AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 1

Thursday 25 May 2023 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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Answer **all** questions in the spaces provided.

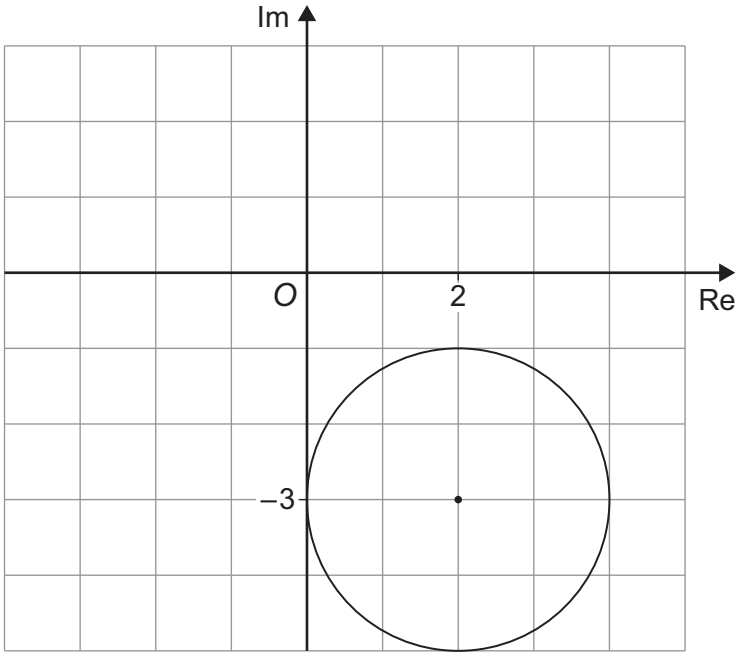
1 Find the number of solutions of the equation $\tanh x = \cosh x$

Circle your answer.

[1 mark]

(0) 1 2 3

2 The diagram below shows a locus on an Argand diagram.



Which of the equations below represents the locus shown above?

Circle your answer.

[1 mark]

($|z - 2 + 3i| = 2$) $|z + 2 - 3i| = 2$ $|z - 2 + 3i| = 4$ $|z + 2 - 3i| = 4$

Do not write outside the box

3 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ represents a transformation.

Which one of the points below is an invariant point under this transformation?

Circle your answer.

[1 mark]

(1, 1) (0, 2) (3, 0) (2, 1)

4 The solution of a second order differential equation is $f(t)$

The differential equation models heavy damping.

Which one of the statements below could be true?

Tick (✓) **one** box.

[1 mark]

$f(t) = 2e^{-t}\cos(3t) + 5e^{-t}\sin(3t)$ ☐

$f(t) = 3e^{-t} + 4te^{-t}$ ☐

$f(t) = 7e^{-t} + 2e^{-2t}$ ✓

$f(t) = 8e^{-t}\cos(3t - 0.1)$ ☐

Turn over for the next question

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5 The function f is defined by

$$\mathrm{f}(r) = 2^r(r-2) \qquad (r \in \mathbb{Z})$$

5 (a) Show that

$$\mathrm{f}(r+1) - \mathrm{f}(r) = r2^r$$

[2 marks]

$f(r+1) - f(r) = 2^{r+1}((r+1) - 2) - 2^r(r-2)$

$= 2^{r+1}(r-1) - 2^r(r-2)$

$= 2^r[2(r-1) - (r-2)]$

$= 2^r(2r - 2 - r + 2)$

$= r2^r$ as required

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5 (b) Use the method of differences to show that

$$\sum_{r=1}^{n} r2^r = 2^{n+1}(n-1) + 2$$

[4 marks]

$\sum_{r=1}^{n} r2^r = \sum_{r=1}^{n} [\, f(r+1) - f(r) \,]$

$r = 1$: $f(2) - f(1) = 0 - (-2)$

$r = 2$: $f(3) - f(2) = 8 - 0$

$r = 3$: $f(4) - f(3) = 32 - 8$

$\vdots$

$r = n - 1$: $f(n) - f(n-1) = 2^n(n-2) - 2^{n-1}(n-3)$

$r = n$: $f(n+1) - f(n) = 2^{n+1}(n-1) - 2^n(n-2)$

Adding, all the intermediate terms cancel:

$\sum_{r=1}^{n} r2^r = f(n+1) - f(1)$

$= 2^{n+1}(n-1) - 2^1(1-2)$

$= 2^{n+1}(n-1) + 2$ as required

Turn over for the next question

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6 The matrix **M** is given by

$$\mathbf{M} = \frac{1}{10}\begin{bmatrix} a & a & -6 \\ 0 & 10 & 0 \\ 9 & 14 & -13 \end{bmatrix}$$

where a is a real number.

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of **M**

The corresponding eigenvalues are λ_1, λ_2, and λ_3 respectively.

It is given that $\lambda_2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix}$,

where c is an integer.

6 (a) (i) Find the value of λ_1

[2 marks]

$Mv_1 = \frac{1}{10}\begin{bmatrix} a & a & -6 \\ 0 & 10 & 0 \\ 9 & 14 & -13 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} a-18 \\ 0 \\ -30 \end{bmatrix} = \lambda_1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Third component: $\frac{-30}{10} = 3\lambda_1 \Rightarrow -3 = 3\lambda_1$

$\lambda_1 = -1$

6 (a) (ii) Find the value of a

[2 marks]

First component: $\frac{a-18}{10} = \lambda_1 \times 1 = -1$

$a - 18 = -10$

$a = 8$

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6 (b) Find the integer c and the value of λ_3

[4 marks]

$$Mv_3 = \frac{1}{10}\begin{bmatrix} 8 & 8 & -6 \\ 0 & 10 & 0 \\ 9 & 14 & -13 \end{bmatrix}\begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8c-6 \\ 0 \\ 9c-13 \end{bmatrix} = \lambda_3\begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix}$$

Third row: $\lambda_3 = \frac{9c-13}{10}$

First row: $\frac{8c-6}{10} = \lambda_3 c = \frac{c(9c-13)}{10}$

$8c - 6 = 9c^2 - 13c$

$9c^2 - 21c + 6 = 0 \Rightarrow 3c^2 - 7c + 2 = 0$

$(3c-1)(c-2) = 0$

$c = \frac{1}{3}$ (reject, c is an integer) or $c = 2$

$\lambda_3 = \frac{9(2)-13}{10} = \frac{5}{10} = \frac{1}{2}$

$c = 2, \quad \lambda_3 = 0.5$

6 (c) Find matrices **U**, **D** and $\mathbf{U}^{-1}$, such that **D** is diagonal and $\mathbf{M} = \mathbf{UDU}^{-1}$

[3 marks]

$$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$\det U = 1(1-0) - 1(0-0) + 2(0-3) = -5$

$$U^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & 1 & -2 \\ 0 & -5 & 0 \\ -3 & 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & -1 & 2 \\ 0 & 5 & 0 \\ 3 & -2 & -1 \end{bmatrix}$$

Do not write outside the box

7 The function f is defined by

$$\mathrm{f}(x) = \left|\sin x + \frac{1}{2}\right| \qquad (0 \leq x \leq 2\pi)$$

Find the set of values of x for which

$$\mathrm{f}(x) \geq \frac{1}{2}$$

Give your answer in set notation.

[5 marks]

$f(x) \geqslant \frac{1}{2} \Rightarrow \sin x + \frac{1}{2} \geqslant \frac{1}{2}$ or $\sin x + \frac{1}{2} \leqslant -\frac{1}{2}$

$\sin x \geqslant 0$ or $\sin x \leqslant -1$

$\sin x \geqslant 0$ for $0 \leqslant x \leqslant \pi$ and at $x = 2\pi$

$\sin x \leqslant -1 \Rightarrow \sin x = -1 \Rightarrow x = \frac{3\pi}{2}$

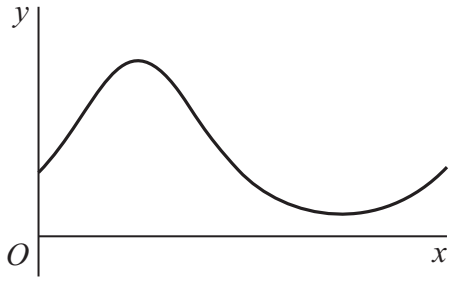
$\{ x : 0 \leqslant x \leqslant \pi \} \cup \{ \frac{3\pi}{2}, 2\pi \}$

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8 The function g is defined by

$$g(x) = e^{\sin x} \qquad (0 \leq x \leq 2\pi)$$

The diagram below shows the graph of $y = g(x)$



8 (a) Find the x-coordinate of each of the stationary points of the graph of $y = g(x)$, giving your answers in exact form.

[1 mark]

$g'(x) = \cos x \, e^{\sin x} = 0 \Rightarrow \cos x = 0$

$x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

8 (b) Use Simpson's rule with 3 ordinates to estimate

$$\int_0^{\pi} g(x)\,dx$$

giving your answer to two decimal places.

[3 marks]

$h = \frac{\pi}{2}$: $x = 0, \frac{\pi}{2}, \pi$ $y = 1, e, 1$

$\int \approx \frac{1}{3} \times \frac{\pi}{2} \times [1 + 1 + 4e]$

$= \frac{\pi}{6}(2 + 4e) = 6.7403...$

$= 6.74$ (2 d.p.)

8 (c) Explain how Simpson's rule could be used to find a more accurate estimate of the integral in part **(b)**.

[1 mark]

Use Simpson's rule with more ordinates (e.g. 5 ordinates),

i.e. more strips of smaller width.

Turn over ▶

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9 The position vectors of the points A, B and C are

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = -\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{c} = -2\mathbf{j}$$

respectively.

9 (a) Find the area of the triangle ABC

[4 marks]

$\overrightarrow{AB} = b - a = -3i - 9j + 0k$

$\overrightarrow{AC} = c - a = -2i - 3j - 2k$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -3 & -9 & 0 \\ -2 & -3 & -2 \end{vmatrix}$$

$= i(18 - 0) - j(6 - 0) + k(9 - 18)$

$= 18i - 6j - 9k$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(18^2 + (-6)^2 + (-9)^2)} = \sqrt{441} = 21$

$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{21}{2} = 10.5$

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9 (b) The points A, B and C all lie in the plane Π

Find an equation of the plane Π, in the form $\mathbf{r} \bullet \mathbf{n} = d$

[2 marks]

Normal: $n = 6i - 2j - 3k$ (parallel to $18i - 6j - 9k$)

$a \cdot n = (2)(6) + (1)(-2) + (2)(-3) = 4$

$r \cdot (6i - 2j - 3k) = 4$

9 (c) The point P has position vector $\mathbf{p} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

Find the exact distance of P from Π

[3 marks]

$p \cdot n = (1)(6) + (4)(-2) + (2)(-3) = -8$

$|n| = \sqrt{(6^2 + (-2)^2 + (-3)^2)} = \sqrt{49} = 7$

Distance of Π from $O = \frac{4}{7}$, of plane through $P = \frac{-8}{7}$

Distance of P from $\Pi = \frac{4}{7} - \frac{-8}{7} = \frac{12}{7}$

Turn over for the next question

Do not write outside the box

10 The matrix **M** is defined as

$$\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 2 & c \end{bmatrix}$$

where c is a real number.

10 (a) The linear transformation T is represented by the matrix **M**

Show that, for one particular value of c, the image under T of every point lies in the plane

$$x + 5y + 3z = 0$$

State the value of c for which this occurs.

[3 marks]

$\begin{bmatrix} 2 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 2 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-y+z \\ -x-y-2z \\ x+2y+cz \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

$x' + 5y' + 3z' = (2x - y + z) + 5(-x - y - 2z) + 3(x + 2y + cz)$

$= (2 - 5 + 3)x + (-1 - 5 + 6)y + (1 - 10 + 3c)z$

$= (3c - 9)z$

This is 0 for every point (x, y, z) when $3c - 9 = 0$, so every image point lies in the plane $x + 5y + 3z = 0$ when $c = 3$

$c = 3$

10 (b) It is given that **M** is a non-singular matrix.

10 (b) (i) State any restrictions on the value of c

[2 marks]

$\det M = 2(-c + 4) + 1(-c + 2) + 1(-2 + 1)$

$= 9 - 3c$

M non-singular $\Rightarrow$ $\det M \neq 0$ $\Rightarrow$ $c \neq 3$

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10 (b) (ii) Find $\mathbf{M}^{-1}$ in terms of c

[4 marks]

$\det M = 9 - 3c$

Minors: $\begin{bmatrix} 4-c & -c+2 & -1 \\ -c-2 & 2c-1 & 5 \\ 3 & -3 & -3 \end{bmatrix}$

Cofactors: $\begin{bmatrix} 4-c & c-2 & -1 \\ c+2 & 2c-1 & -5 \\ 3 & 3 & -3 \end{bmatrix}$

$M^{-1} = \frac{1}{9-3c}\begin{bmatrix} 4-c & c+2 & 3 \\ c-2 & 2c-1 & 3 \\ -1 & -5 & -3 \end{bmatrix}$

10 (b) (iii) Using your answer from part **(b)(ii)**, solve

$$2x - y + z = -3$$
$$-x - y - 2z = -6$$
$$x + 2y + 4z = 13$$

[3 marks]

$c = 4$: $\det M = 9 - 12 = -3$

$M^{-1} = \frac{1}{-3}\begin{bmatrix} 0 & 6 & 3 \\ 2 & 7 & 3 \\ -1 & -5 & -3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1}\begin{bmatrix} -3 \\ -6 \\ 13 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 0-36+39 \\ -6-42+39 \\ 3+30-39 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 3 \\ -9 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

$x = -1, \quad y = 3, \quad z = 2$

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11 The function f is defined by

$$\mathrm{f}(x) = 4x^3 - 8x^2 - 51x - 45 \qquad (x \in \mathbb{R})$$

11 (a) (i) Fully factorise $\mathrm{f}(x)$

[2 marks]

$f(5) = 500 - 200 - 255 - 45 = 0 \Rightarrow (x - 5)$ is a factor

$f(x) = (x - 5)(4x^2 + 12x + 9)$

$= (x - 5)(2x + 3)^2$

11 (a) (ii) Hence, solve the inequality $\mathrm{f}(x) < 0$

[2 marks]

$(2x + 3)^2 \geq 0$, so $f(x) < 0$ when $x - 5 < 0$ and $2x + 3 \neq 0$

$x < 5, \quad x \neq -\frac{3}{2}$

$x < -\frac{3}{2} \quad$ or $\quad -\frac{3}{2} < x < 5$

Do not write outside the box

11 (b) The graph of $y = f(x)$ is translated by the vector $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

The new graph is then reflected in the x-axis, to give the graph of $y = g(x)$

Solve the inequality $g(x) \leq 0$

[3 marks]

Translation: $y = f(x - 7)$

Reflection in x-axis: $g(x) = -f(x - 7)$

$= -(x - 12)(2x - 11)^2$

$g(x) = 0 \Rightarrow x = \frac{11}{2}$ or $x = 12$

$g(x) \leq 0 \Leftrightarrow (x - 12)(2x - 11)^2 \geq 0$

$x = \frac{11}{2}$ or $x \geq 12$

Turn over for the next question

Do not write outside the box

12 (a) Starting from the identities for $\sinh 2x$ and $\cosh 2x$, prove the identity

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

[2 marks]

$\tanh 2x = \frac{\sinh 2x}{\cosh 2x} = \frac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$

Dividing numerator and denominator by $\cosh^2 x$:

$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$ as required

12 (b) (i) The function f is defined by

$$\mathrm{f}(x) = \tanh x \qquad (x > 0)$$

State the range of f

[1 mark]

$0 < f(x) < 1$

12 (b) (ii) Use part **(a)** and part **(b)(i)** to prove that $\tanh 2x > \tanh x$ if $x > 0$

[3 marks]

For $x > 0$, $0 < \tanh x < 1$

$\Rightarrow 0 < \tanh^2 x < 1 \Rightarrow 1 < 1 + \tanh^2 x < 2$

$\Rightarrow \frac{2}{1+\tanh^2 x} > 1$

Since $\tanh x > 0$, multiplying both sides by $\tanh x$:

$\frac{2\tanh x}{1+\tanh^2 x} > \tanh x$

$\therefore \tanh 2x > \tanh x$ for $x > 0$ (using part (a))

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13 Use l'Hôpital's rule to prove that

$$\lim_{x \to \pi} \left(\frac{x \sin 2x}{\cos\left(\frac{x}{2}\right)} \right) = -4\pi$$

[5 marks]

Let $f(x) = x \sin 2x$ and $g(x) = \cos\frac{x}{2}$

$f(\pi) = \pi \sin 2\pi = 0$ and $g(\pi) = \cos\frac{\pi}{2} = 0$

Both are 0 at $x = \pi$, so l'Hôpital's rule can be used.

$f'(x) = \sin 2x + 2x \cos 2x$

$g'(x) = -\frac{1}{2} \sin\frac{x}{2}$

$\lim_{x\to\pi} \frac{f(x)}{g(x)} = \lim_{x\to\pi} \frac{f'(x)}{g'(x)}$

$f'(\pi) = \sin 2\pi + 2\pi \cos 2\pi = 0 + 2\pi = 2\pi$

$g'(\pi) = -\frac{1}{2} \sin\frac{\pi}{2} = -\frac{1}{2}$

$\lim_{x\to\pi} \left(\frac{x \sin 2x}{\cos(x/2)}\right) = \frac{2\pi}{-1/2} = -4\pi$ as required

Turn over for the next question

Do not write outside the box

14 The curve C has polar equation

$$r = \frac{4}{5 + 3\cos\theta} \qquad (-\pi < \theta \leq \pi)$$

14 (a) Show that r takes values in the range $\frac{1}{k} \leq r \leq k$, where k is an integer.

[2 marks]

$-1 \leq \cos\theta \leq 1 \Rightarrow 2 \leq 5 + 3\cos\theta \leq 8$

r is maximum when $\cos\theta = -1$: $r = \frac{4}{2} = 2$

r is minimum when $\cos\theta = 1$: $r = \frac{4}{8} = \frac{1}{2}$

So $\frac{1}{2} \leq r \leq 2$, i.e. $k = 2$

14 (b) Find the Cartesian equation of C in the form $y^2 = \mathrm{f}(x)$

[4 marks]

$5r + 3r\cos\theta = 4$

$x = r\cos\theta, \quad r = \sqrt{(x^2 + y^2)}$

$5\sqrt{(x^2 + y^2)} = 4 - 3x$

$25(x^2 + y^2) = (4 - 3x)^2 = 16 - 24x + 9x^2$

$25y^2 = 16 - 24x - 16x^2$

$y^2 = \frac{16 - 24x - 16x^2}{25}$

Do not write outside the box

14 (c) The ellipse E has equation

$$y^2 + \frac{16x^2}{25} = 1$$

Find the transformation that maps the graph of E onto C

[4 marks]

C: $16x^2 + 24x + 25y^2 = 16$

$16(x^2 + \frac{3}{2}x) + 25y^2 = 16$

$16(x + \frac{3}{4})^2 - 9 + 25y^2 = 16$

$16(x + \frac{3}{4})^2 + 25y^2 = 25$

$y^2 + \frac{16}{25}(x + \frac{3}{4})^2 = 1$

E: $y^2 + \frac{16x^2}{25} = 1$, so x is replaced by $x + \frac{3}{4}$

E maps onto C by a translation by the vector $\begin{bmatrix} -3/4 \\ 0 \end{bmatrix}$

Turn over for the next question

Do not write outside the box

15 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = \cos 2x + 5x$$

[9 marks]

Auxiliary equation: $m^2 - 3m - 4 = 0$

$(m - 4)(m + 1) = 0 \Rightarrow m = 4, -1$

CF: $y = Ae^{4x} + Be^{-x}$

PI: try $y = p\cos 2x + q\sin 2x + rx + s$

$y' = -2p\sin 2x + 2q\cos 2x + r$

$y'' = -4p\cos 2x - 4q\sin 2x$

Substituting into the differential equation:

$(-4p\cos 2x - 4q\sin 2x) - 3(-2p\sin 2x + 2q\cos 2x + r)$

$- 4(p\cos 2x + q\sin 2x + rx + s) = \cos 2x + 5x$

$\cos 2x$: $-4p - 6q - 4p = 1 \Rightarrow -8p - 6q = 1$

$\sin 2x$: $-4q + 6p - 4q = 0 \Rightarrow 6p - 8q = 0$

$p = \frac{4q}{3} \Rightarrow -\frac{32q}{3} - 6q = 1 \Rightarrow -\frac{50q}{3} = 1$

$q = -\frac{3}{50}, \quad p = -\frac{2}{25}$

x: $-4r = 5 \Rightarrow r = -\frac{5}{4}$

constant: $-3r - 4s = 0 \Rightarrow s = -\frac{3r}{4} = \frac{15}{16}$

General solution:

$y = Ae^{4x} + Be^{-x} - \frac{2}{25}\cos 2x - \frac{3}{50}\sin 2x - \frac{5}{4}x + \frac{15}{16}$

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Do not write outside the box

16 (a) Show that

$$\int_{0.5}^{4} \frac{1}{t} \ln t \, \mathrm{d}t = a(\ln 2)^2$$

where a is a rational number to be found.

[4 marks]

Let $u = \ln t$, $du = \frac{1}{t} dt$

$\int_{0.5}^{4} \frac{1}{t} \ln t \, dt = \int_{\ln 0.5}^{\ln 4} u \, du$

$= \left[\frac{u^2}{2} \right]_{\ln 0.5}^{\ln 4} = \frac{(\ln 4)^2}{2} - \frac{(\ln 0.5)^2}{2}$

$= \frac{(2 \ln 2)^2}{2} - \frac{(-\ln 2)^2}{2} = 2(\ln 2)^2 - \frac{1}{2}(\ln 2)^2$

$= \frac{3}{2}(\ln 2)^2$ so $a = \frac{3}{2}$

Do not write outside the box

16 (b) A curve C is defined parametrically for $t > 0$ by

$$x = 2t \qquad y = \frac{1}{2}t^2 - \ln t$$

The arc formed by the graph of C from $t = 0.5$ to $t = 4$ is rotated through 2π radians about the x-axis to generate a surface with area S

Find the exact value of S, giving your answer in the form

$$S = \pi\left(b + c\ln 2 + d(\ln 2)^2\right)$$

where b, c and d are rational numbers to be found.

[7 marks]

$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = t - \frac{1}{t}$

$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 4 + t^2 - 2 + \frac{1}{t^2} = t^2 + 2 + \frac{1}{t^2} = (t + \frac{1}{t})^2$

$S = 2\pi\int_{0.5}^{4} y\sqrt{((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)}\,dt = 2\pi\int_{0.5}^{4} (\frac{1}{2}t^2 - \ln t)(t + \frac{1}{t})\,dt$

$S = 2\pi\int_{0.5}^{4} (\frac{1}{2}t^3 + \frac{1}{2}t - t\ln t - \frac{1}{t}\ln t)\,dt$

$\int t\ln t\,dt = \frac{1}{2}t^2\ln t - \int \frac{1}{2}t\,dt = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2$

and from part (a): $\int_{0.5}^{4} \frac{1}{t}\ln t\,dt = \frac{3}{2}(\ln 2)^2$

$S = 2\pi\{[\frac{1}{8}t^4 + \frac{1}{4}t^2 - \frac{1}{2}t^2\ln t + \frac{1}{4}t^2]_{0.5}^{4} - \frac{3}{2}(\ln 2)^2\}$

$[\frac{1}{8}t^4 + \frac{1}{2}t^2 - \frac{1}{2}t^2\ln t]_{0.5}^{4} = (32 + 8 - 8\ln 4)$

$- (\frac{1}{128} + \frac{1}{8} - \frac{1}{8}\ln\frac{1}{2})$

$= 40 - 16\ln 2 - \frac{17}{128} - \frac{1}{8}\ln 2 = \frac{5103}{128} - \frac{129}{8}\ln 2$

$S = 2\pi(\frac{5103}{128} - \frac{129}{8}\ln 2 - \frac{3}{2}(\ln 2)^2)$

$S = \pi(\frac{5103}{64} - \frac{129}{4}\ln 2 - 3(\ln 2)^2)$

$b = \frac{5103}{64}, \quad c = -\frac{129}{4}, \quad d = -3$

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END OF QUESTIONS

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