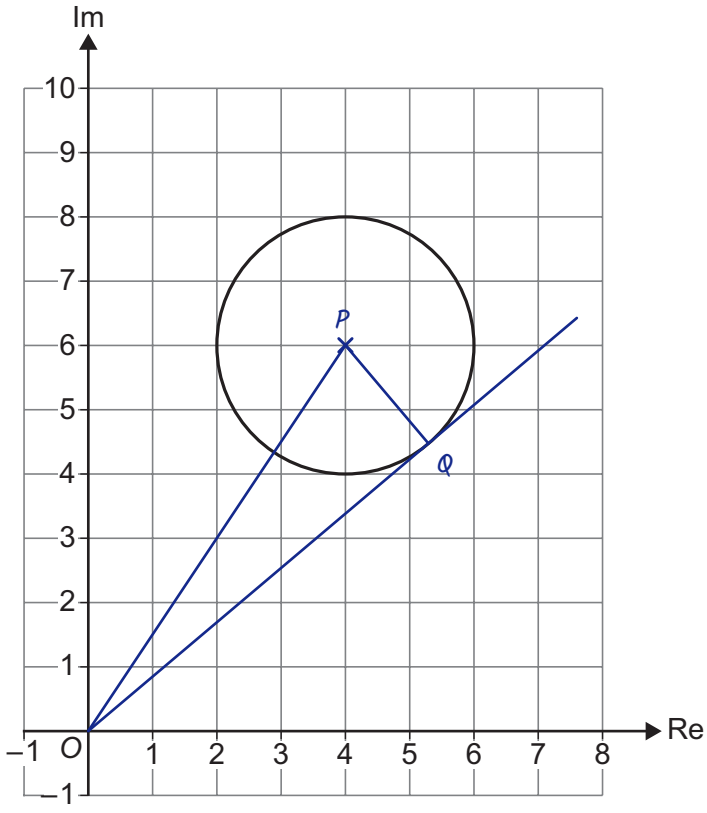
$= \pi\left[\left(10 + 4\ln\frac{1}{2} + 2\right) - \left(4\ln 3 + \frac{1}{3}\right)\right]$

$= \pi\left(\frac{35}{3} - 4\ln 6\right) = 14.1360...$

$V = 14.1$ (3 s.f.)

Do not write outside the box

17 The Argand diagram below shows a circle C



17 (a) Write down the equation of the locus of C in the form

$$|z - w| = a$$

where w is a complex number whose real and imaginary parts are integers, and a is an integer.

[2 marks]

Centre (4, 6), so w = 4 + 6i; radius 2, so a = 2

|z − (4 + 6i)| = 2

Do not write outside the box

17 (b) It is given that z_1 is a complex number representing a point on C. Of all the complex numbers which represent points on C, z_1 has the least argument.

17 (b) (i) Find $|z_1|$

Give your answer in an exact form.

[3 marks]

Let P be the centre (4, 6) and Q the point representing z_1.

z_1 has the least argument, so the half-line OQ is a tangent to C at Q, and angle OQP = 90°.

$OP^2 = 4^2 + 6^2 = 52$, $PQ^2 = 2^2 = 4$

$OQ^2 = OP^2 - PQ^2 = 52 - 4 = 48$ (Pythagoras)

$|z_1| = OQ = \sqrt{48} = 4\sqrt{3}$

Do not write outside the box

17 (b) (ii) Show that $\arg z_1 = \arcsin\left(\frac{6\sqrt{3}-2}{13}\right)$

[4 marks]

Let R be a point on the positive real axis. Then

$\arg z_1 = \angle POR - \angle POQ$

$\sin \angle POQ = \frac{PQ}{OP} = \frac{2}{\sqrt{52}} = \frac{1}{\sqrt{13}}$, $\cos \angle POQ = \frac{OQ}{OP} = \frac{4\sqrt{3}}{\sqrt{52}} = \frac{2\sqrt{3}}{\sqrt{13}}$

$\sin \angle POR = \frac{6}{\sqrt{52}} = \frac{3}{\sqrt{13}}$, $\cos \angle POR = \frac{4}{\sqrt{52}} = \frac{2}{\sqrt{13}}$

$\sin(\arg z_1) = \sin \angle POR \cos \angle POQ - \cos \angle POR \sin \angle POQ$

$= \frac{3}{\sqrt{13}} \times \frac{2\sqrt{3}}{\sqrt{13}} - \frac{2}{\sqrt{13}} \times \frac{1}{\sqrt{13}}$

$= \frac{6\sqrt{3}}{13} - \frac{2}{13} = \frac{6\sqrt{3}-2}{13}$

z_1 is in the first quadrant, so $\arg z_1$ is acute, hence

$\arg z_1 = \arcsin\left(\frac{6\sqrt{3}-2}{13}\right)$ as required

Do not write outside the box

18 In this question you may use results from the formulae booklet without proof.

Use the binomial series for $(1 + x)^n$ and the Maclaurin's series for $\sin x$ to find the series expansion for $\dfrac{1}{(1 + \sin \theta)^4}$ up to and including the term in θ^3

[4 marks]

$(1 + x)^{-4} = 1 + (-4)x + \frac{(-4)(-5)}{2!}x^2 + \frac{(-4)(-5)(-6)}{3!}x^3 + \ldots$

$= 1 - 4x + 10x^2 - 20x^3 + \ldots$

$\sin \theta = \theta - \frac{\theta^3}{3!} + \ldots = \theta - \frac{\theta^3}{6} + \ldots$ Let $x = \sin \theta$:

$(1 + \sin \theta)^{-4} = 1 - 4 \sin \theta + 10 \sin^2\theta - 20 \sin^3\theta + \ldots$

$= 1 - 4(\theta - \frac{\theta^3}{6}) + 10(\theta - \frac{\theta^3}{6})^2 - 20(\theta - \frac{\theta^3}{6})^3 + \ldots$

$= 1 - 4\theta + \frac{2\theta^3}{3} + 10\theta^2 - 20\theta^3 + \ldots$ (ignoring θ^4 and higher)

$\frac{1}{(1 + \sin \theta)^4} = 1 - 4\theta + 10\theta^2 - \frac{58}{3}\theta^3 + \ldots$

Turn over for the next question

Do not write outside the box

19 Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 45y = 21e^{5x} - 0.3x + 27x^2$$

given that $y = \frac{37}{225}$ and $\frac{dy}{dx} = 0$ when $x = 0$

[10 marks]

Auxiliary equation: $\lambda^2 + 4\lambda - 45 = 0$

$(\lambda + 9)(\lambda - 5) = 0 \Rightarrow \lambda = 5$ or $\lambda = -9$

CF: $y = Ae^{-9x} + Be^{5x}$

e^{5x} is in the CF, so PI: $y = Cxe^{5x} + D + Ex + Fx^2$

$y' = Ce^{5x} + 5Cxe^{5x} + E + 2Fx$

$y'' = 10Ce^{5x} + 25Cxe^{5x} + 2F$

Substituting into the differential equation:

$10Ce^{5x} + 25Cxe^{5x} + 2F + 4(Ce^{5x} + 5Cxe^{5x} + E + 2Fx)$

$- 45(Cxe^{5x} + D + Ex + Fx^2) = 21e^{5x} - 0.3x + 27x^2$

e^{5x}: $10C + 4C = 21 \Rightarrow C = \frac{3}{2}$

x^2: $-45F = 27 \Rightarrow F = -\frac{3}{5}$

x: $8F - 45E = -0.3 \Rightarrow -4.8 - 45E = -0.3 \Rightarrow E = -\frac{1}{10}$

const: $2F + 4E - 45D = 0 \Rightarrow -1.2 - 0.4 = 45D \Rightarrow D = -\frac{8}{225}$

General solution:

$y = Ae^{-9x} + Be^{5x} + \frac{3}{2}xe^{5x} - \frac{8}{225} - \frac{1}{10}x - \frac{3}{5}x^2$

$\frac{dy}{dx} = -9Ae^{-9x} + 5Be^{5x} + \frac{3}{2}e^{5x} + \frac{15}{2}xe^{5x} - \frac{1}{10} - \frac{6}{5}x$

$x = 0, y = \frac{37}{225}$: $A + B - \frac{8}{225} = \frac{37}{225} \Rightarrow A + B = \frac{1}{5}$

$x = 0, \frac{dy}{dx} = 0$: $-9A + 5B + \frac{3}{2} - \frac{1}{10} = 0 \Rightarrow 9A - 5B = \frac{7}{5}$

Solving: $14A = \frac{12}{5} \Rightarrow A = \frac{6}{35}, \quad B = \frac{1}{35}$

$y = \frac{6}{35}e^{-9x} + \frac{1}{35}e^{5x} + \frac{3}{2}xe^{5x} - \frac{8}{225} - \frac{1}{10}x - \frac{3}{5}x^2$

Do not write outside the box

Turn over ▶

Do not write outside the box

20 The integral I_n is defined by

$$I_n = \int_0^{\frac{\pi}{4}} \cos^n x \, dx \qquad (n \geq 0)$$

20 (a) Show that

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{1}{n\left(2^{\frac{n}{2}}\right)} \qquad (n \geq 2)$$

[6 marks]

$I_n = \int_0^{\pi/4} \cos^{n-1}x \cos x \, dx$ (parts)

$u = \cos^{n-1}x, \quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = -(n-1)\cos^{n-2}x \sin x, \quad v = \sin x$

$I_n = [\sin x \cos^{n-1}x]_0^{\pi/4} + (n-1)\int_0^{\pi/4} \cos^{n-2}x \sin^2 x \, dx$

$= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}})^{n-1} - 0 + (n-1)\int_0^{\pi/4} \cos^{n-2}x (1 - \cos^2 x) \, dx$

$= (\frac{1}{\sqrt{2}})^n + (n-1)\int_0^{\pi/4} \cos^{n-2}x \, dx - (n-1)\int_0^{\pi/4} \cos^n x \, dx$

$I_n = \frac{1}{2^{n/2}} + (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = nI_n = \frac{1}{2^{n/2}} + (n-1)I_{n-2}$

$I_n = (\frac{n-1}{n})I_{n-2} + \frac{1}{n(2^{n/2})}$ $(n \geq 2)$ as required

Do not write outside the box

20 (b) Use the result from part **(a)** to show that

$$\int_0^{\frac{\pi}{4}} \cos^6 x \, \mathrm{d}x = \frac{a\pi + b}{192}$$

where a and b are integers to be found.

[3 marks]

$I_0 = \int_0^{\pi/4} 1 \, dx = \frac{\pi}{4}$

$I_2 = \frac{1}{2} I_0 + \frac{1}{2(2^1)} = \frac{\pi}{8} + \frac{1}{4}$

$I_4 = \frac{3}{4} I_2 + \frac{1}{4(2^2)} = \frac{3\pi}{32} + \frac{3}{16} + \frac{1}{16} = \frac{3\pi}{32} + \frac{1}{4}$

$I_6 = \frac{5}{6} I_4 + \frac{1}{6(2^3)} = \frac{15\pi}{192} + \frac{5}{24} + \frac{1}{48}$

$= \frac{15\pi}{192} + \frac{44}{192} = \frac{15\pi + 44}{192}$

$\int_0^{\pi/4} \cos^6 x \, dx = \frac{15\pi + 44}{192}, \quad a = 15, \quad b = 44$

Turn over ▶

Do not write outside the box

END OF QUESTIONS

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Copyright information

For confidentiality purposes, all acknowledgements of third-party copyright material are published in a separate booklet. This booklet is published after each live examination series and is available for free download from www.aqa.org.uk.

Permission to reproduce all copyright material has been applied for. In some cases, efforts to contact copyright-holders may have been unsuccessful and AQA will be happy to rectify any omissions of acknowledgements. If you have any queries please contact the Copyright Team.

Copyright © 2024 AQA and its licensors. All rights reserved.