AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 3 Discrete

Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification.
- You must ensure you have the other optional Question Paper/Answer Book for which you are entered (**either** Mechanics **or** Statistics). You will have 2 hours to complete **both** papers.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

For Examiner's Use	
Question	Mark
1	
2	
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8	
9	
10	
TOTAL	

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 50.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

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ANSWER IN THE SPACES PROVIDED

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Answer **all** questions in the spaces provided.

1 The graph *G* has a subgraph isomorphic to K_5, the complete graph with 5 vertices.

Which of the following statements about *G* must be true?

Tick (✓) **one** box.

[1 mark]

G is not connected	☐
G is not Hamiltonian	☐
G is not planar	☑
G is not simple	☐

2 Graph *A* is a connected planar graph with 12 vertices, 18 edges and n faces.

Find the value of n

Circle your answer.

[1 mark]

4 (8) 28 32

Turn over for the next question

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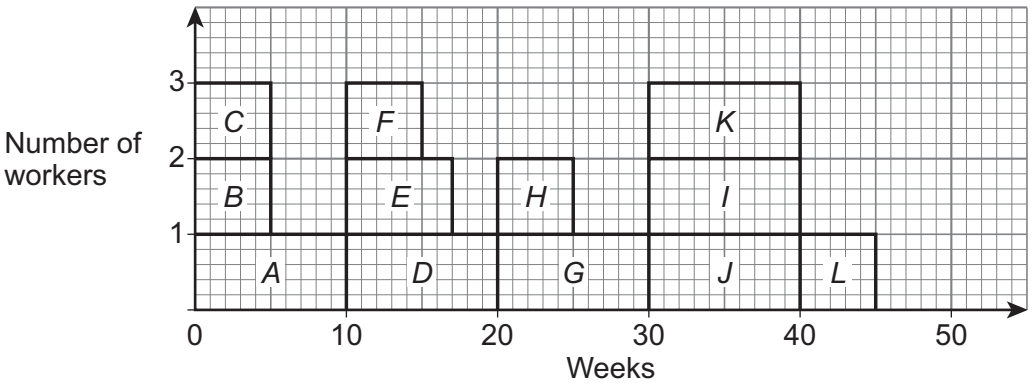
3 A company undertakes a project which consists of 12 activities, *A*, *B*, *C*, …, *L*

Each activity requires one worker.

The resource histogram below shows the duration of each activity.

Each activity begins at its earliest start time.

The path *ADGJL* is critical.



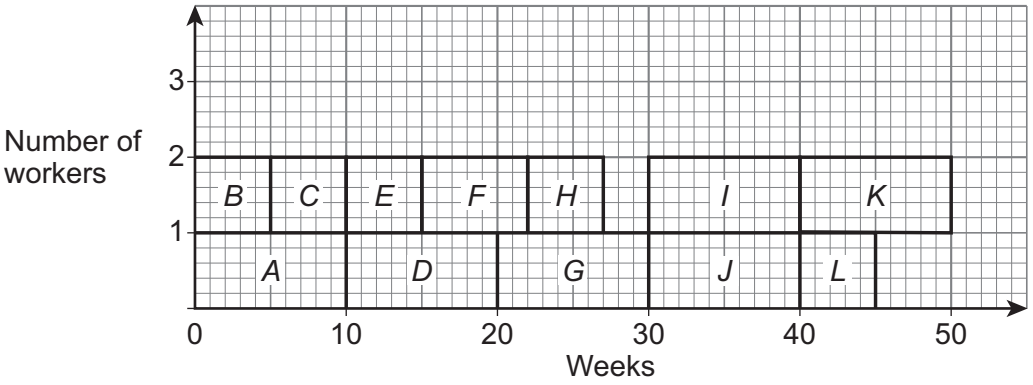
The company only has two workers available to work on the project.

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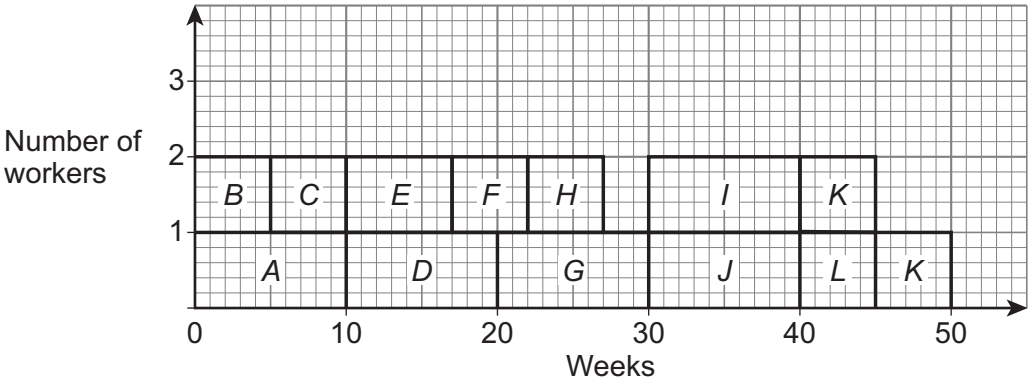
Which of the following could be a correctly levelled histogram?

Tick (✓) **one** box.

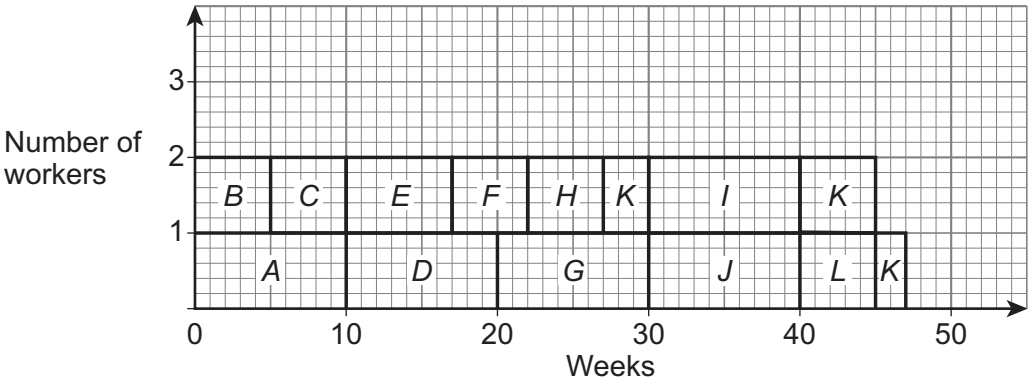
[1 mark]



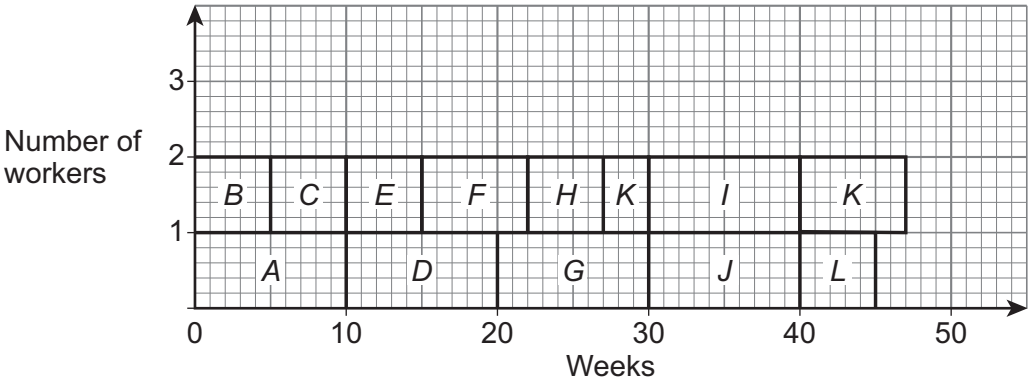
☐



☑



☐



☐

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4 Ben and Jadzia play a zero-sum game.

The game is represented by the following pay-off matrix for Ben.

		Jadzia			
	Strategy	**X**	**Y**	**Z**	row min
Ben	**A**	−3	2	3	−3
	B	6	0	−4	−4
	C	7	−1	1	−1 ←
	D	6	−2	1	−2
	col max	7	2 ←	3	

4 (a) State, with a reason, which strategy Ben should never play.

[1 mark]

Strategy D, because it is dominated by strategy C (6 ≤ 7, −2 ≤ −1, 1 ≤ 1), so Ben should never play D.

4 (b) Determine whether or not the game has a stable solution.

Fully justify your answer.

[3 marks]

Row minima: A −3, B −4, C −1, (D −2)

max(row minima) = −1 (Ben's play-safe is C)

Column maxima: X 7, Y 2, Z 3

min(column maxima) = 2 (Jadzia's play-safe is Y)

max(row minima) = −1 ≠ 2 = min(column maxima)

so there is no saddle point, and therefore the game does not have a stable solution.

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4 (c) Ben knows that Jadzia will always play her play-safe strategy.

Explain how Ben can maximise his expected pay-off.

[2 marks]

Jadzia's play-safe strategy is Y (smallest column maximum, 2).

Against Y, Ben's pay-offs are A: 2, B: 0, C: −1, D: −2

The largest is 2, so Ben should always play strategy A

(giving him a pay-off of 2 every time).

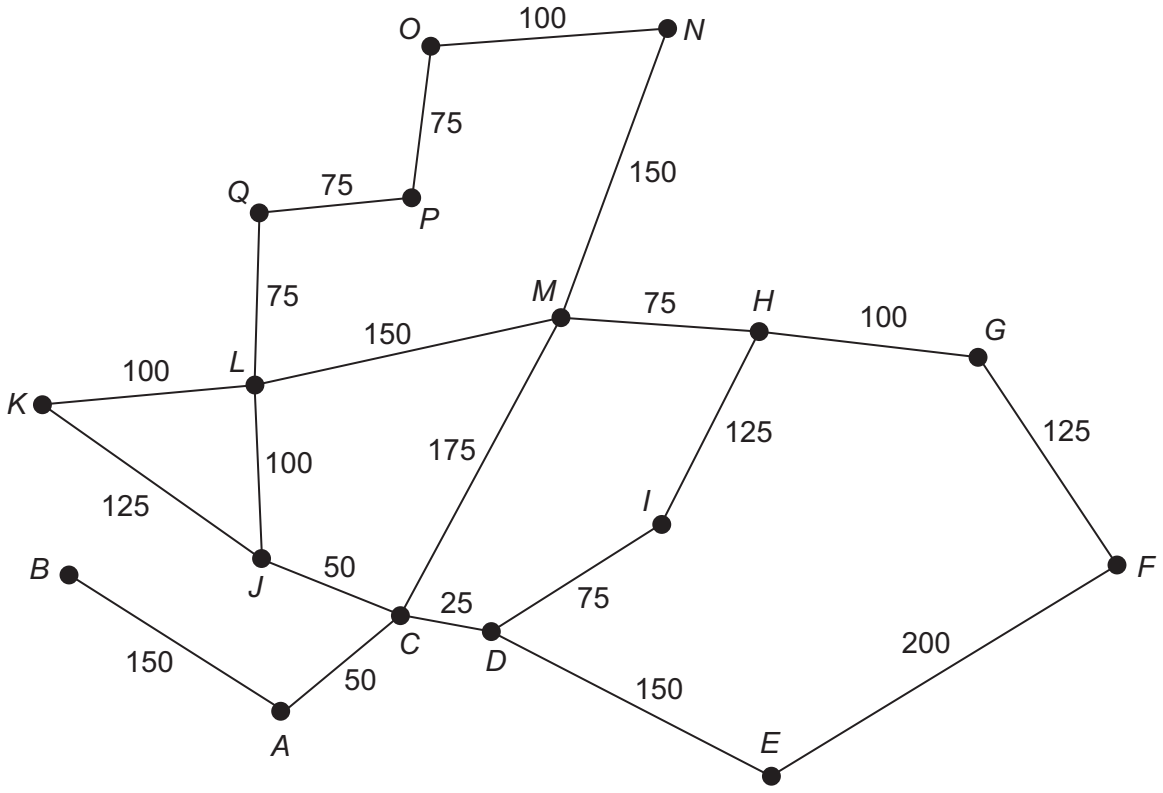
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5 A council wants to convert all of the street lighting in a village to use LED lighting.

The network below shows each street in the village. Each node represents a junction and the weight of each arc represents the length, in metres, of the street.

The street lights are only positioned on one side of each street in the village.



The total length of all of the streets in the village is 2250 metres.

In order to determine the total number of street lights in the village, a council worker is required to walk along every street in the village at least once, starting and finishing at the same junction.

The shortest possible distance the council worker can walk in order to determine the total number of street lights in the village is x metres.

5 (a) Find the value of x

Fully justify your answer.

[4 marks]

Odd nodes: B (1), D (3), H (3), J (3); route inspection

Shortest distances between odd nodes:

BD = 225 (BACD) HJ = 275 (HIDCJ)

BH = 425 (BACDIH) DJ = 75 (DCJ)

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BJ = 250 (BACJ) DH = 200 (DIH)

Pairings: BD + HJ = 225 + 275 = 500

BH + DJ = 425 + 75 = 500

BJ + DH = 250 + 200 = 450 (minimum)

So repeat BJ and DH, an extra 450 m

x = 2250 + 450

x = 2700

5 (b) A new council regulation requires that the mean distance along a street between adjacent LED street lights in a village be less than 25 metres.

The council worker counted 91 different street lights on their journey around the village.

Determine whether or not the village will meet the new council regulation.

[2 marks]

91 street lights spread over a total of 2250 m of street

mean distance between adjacent lights ≈ 2250 ÷ 91

= 24.725... ≈ 24.7 m

24.7 < 25, so the village will meet the new

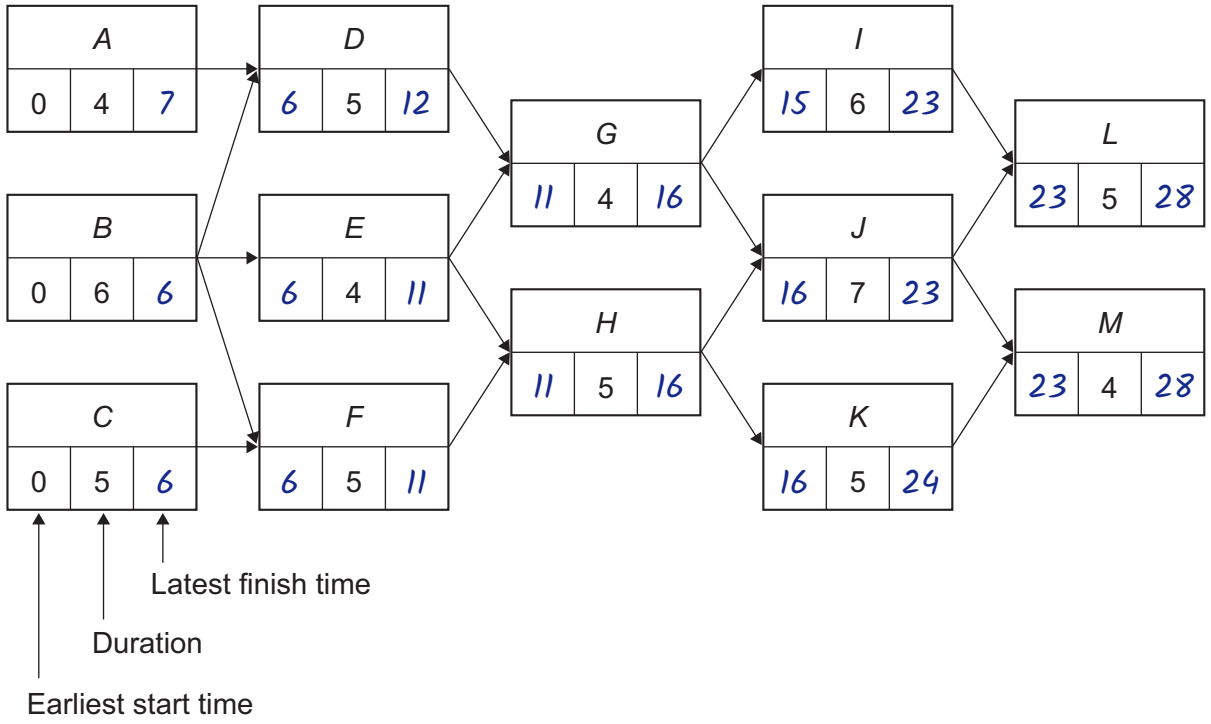
council regulation.

Turn over for the next question

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6 Bill Durrh Ltd undertake a construction project.

The activity network for the project is shown below. The duration of each activity is given in weeks.



6 (a) (i) Find the earliest start time and the latest finish time for each activity and show these values on the activity network above.

[3 marks]

6 (a) (ii) Identify all of the critical activities.

[1 mark]

B, F, H, J, L

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6 (b) The manager of Bill Durrh Ltd recruits some additional temporary workers in order to reduce the duration of one activity by 2 weeks.

The manager wants to reduce the minimum completion time of the project by the largest amount.

State, with a reason, which activity the manager should choose.

[2 marks]

Activity J.

J is critical, and every path of 27 or more weeks passes through J (BFHJL = 28, BDGJL = 27, BFHJM = 27, CFHJL = 27), so reducing J by 2 weeks reduces the minimum completion time by 2 weeks (28 → 26). Reducing any other critical activity by 2 weeks only reduces the minimum completion time by 1 week.

Turn over for the next question

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7 The group G has binary operation $*$ and order p, where p is a prime number.

7 (a) Determine the number of distinct subgroups of G

Fully justify your answer.

[2 marks]

By Lagrange's theorem the order of any subgroup of G divides the order of G, which is p.

p is prime, so its only factors are 1 and p.

Subgroups: {e} of order 1, and G itself of order p.

So G has exactly 2 distinct subgroups.

7 (b) G contains an element g which has period p

7 (b) (i) State the general name given to elements such as g

[1 mark]

g is a generator (of G)

7 (b) (ii) State the name of a group that is isomorphic to G

[1 mark]

The cyclic group of order p, C_p

7 (c) G contains an element g^r, where $r < p$

Find, in terms of g, r and p, the inverse of g^r

[2 marks]

Let the inverse be x, so $g^r * x = e$

g has period p, so $g^p = e$

$g^r * g^{p-r} = g^{r+p-r} = g^p = e$

$\therefore$ the inverse of g^r is g^{p-r}

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7 (d) In the case when $p = 5$ and the binary operation $*$ represents addition modulo 5, G contains the elements 0, 1, 2, 3 and 4

7 (d) (i) Explain why G is closed.

[1 mark]

For any a, b in G, a + b (mod 5) is always 0, 1, 2, 3 or 4,

all of which are in G, so G is closed.

7 (d) (ii) Complete the Cayley table for $(G, *)$

[1 mark]

*	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Turn over for the next question

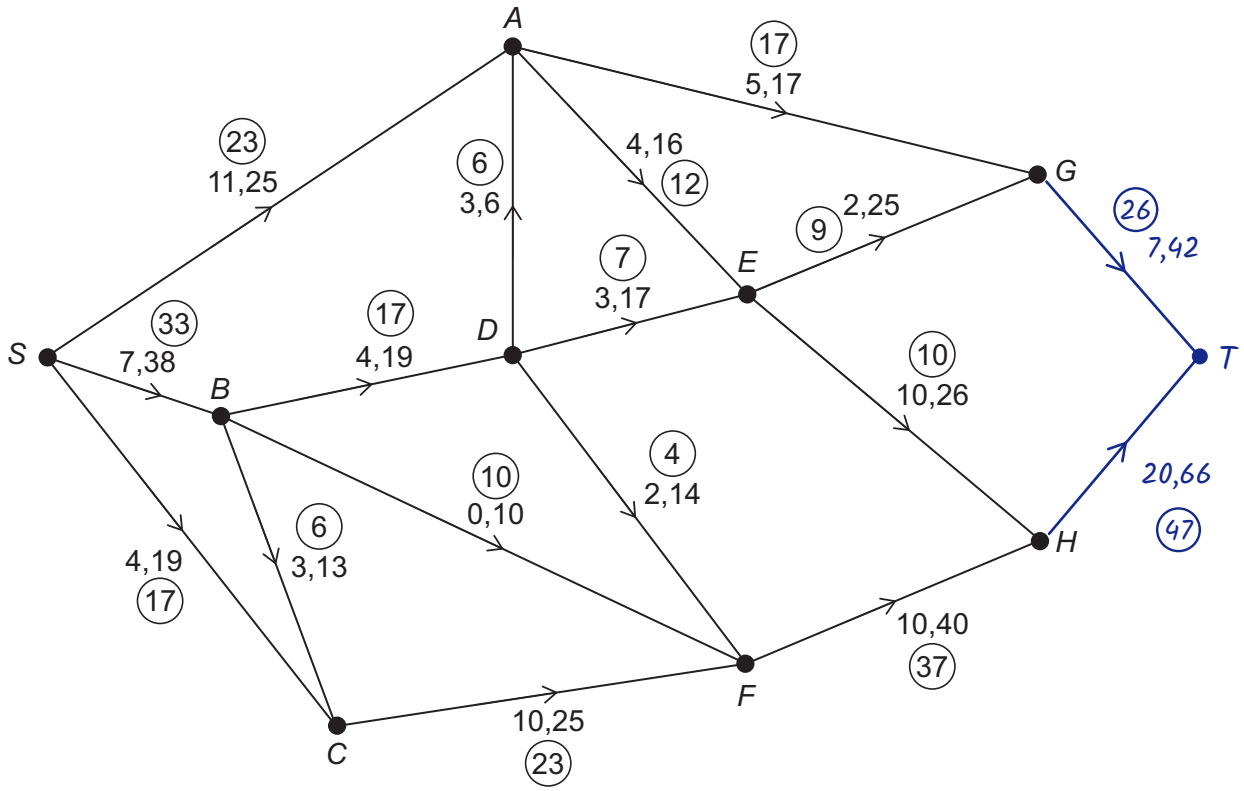
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8 **Figure 1** shows a network of gas pipes.

The numbers on each arc represent the lower and upper capacity for each pipe in $m^3 s^{-1}$

The numbers in the circles represent an initial feasible flow of $73 m^3 s^{-1}$ through the network.

Figure 1



8 (a) On **Figure 1** above, add a supersink T to the network.

[2 marks]

8 (b) Using flow augmentation, find the maximum flow through the network.

You must indicate any flow augmenting paths clearly in the table below.

You may use **Figure 2**, on the page opposite, in your solution.

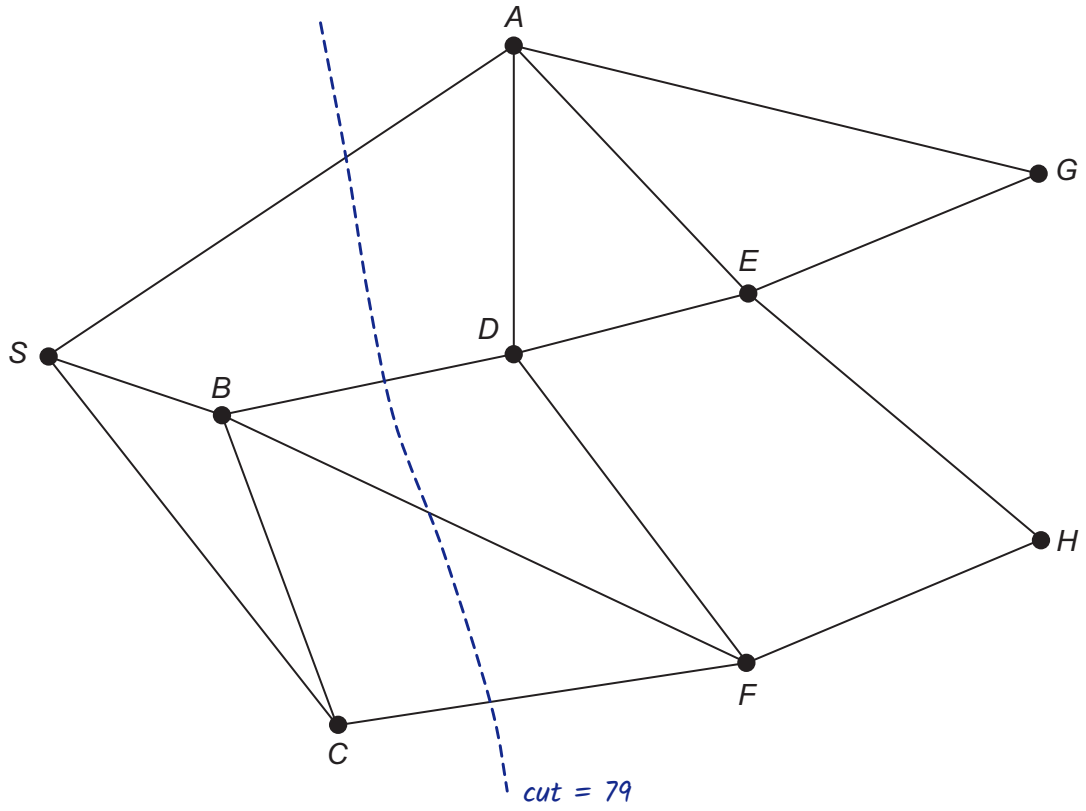
[4 marks]

Augmenting Path	Flow
SAEGT	2
SBDEGT	2
SCFHT	2

Maximum flow 79 $m^3 s^{-1}$

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Figure 2



8 (c) Prove that the flow found in part **(b)** is the maximum flow through the network.

[2 marks]

Cut {S, B, C} | {A, D, E, F, G, H, T} (shown on Figure 2)

value = SA + BD + BF + CF = 25 + 19 + 10 + 25 = 79

Flow = 79 = value of this cut, so by the maximum flow-minimum cut theorem, the flow of 79 $m^3 s^{-1}$ is the maximum.

8 (d) A trainee engineer claims that increasing the upper capacity of the pipe *AG* will increase the maximum flow through the network, as the flow through this pipe cannot currently be increased.

Comment on the validity of the trainee's claim.

[2 marks]

Not valid. Both arcs into A, SA (25) and DA (6), are saturated, so the flow into A is already at its maximum. AG is therefore not limiting the flow: increasing its upper capacity would not increase the maximum flow, so the trainee's claim is incorrect.

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9 The binary operation $\oplus$ acts on the positive integers x and y such that

$$x \oplus y = x + y + 8 \pmod{k^2 - 16k + 74}$$

where k is a positive integer.

9 (a) (i) Show that $\oplus$ is commutative.

[1 mark]

$x \oplus y = x + y + 8 \pmod{k^2 - 16k + 74}$

$y \oplus x = y + x + 8 \pmod{k^2 - 16k + 74}$

Addition is commutative, so $x + y + 8 = y + x + 8$

$\therefore x \oplus y = y \oplus x$, so $\oplus$ is commutative.

9 (a) (ii) Determine whether or not $\oplus$ is associative.

Fully justify your answer.

[2 marks]

Let $m = k^2 - 16k + 74$

$(x \oplus y) \oplus z = (x + y + 8) + z + 8$

$= x + y + z + 16 \pmod{m}$

$x \oplus (y \oplus z) = x + (y + z + 8) + 8$

$= x + y + z + 16 \pmod{m}$

So $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all positive integers x, y, z

$\therefore \oplus$ is associative.

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9 (b) Find the values of k for which 3 is an identity element for the set of positive integers under $\oplus$

[3 marks]

For 3 to be an identity, $3 \oplus x = x \oplus 3 = x$ for all x

$3 \oplus x = 3 + x + 8 = x + 11 \pmod{k^2 - 16k + 74}$

$x + 11 \equiv x \Rightarrow 11 \equiv 0 \pmod{k^2 - 16k + 74}$, so the modulus divides 11

$k^2 - 16k + 74 = (k - 8)^2 + 10 \geq 10$, so the modulus is 11

$k^2 - 16k + 74 = 11$

$k^2 - 16k + 63 = 0$

$(k - 7)(k - 9) = 0$

$k = 7$ or $k = 9$

Turn over for the next question

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10 Kira and Julian play a zero-sum game that does not have a stable solution.

Kira has three strategies to choose from: $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$

To determine her optimal mixed strategy, Kira begins by defining the following variables:

$v =$ value of the game for Kira

$p_1 =$ probability of Kira playing strategy $\mathbf{K}_1$

$p_2 =$ probability of Kira playing strategy $\mathbf{K}_2$

$p_3 =$ probability of Kira playing strategy $\mathbf{K}_3$

Kira then formulates the following linear programming problem.

Maximise v

subject to
$$7p_1 + p_2 + 8p_3 \geq v$$
$$3p_1 + 7p_2 + 2p_3 \geq v$$
$$9p_1 + 2p_2 + 4p_3 \geq v$$

and
$$p_1 + p_2 + p_3 \leq 1$$
$$p_1, p_2, p_3 \geq 0$$

10 (a) (i) Explain why the condition $p_1 + p_2 + p_3 \leq 1$ is necessary in Kira's linear programming problem.

[1 mark]

p_1, p_2, p_3 are probabilities, and the sum of the probabilities cannot be greater than 1; this condition ensures this.

10 (a) (ii) Explain why the condition $p_1, p_2, p_3 \geq 0$ is necessary in Kira's linear programming problem.

[1 mark]

Probabilities cannot be negative; this condition ensures that p_1, p_2 and p_3 are all non-negative.

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10 (b) Julian has three strategies to choose from: $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$

Complete the following pay-off matrix which represents the game for Kira.

[3 marks]

		Julian		
	Strategy	$\mathbf{J}_1$	$\mathbf{J}_2$	$\mathbf{J}_3$
Kira	$\mathbf{K}_1$	7	3	9
	$\mathbf{K}_2$	1	7	2
	$\mathbf{K}_3$	8	2	4

Each constraint gives one column (Julian's strategy):

J_1: 7, 1, 8 J_2: 3, 7, 2 J_3: 9, 2, 4

END OF QUESTIONS

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