AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

I declare this is my own work.

A-level
FURTHER MATHEMATICS

Paper 2

Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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11	
12	
13	
14	
TOTAL	

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Answer **all** questions in the spaces provided.

1 Find the imaginary part of

$$\frac{5+\mathrm{i}}{1-\mathrm{i}}$$

Circle your answer.

[1 mark]

-3 -2 2 (3)

2 Find the mean value of the function $\mathrm{f}(x) = 10x^4$ between $x = 0$ and $x = a$

Circle your answer.

[1 mark]

$10a^3$ $40a^3$ ($2a^4$) $4a^5$

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3 The roots of the equation $x^2 - px - 6 = 0$ are α and β

Find $\alpha^2 + \beta^2$ in terms of p

Circle your answer.

[1 mark]

$p^2 - 6$ $p^2 + 6$ $p^2 - 12$ $p^2 + 12$

4 Which of the following graphs intersects the graph of $y = \sinh x$ at exactly one point?

Circle your answer.

[1 mark]

$y = \text{cosech}\, x$ $y = \cosh x$ $y = \coth x$ $y = \text{sech}\, x$

Turn over for the next question

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5 Prove by induction that, for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$$

[4 marks]

$n = 1$: LHS $= 1^3 = 1$, RHS $= (\frac{1}{2} \times 1 \times 2)^2 = 1^2 = 1$

LHS = RHS so the result is true for $n = 1$

Assume true for $n = k$: Σr^3 ($r = 1$ to k) $= (\frac{1}{2}k(k + 1))^2$

Then Σr^3 ($r = 1$ to $k + 1$) $= (\frac{1}{2}k(k + 1))^2 + (k + 1)^3$

$= \frac{1}{4}(k + 1)^2[k^2 + 4(k + 1)]$

$= \frac{1}{4}(k + 1)^2(k^2 + 4k + 4)$

$= \frac{1}{4}(k + 1)^2(k + 2)^2$

$= (\frac{1}{2}(k + 1)(k + 2))^2$

$= (\frac{1}{2}(k + 1)((k + 1) + 1))^2$

which is the result with $n = k + 1$.

The result is true for $n = 1$, and if it is true for $n = k$ then it is also true for $n = k + 1$. Hence, by induction, $\Sigma r^3 = (\frac{1}{2}n(n + 1))^2$ is true for all integers $n \geq 1$.

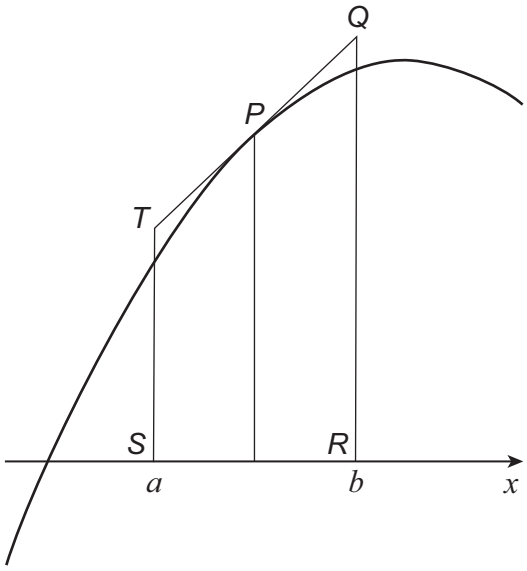
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6 The diagram below shows part of the graph of $y = \mathrm{f}(x)$

The line *TPQ* is a tangent to the graph of $y = \mathrm{f}(x)$ at the point $P\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right)$

The points $S(a, 0)$ and *T* lie on the line $x = a$

The points *Q* and $R(b, 0)$ lie on the line $x = b$



Sharon uses the mid-ordinate rule with one strip to estimate the value of the integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$

By considering the area of the trapezium *QRST*, state, giving reasons, whether you would expect Sharon's estimate to be an under-estimate or an over-estimate.

[3 marks]

P is the midpoint of TQ, so (ST + RQ)/2 = f((a + b)/2)

Area of trapezium QRST = (b − a) × ½(ST + RQ) = (b − a) f((a + b)/2)

This is the same as the area of the rectangle from the mid-ordinate rule with one strip, so it equals Sharon's estimate.

The curve lies below the tangent TQ between x = a and x = b, so the trapezium contains all of the area represented by the integral: area of trapezium > ∫ f(x) dx.

Hence Sharon's estimate is an over-estimate.

Do not write outside the box

7 The function f is defined by

$$\mathrm{f}(x) = \frac{ax - 5}{2x + b} \qquad x \in \mathbb{R},\ x \neq \frac{9}{2}$$

where a and b are integers.

The graph of $y = \mathrm{f}(x)$ has asymptotes $x = \frac{9}{2}$ and $y = 3$

7 (a) Find the value of a and the value of b

[2 marks]

Vertical asymptote where $2x + b = 0$: $2(9/2) + b = 0 \Rightarrow b = -9$

Horizontal asymptote $y = a/2 = 3 \Rightarrow a = 6$

$a = 6, \quad b = -9$

7 (b) Solve the inequality

$$\mathrm{f}(x) \leq x + 2$$

Fully justify your answer.

[6 marks]

$\frac{6x - 5}{2x - 9} \leq x + 2 \qquad (x \neq \frac{9}{2})$

Multiply by $(2x - 9)^2 > 0$:

$(6x - 5)(2x - 9) \leq (x + 2)(2x - 9)^2$

$0 \leq (2x - 9)[(x + 2)(2x - 9) - (6x - 5)]$

$0 \leq (2x - 9)(2x^2 - 5x - 18 - 6x + 5)$

$0 \leq (2x - 9)(2x^2 - 11x - 13)$

$0 \leq (2x - 9)(2x - 13)(x + 1)$

Critical values: $x = -1, \ x = \frac{9}{2}, \ x = \frac{13}{2}$

Cubic with positive leading coefficient, so it is ≥ 0

for $-1 \leq x \leq \frac{9}{2}$ or $x \geq \frac{13}{2}$

but $x = \frac{9}{2}$ is excluded as f is not defined there.

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Q7(b) continued

Check: $x = 0$: $f(0) = \frac{5}{9} \leq 2$ ✓; $x = 5$: $f(5) = 25 > 7$ ✗;

$x = 7$: $f(7) = 7.4 \leq 9$ ✓; $x = -2$: $f(-2) = \frac{17}{13} > 0$ ✗

Solution: $-1 \leq x < \frac{9}{2}$ or $x \geq \frac{13}{2}$

Turn over for the next question

Do not write outside the box

8 (a) The function f is defined as $f(x) = \sec x$

8 (a) (i) Show that $f^{(4)}(0) = 5$

[4 marks]

$f(x) = \sec x$

$f'(x) = \sec x \tan x$

$f''(x) = \sec x (\sec^2 x) + \tan x (\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$

$= \sec^3 x + \sec x (\sec^2 x - 1) = 2\sec^3 x - \sec x$

$f'''(x) = 6\sec^2 x (\sec x \tan x) - \sec x \tan x$

$= 6\sec^3 x \tan x - \sec x \tan x$

$f^{(4)}(x) = 18\sec^2 x (\sec x \tan x) \tan x + 6\sec^3 x \sec^2 x$

$- (\sec x \tan x) \tan x - \sec x \sec^2 x$

$= 18\sec^3 x \tan^2 x + 6\sec^5 x - \sec x \tan^2 x - \sec^3 x$

$\sec 0 = 1, \quad \tan 0 = 0:$

$f^{(4)}(0) = 0 + 6 - 0 - 1 = 5$ as required

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8 (a) (ii) Hence find the first three non-zero terms of the Maclaurin series for $f(x) = \sec x$

[2 marks]

$f(0) = 1,\ f'(0) = 0,\ f''(0) = 2 - 1 = 1,\ f'''(0) = 0,\ f^{(4)}(0) = 5$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$

$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots$

8 (b) Prove that

$$\lim_{x \to 0}\left(\frac{\sec x - \cosh x}{x^4}\right) = \frac{1}{6}$$

[4 marks]

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots,\quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$

$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$

$\sec x - \cosh x = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + O(x^6)\right) - \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)$

$= \frac{4x^4}{24} + O(x^6) = \frac{x^4}{6} + O(x^6)$

$\frac{\sec x - \cosh x}{x^4} = \frac{1}{6} + O(x^2)$ (higher powers of x)

As $x \to 0$, the $O(x^2)$ terms $\to 0$, so

$\lim_{x \to 0} \frac{\sec x - \cosh x}{x^4} = \frac{1}{6}$

Turn over ▶

Do not write outside the box

9 (a) A curve passes through the point (5, 12.3) and satisfies the differential equation

$$\frac{dy}{dx} = (x^2 - 9)^{\frac{1}{2}} + \frac{2xy}{x^2 - 9} \qquad x > 3$$

Use Euler's step by step method once, and then the midpoint formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r), \quad x_{r+1} = x_r + h$$

once, each with a step length of 0.1, to estimate the value of y when $x = 5.2$

Give your answer to six significant figures.

[4 marks]

$f(x, y) = (x^2 - 9)^{½} + \frac{2xy}{x^2 - 9}$, $x_0 = 5$, $y_0 = 12.3$, $h = 0.1$

$f(5, 12.3) = \sqrt{16} + \frac{2 \times 5 \times 12.3}{16} = 4 + 7.6875 = 11.6875$

Euler: $y_1 = 12.3 + 0.1 \times 11.6875 = 13.46875$, $x_1 = 5.1$

$f(5.1, 13.46875) = \sqrt{17.01} + \frac{2 \times 5.1 \times 13.46875}{17.01} = 12.2008172...$

Midpoint: $y_2 = y_0 + 2h\, f(x_1, y_1)$

$= 12.3 + 0.2 \times 12.2008172...$

$= 14.7401634...$

$y \approx 14.7402$ (6 s.f.)

9 (b) (i) Find the general solution of the differential equation

$$\frac{dy}{dx} = (x^2 - 9)^{\frac{1}{2}} + \frac{2xy}{x^2 - 9} \qquad (x > 3)$$

[6 marks]

$\frac{dy}{dx} - \frac{2x}{x^2 - 9} y = (x^2 - 9)^{½}$

$\int -\frac{2x}{x^2 - 9}\, dx = -\ln(x^2 - 9)$

Integrating factor $= e^{-\ln(x^2 - 9)} = \frac{1}{x^2 - 9}$

$\frac{1}{x^2 - 9} \frac{dy}{dx} - \frac{2x}{(x^2 - 9)^2} y = (x^2 - 9)^{-½}$

$\frac{d}{dx}\left(\frac{y}{x^2 - 9}\right) = \frac{1}{\sqrt{(x^2 - 9)}}$

$\frac{y}{x^2 - 9} = \int \frac{1}{\sqrt{(x^2 - 9)}}\, dx = \cosh^{-1}\left(\frac{x}{3}\right) + c$

Do not write outside the box

Q9(b)(i) continued

$y = (x^2 - 9)(\cosh^{-1}(\frac{x}{3}) + c)$

9 (b) (ii) Given that y satisfies the differential equation in part **(b)(i)** and that $y = 12.3$ when $x = 5$, find the value of y when $x = 5.2$

Give your answer to six significant figures.

[3 marks]

$x = 5, y = 12.3$: $\frac{12.3}{16} = \cosh^{-1}(\frac{5}{3}) + c$

$c = 0.76875 - \cosh^{-1}(\frac{5}{3}) = 0.76875 - \ln 3 = -0.3298622...$

$x = 5.2$: $y = (5.2^2 - 9)(\cosh^{-1}(\frac{5.2}{3}) - 0.3298622...)$

$= 18.04 \times (1.1471222... - 0.3298622...)$

$= 14.7433690...$

$y = 14.7434$ (6 s.f.)

9 (c) Comment on the accuracy of your answer to part **(a)**.

[1 mark]

14.7402 and 14.7434 agree to 4 s.f. (14.74), so the estimate in (a) is very accurate (error about 0.02%).

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10 The curve C_1 has equation

$$\frac{x^2}{25} - \frac{y^2}{4} = 1$$

The curve C_2 has equation

$$x^2 - 25y^2 - 6x - 200y - 416 = 0$$

10 (a) Find a sequence of transformations that maps the graph of C_1 onto the graph of C_2

[4 marks]

C_2: $(x^2 - 6x) - 25(y^2 + 8y) = 416$

$(x - 3)^2 - 9 - 25(y + 4)^2 + 400 = 416$

$(x - 3)^2 - 25(y + 4)^2 = 25$

$\frac{(x - 3)^2}{25} - (y + 4)^2 = 1$

C_1: $\frac{x^2}{25} - \frac{y^2}{4} = 1$

Stretch scale factor ½ parallel to the y-axis (replace y by 2y):

$\frac{x^2}{25} - \frac{(2y)^2}{4} = 1 \Rightarrow \frac{x^2}{25} - y^2 = 1$

Translation by vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ (replace x by x − 3, y by y + 4):

$\frac{(x - 3)^2}{25} - (y + 4)^2 = 1$, which is C_2

So: a stretch, scale factor ½, parallel to the y-axis, followed by a translation by the vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

Do not write outside the box

10 (b) Find the equations of the asymptotes to C_2

Give your answers in the form $ax + by + c = 0$ where a, b and c are integers.

[3 marks]

Asymptotes of C_1: $y = \pm\frac{2}{5}x$

Stretch s.f. ½ parallel to y-axis: $2y = \pm\frac{2}{5}x \Rightarrow y = \pm\frac{1}{5}x \Rightarrow 5y = \pm x$

Translation $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$: $5(y + 4) = \pm(x - 3)$

$5y + 20 = x - 3 \Rightarrow x - 5y - 23 = 0$

$5y + 20 = -x + 3 \Rightarrow x + 5y + 17 = 0$

Asymptotes: $x - 5y - 23 = 0$ and $x + 5y + 17 = 0$

Turn over for the next question

Do not write outside the box

11 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{13}{2} \end{bmatrix}$$

[5 marks]

$\det(M - \lambda I) = 0$: $\left(\frac{5}{2} - \lambda\right)\left(\frac{13}{2} - \lambda\right) - \frac{9}{4} = 0$

$\lambda^2 - 9\lambda + \frac{65}{4} - \frac{9}{4} = 0$

$\lambda^2 - 9\lambda + 14 = 0$

$(\lambda - 2)(\lambda - 7) = 0 \Rightarrow \lambda = 2, \lambda = 7$

$\lambda = 2$: $\begin{bmatrix} 1/2 & -3/2 \\ -3/2 & 9/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{1}{2}x - \frac{3}{2}y = 0 \Rightarrow x = 3y$

eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = 7$: $\begin{bmatrix} -9/2 & -3/2 \\ -3/2 & -1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\frac{9}{2}x - \frac{3}{2}y = 0 \Rightarrow y = -3x$

eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\lambda = 2$: eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\lambda = 7$: eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (or multiples)

Do not write outside the box

11 (b) (i) Describe how the directions of the invariant lines of the transformation represented by **M** are related to each other.

Fully justify your answer.

[2 marks]

The invariant lines are in the directions of the eigenvectors.

$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 3 \times (-1) + 1 \times 3 = 0$

so the directions of the invariant lines are perpendicular.

11 (b) (ii) Describe fully the transformation represented by **M**

[2 marks]

Stretch, scale factor 2, parallel to $y = \frac{1}{3}x$ [direction $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$]

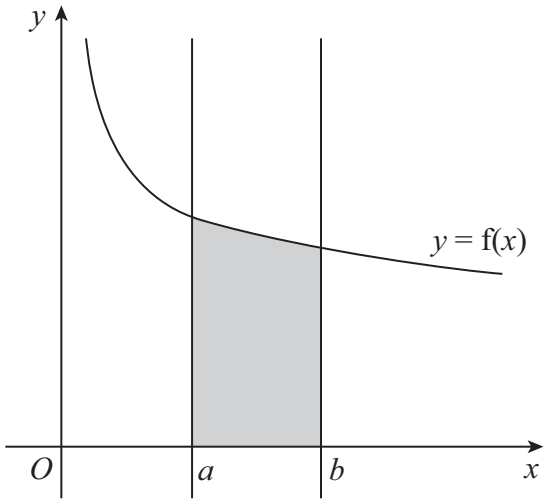
and stretch, scale factor 7, parallel to $y = -3x$ [direction $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$]

(a two-way stretch)

Turn over for the next question

Do not write outside the box

12 The shaded region shown in the diagram below is bounded by the x-axis, the curve $y = \mathrm{f}(x)$, and the lines $x = a$ and $x = b$



The shaded region is rotated through 2π radians about the x-axis to form a solid.

12 (a) Show that the volume of this solid is

$$\pi \int_a^b (\mathrm{f}(x))^2 \, \mathrm{d}x$$

[4 marks]

Split the region into thin vertical strips of width δx. The strip at x has height $y = f(x)$.

When a strip is rotated through 2π about the x-axis it forms a thin disc, approximately a cylinder of radius y and thickness δx:

Volume of disc ≈ $\pi y^2 \, \delta x$

Total volume ≈ $\Sigma \, \pi y^2 \, \delta x$ (sum from x = a to x = b)

Volume of solid = $\lim_{\delta x \to 0} \Sigma \, \pi y^2 \, \delta x$

$$= \pi \int_a^b y^2 \, dx = \pi \int_a^b (f(x))^2 \, dx$$

Do not write outside the box

12 (b) In the case where $a = 1$, $b = 2$ and

$$f(x) = \frac{x+3}{(x+1)\sqrt{x}}$$

show that the volume of the solid is

$$\pi\left(\ln\left(\frac{2^m}{3^n}\right) - \frac{2}{3}\right)$$

where m and n are integers.

[7 marks]

$(f(x))^2 = \frac{(x+3)^2}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$(x+3)^2 \equiv A(x+1)^2 + Bx(x+1) + Cx$

$x = 0$: $9 = A \Rightarrow A = 9$

$x = -1$: $4 = -C \Rightarrow C = -4$

x^2 coefficients: $1 = A + B \Rightarrow B = -8$

$(f(x))^2 = \frac{9}{x} - \frac{8}{x+1} - \frac{4}{(x+1)^2}$

$V = \pi\int_1^2 \left(\frac{9}{x} - \frac{8}{x+1} - \frac{4}{(x+1)^2}\right) dx$

$= \pi\left[9\ln x - 8\ln(x+1) + \frac{4}{x+1}\right]_1^2$

$= \pi\left[\left(9\ln 2 - 8\ln 3 + \frac{4}{3}\right) - (9\ln 1 - 8\ln 2 + 2)\right]$

$= \pi\left(17\ln 2 - 8\ln 3 - \frac{2}{3}\right)$

$= \pi\left(\ln\left(\frac{2^{17}}{3^8}\right) - \frac{2}{3}\right)$

which is of the required form with $m = 17$, $n = 8$

Do not write outside the box

13 (a) The matrix **A** represents a reflection in the line $y = mx$, where m is a constant.

Show that $\mathbf{A} = \left(\frac{1}{m^2+1}\right)\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$

You may use the result in the formulae booklet.

[5 marks]

Reflection in $y = (\tan\theta)x$: $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ where $m = \tan\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta(1 - \tan^2\theta) = \cos^2\theta(1 - m^2)$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\tan\theta\cos^2\theta = 2m\cos^2\theta$

$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{m^2+1}$

so $\cos 2\theta = \frac{1-m^2}{m^2+1}$, $\sin 2\theta = \frac{2m}{m^2+1}$, $-\cos 2\theta = \frac{m^2-1}{m^2+1}$

$A = \frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$

Do not write outside the box

13 (b) The matrix **B** is defined as $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Show that $(\mathbf{BA})^2 = k\mathbf{I}$

where $\mathbf{I}$ is the 2×2 identity matrix and k is an integer.

[3 marks]

$B = 3I, \quad BA = \frac{3}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$

$(BA)^2 = \frac{9}{(m^2+1)^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$

$= \frac{9}{(m^2+1)^2}\begin{bmatrix} (1-m^2)^2+4m^2 & 2m(1-m^2)+2m(m^2-1) \\ 2m(1-m^2)+2m(m^2-1) & 4m^2+(m^2-1)^2 \end{bmatrix}$

$= \frac{9}{(m^2+1)^2}\begin{bmatrix} 1+2m^2+m^4 & 0 \\ 0 & 1+2m^2+m^4 \end{bmatrix}$

$= \frac{9}{(m^2+1)^2}\begin{bmatrix} (m^2+1)^2 & 0 \\ 0 & (m^2+1)^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$(BA)^2 = 9I, \quad$ so $\quad k = 9$

Question 13 continues on the next page

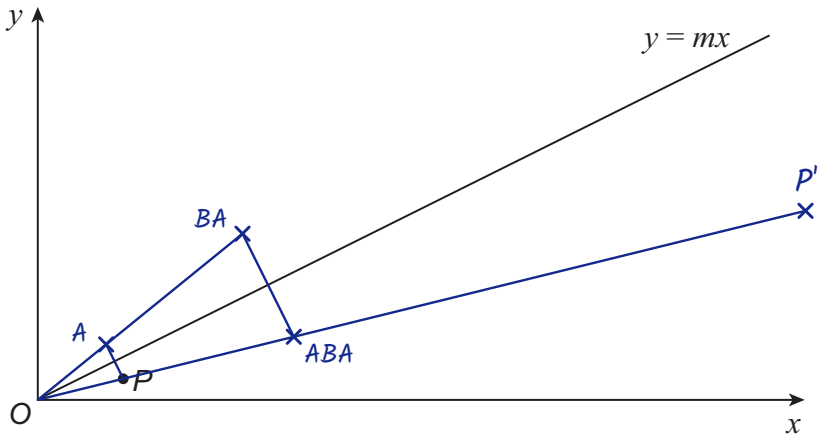
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13 (c) (i) The diagram below shows a point P and the line $y = mx$

Draw four lines on the diagram to demonstrate the result proved in part **(b)**.

Label as P' the image of P under the transformation represented by $(\mathbf{BA})^2$

[2 marks]



13 (c) (ii) Explain how your completed diagram shows the result proved in part **(b)**.

[2 marks]

The lines show the effect on P of A (reflection in y = mx), then B (enlargement, scale factor 3, centre O), then A again, then B again. The end point P' lies on the line OP with OP' = 9 × OP, so $(BA)^2$ is an enlargement scale factor 9, centre O, i.e. $(BA)^2 = 9I$.

Do not write outside the box

13 (d) The matrix **C** is defined as $\mathbf{C} = \begin{bmatrix} \frac{12}{5} & \frac{9}{5} \\ \frac{9}{5} & -\frac{12}{5} \end{bmatrix}$

Find the value of m such that $\mathbf{C} = \mathbf{BA}$

Fully justify your answer.

[4 marks]

$C = BA$: $\begin{bmatrix} 12/5 & 9/5 \\ 9/5 & -12/5 \end{bmatrix} = \frac{3}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$

Top right: $\frac{6m}{m^2+1} = \frac{9}{5}$

$30m = 9m^2 + 9$

$3m^2 - 10m + 3 = 0$

$(3m-1)(m-3) = 0 \Rightarrow m = 3$ or $m = \frac{1}{3}$

Top left: $\frac{3(1-m^2)}{m^2+1} = \frac{12}{5}$

$m = 3$: $\frac{3(1-9)}{10} = -\frac{12}{5} \neq \frac{12}{5}$ so reject $m = 3$

$m = \frac{1}{3}$: $\frac{3 \times 8/9}{10/9} = \frac{24}{10} = \frac{12}{5}$ ✓

(and bottom right $\frac{3(m^2-1)}{m^2+1} = -\frac{12}{5}$ ✓)

$m = \frac{1}{3}$

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14 On an isolated island some rabbits have been accidently introduced.

In order to eliminate them, conservationists have introduced some birds of prey.

At time t years $(t \geq 0)$ there are x rabbits and y birds of prey.

At time $t = 0$ there are 1755 rabbits and 30 birds of prey.

When $t > 0$ it is assumed that:

- the rabbits will reproduce at a rate of a% per year
- each bird of prey will kill, on average, b rabbits per year
- the death rate of the birds of prey is c birds per year
- the number of birds of prey will increase at a rate of d% of the rabbit population per year.

This system is represented by the coupled differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.4x - 13y \qquad (1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.01x - 1.95 \qquad (2)$$

14 (a) State the value of a, the value of b, the value of c and the value of d

[2 marks]

$a = 40, \quad b = 13, \quad c = 1.95, \quad d = 1$

14 (b) Solve the coupled differential equations to find both x and y in terms of t

[9 marks]

From (1): $13y = 0.4x - \dot{x} \quad \Rightarrow \quad y = \frac{0.4x - \dot{x}}{13}$

Differentiate: $\dot{y} = \frac{0.4\dot{x} - \ddot{x}}{13}$

Sub into (2): $\frac{0.4\dot{x} - \ddot{x}}{13} = 0.01x - 1.95$

$0.4\dot{x} - \ddot{x} = 0.13x - 25.35$

$\ddot{x} - 0.4\dot{x} + 0.13x = 25.35$

Auxiliary equation: $m^2 - 0.4m + 0.13 = 0$

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Q14(b) continued

$m = \frac{0.4 \pm \sqrt{(0.16 - 0.52)}}{2} = 0.2 \pm 0.3i$

CF: $x = e^{0.2t}(A \cos 0.3t + B \sin 0.3t)$

PI: $x = k \Rightarrow 0.13k = 25.35 \Rightarrow k = 195$

$x = e^{0.2t}(A \cos 0.3t + B \sin 0.3t) + 195$

$\dot{x} = e^{0.2t}[(0.2A + 0.3B) \cos 0.3t + (0.2B - 0.3A) \sin 0.3t]$

$y = \frac{0.4x - \dot{x}}{13}$

$= \frac{1}{13}e^{0.2t}[(0.2A - 0.3B) \cos 0.3t + (0.3A + 0.2B) \sin 0.3t] + \frac{78}{13}$

$= \frac{1}{13}e^{0.2t}[(0.2A - 0.3B) \cos 0.3t + (0.3A + 0.2B) \sin 0.3t] + 6$

$t = 0$: $x = 1755 \Rightarrow A + 195 = 1755 \Rightarrow A = 1560$

$t = 0$: $y = 30 \Rightarrow \frac{0.2A - 0.3B}{13} + 6 = 30 \Rightarrow 312 - 0.3B = 312 \Rightarrow B = 0$

$x = 1560e^{0.2t} \cos(0.3t) + 195$

$y = 24e^{0.2t} \cos(0.3t) + 36e^{0.2t} \sin(0.3t) + 6$

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14 (c) Given that x and y are both positive for $0 \leq t \leq 5$, use your answer to part **(b)** to show that the conservationists' plan will succeed.

[3 marks]

$t = 5$: $x = 1560e^{1} \cos(1.5) + 195 = 495.0 > 0$

$t = 5.4$: $x = 1560e^{1.08} \cos(1.62) + 195 = -30.9 < 0$

x is continuous, so $x = 0$ for some t between 5 and 5.4: solving gives $t =$ $5.378... \approx 5.38$

When $t = 5.38$: $y = 24e^{1.076} \cos(1.614) + 36e^{1.076} \sin(1.614) + 6 \approx 108$

So $y > 0$ (about 108 birds of prey) when the rabbit population reaches zero. The rabbits die out first, so the conservationists' plan will succeed.

END OF QUESTIONS

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