AQA

Please write clearly in block capitals.

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A-level
FURTHER MATHEMATICS

Paper 3 Discrete

Thursday 13 June 2019 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)
- You must ensure you have the other optional Question Paper/Answer Book for which you are entered (**either** Mechanics **or** Statistics). You will have 2 hours to complete **both** papers.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

For Examiner's Use	
Question	Mark
1	
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7	
8	
TOTAL	

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 50.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

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Answer **all** questions in the spaces provided.

1 Deanna and Will play a zero-sum game.

The game is represented by the following pay-off matrix for Deanna.

		Will		
	Strategy	**X**	**Y**	**Z**
Deanna	**A**	−1	0	2
	B	−2	−1	3
	C	5	−2	−3
	D	6	−2	0

Which strategy is Deanna's play-safe strategy?

Circle your answer.

[1 mark]

A **B** **C** **D**

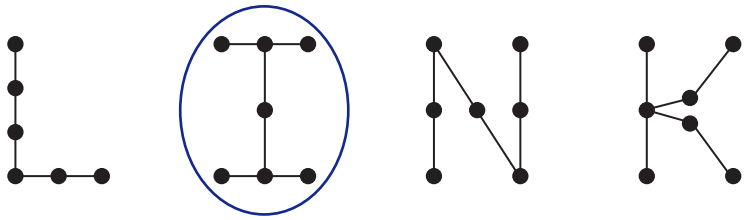
2 The graph *D* is shown in the diagram below.



Which of the graphs below is a subdivision of *D*?

Circle your answer.

[1 mark]



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3 The Simplex tableau below is optimal.

P	x	y	z	r	s	value
1	$k^2 + k - 6$	0	0	$k - 1$	1	20
0	0	0	1	1.5	0	6
0	0	1	0	0	0.5	86

3 (a) Deduce the range of values that k must satisfy.

[3 marks]

Optimal, so no negative entries in the objective row:

$k^2 + k - 6 \geq 0$ and $k - 1 \geq 0$

$(k + 3)(k - 2) \geq 0 \Rightarrow$ critical values $k = 2$, $k = -3$

$\Rightarrow k \leq -3$ or $k \geq 2$

$k - 1 \geq 0 \Rightarrow k \geq 1$

Both inequalities must be satisfied simultaneously, so

$k \geq 2$

3 (b) Write down the value of the variable s which corresponds to the optimal value of P.

[1 mark]

s is non-basic (its column is not a unit column), so

$s = 0$

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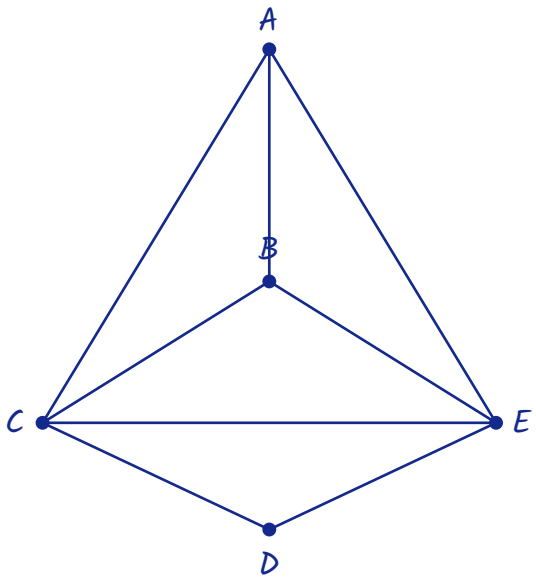
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4 The connected planar graph P has the adjacency matrix

	A	B	C	D	E
A	0	1	1	0	1
B	1	0	1	0	1
C	1	1	0	1	1
D	0	0	1	0	1
E	1	1	1	1	0

4 (a) Draw P

[1 mark]



4 (b) Using Euler's formula for connected planar graphs, show that P has exactly 5 faces.

[2 marks]

Euler's formula: $v - e + f = 2$

$v = 5$ vertices

$e = 8$ edges (sum of entries in the matrix = 16, $16 \div 2 = 8$)

$f = 2 + e - v = 2 + 8 - 5$

$f = 5$

Therefore the graph P has exactly 5 faces.

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4 (c) Ore's theorem states that a simple graph with n vertices is Hamiltonian if, for every pair of vertices X and Y which are not adjacent,

$$\text{degree of } X + \text{degree of } Y \geq n$$

where $n \geq 3$

Using Ore's theorem, prove that the graph P is Hamiltonian.

Fully justify your answer.

[3 marks]

P is simple with n = 5 vertices (n ≥ 3).

Pairs of vertices which are not adjacent: A & D and B & D

deg A = 3, deg B = 3, deg C = 4, deg D = 2, deg E = 4

A & D: 3 + 2 = 5 ≥ 5

B & D: 3 + 2 = 5 ≥ 5

So for every pair of non-adjacent vertices X and Y,

degree of X + degree of Y ≥ 5 = n.

Hence, by Ore's theorem, the graph P is Hamiltonian.

(e.g. Hamiltonian cycle A B C D E A)

Turn over for the next question

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5 The set S is defined as

$$S = \{A, B, C, D\}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The group G is formed by S under matrix multiplication.

The group H is defined as $H = (\langle \mathrm{i} \rangle, \times)$, where $\mathrm{i}^2 = -1$

5 (a) (i) Prove that B is a generator of G.

Fully justify your answer.

[3 marks]

$B^2 = BB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = C$

$B^3 = B^2B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = D$

$B^4 = B^3B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$

So $B^1 = B$, $B^2 = C$, $B^3 = D$ and $B^4 = A$ (the identity).

Each of B, B^2, B^3, B^4 is a different element of G, so repeated multiplication of B by itself produces every element of G.

Hence B is a generator of G.

5 (a) (ii) Show that $G \cong H$.

Fully justify your answer.

[4 marks]

From (a)(i), $G = \langle B \rangle$ is a cyclic group of order 4.

$H = \langle i \rangle$: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

so $H = \{1, i, -1, -i\}$ is also cyclic of order 4.

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Mapping $\varphi : G \to H$ defined by $\varphi(B^n) = i^n$:

$B \to i$

$B^2 = C \to i^2 = -1$

$B^3 = D \to i^3 = -i$

$B^4 = A \to i^4 = 1$

This is a one-to-one mapping between the elements of G and H,

and $\varphi(B^m B^n) = i^{m+n} = i^m i^n = \varphi(B^m)\varphi(B^n)$, so the

group structure is preserved by the mapping.

Therefore G is isomorphic to H, i.e. $G \cong H$.

5 (b) (i) Explain why *H* has no subgroups of order 3

Fully justify your answer.

[2 marks]

The order of H is 4.

By Lagrange's theorem the order of any subgroup must divide

the order of the group. 4 is not divisible by 3, so H cannot

have a subgroup of order 3.

5 (b) (ii) Find **all** of the subgroups of *H*.

[3 marks]

By Lagrange, subgroups can only have order 1, 2 or 4.

Order 1: $\{1\}$ (the trivial subgroup)

Order 2: $\{1, -1\}$ since $(-1)^2 = 1$

(i and $-i$ have order 4, so they are not in a subgroup of order 2)

Order 4: $H = \{1, -1, i, -i\}$

The subgroups of H are:

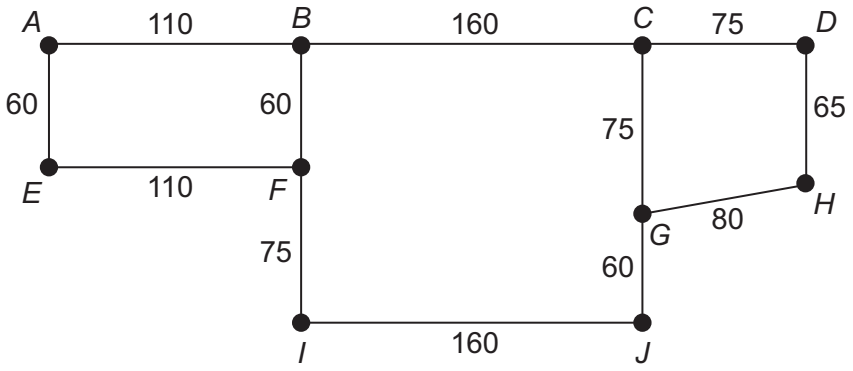
$\{1\}$, $\{1, -1\}$ and $\{1, -1, i, -i\}$

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6 A council wants to monitor how long cars are being parked for in short-stay parking bays in a town centre. They employ a traffic warden to walk along the streets in the town centre and issue fines to drivers who park for longer than the stated time.

The network below shows streets in the town centre which have short-stay parking bays. Each node represents a street corner and the weight of each arc represents the length, in metres, of the street.

The short-stay parking bays are positioned along only one side of each street.



The council assumes that the traffic warden will walk at an average speed of 4.8 kilometres per hour when not issuing fines.

6 (a) To monitor all of the parking bays, the traffic warden needs to walk along every street in the town centre at least once, starting and finishing at the same street corner.

Find the shortest possible time, to the nearest minute, it can take the traffic warden to monitor all of the parking bays.

Fully justify your answer.

[5 marks]

Route inspection problem. Odd nodes: B, C, F, G

Shortest distances: B–C = 160, B–F = 60, B–G = 235,

C–F = 220, C–G = 75, F–G = 295

Pairings:

(B–C)(F–G) = 160 + 295 = 455

(B–F)(C–G) = 60 + 75 = 135 ← minimum

(B–G)(C–F) = 235 + 220 = 455

So repeat B–F and C–G (extra 135 m).

Total length of all streets = 1090 m

Minimum distance = 1090 + 135 = 1225 m

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Speed = 4.8 km/h = 4800 m per hour

Time = (1225 ÷ 4800) × 60

= 15.3125 minutes

Shortest possible time = 15 minutes (to the nearest minute)

6 (b) Explain why the actual time for the traffic warden to walk along every street in the town centre at least once may be different to the value found in part **(a)**.

[1 mark]

The warden may have to stop to issue fines, which reduces the average speed, so the actual time taken to walk along every street will be longer than 15 minutes.

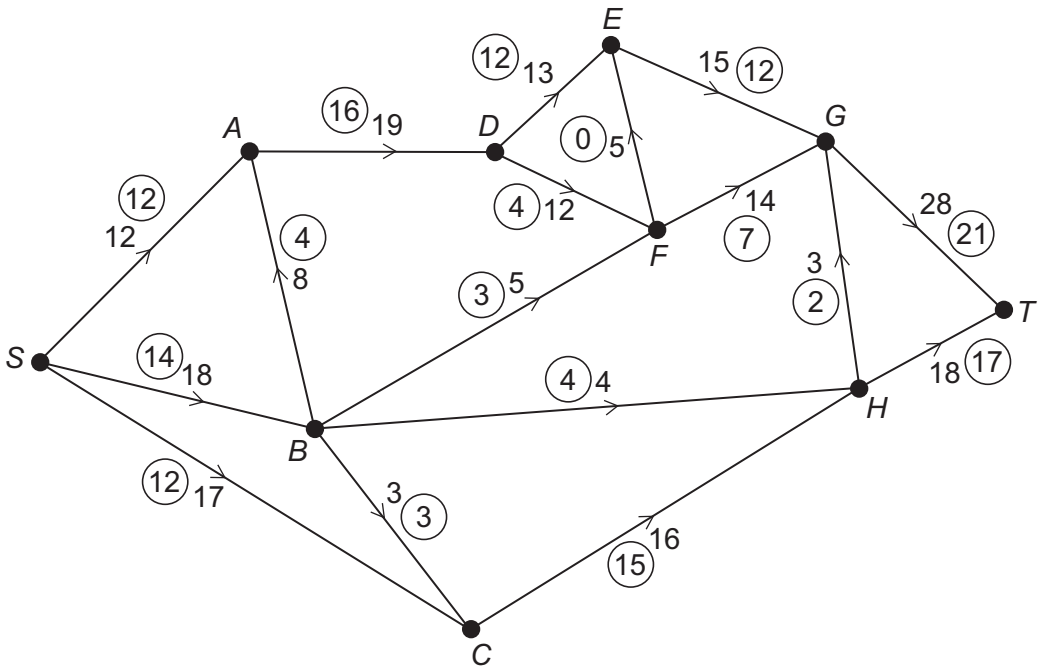
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7 **Figure 1** shows a system of water pipes in a manufacturing complex.

The number on each arc represents the upper capacity for each pipe in litres per second. The numbers in the circles represent an initial feasible flow of 38 litres of water per second.

Figure 1



7 (a) (i) Calculate the value of the cut $\{S, A, B, C\}$ $\{D, E, F, G, H, T\}$.

[1 mark]

Arcs from {S, A, B, C} to {D, E, F, G, H, T}: AD, BF, BH, CH

Value of cut = 19 + 5 + 4 + 16 = 44 litres per second

7 (a) (ii) Explain, in the context of the question, what can be deduced from your answer to part **(a)(i)**.

[1 mark]

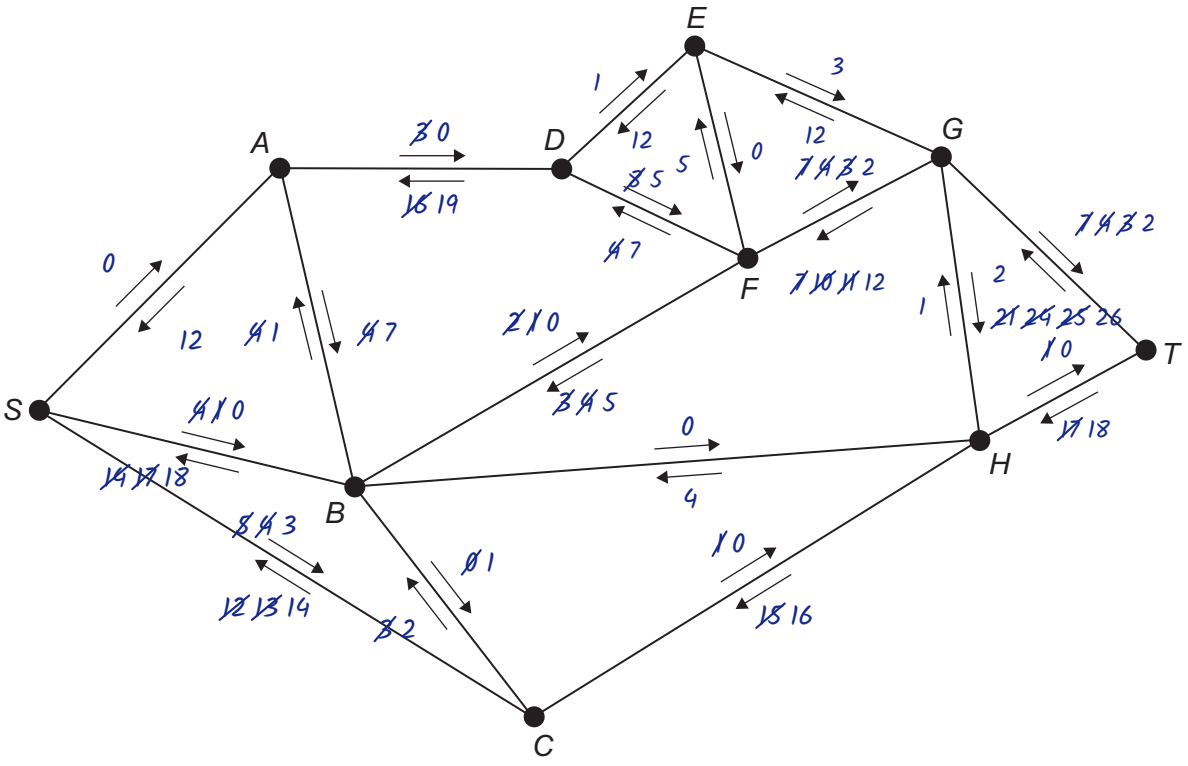
The maximum flow of water through the system of pipes is less than or equal to 44 litres per second.

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7 (b) (i) Using the initial feasible flow shown in **Figure 1**, indicate on **Figure 2** potential increases and decreases in the flow along each arc.

[2 marks]

Figure 2



7 (b) (ii) Use flow augmentation on **Figure 2** to find the maximum flow through the manufacturing complex.

You must indicate any flow augmenting paths clearly in the table and modify the potential increases and decreases of the flow on **Figure 2**.

[4 marks]

Augmenting Path	Flow
S C H T	1
S B A D F G T	3
S B F G T	1
S C B F G T	1

Maximum Flow = 38 + 6 = 44 litres per second

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7 (c) The management of the manufacturing complex want to increase the maximum amount of water which can flow through the system of pipes. To do this they decide to upgrade one of the water pipes by replacing it with a larger capacity pipe.

Explain which pipe should be upgraded.

Deduce what effect this upgrade will have on the maximum amount of water which can flow through the system of pipes.

[2 marks]

The minimum cut has value 44 and is made up of pipes AD, BF, BH and CH, so the upgraded pipe must be one of these.

Pipe BF should be upgraded (upgrading AD, BH or CH would only increase the maximum flow to 45 litres per second).

With BF upgraded, the new minimum cut is GT + HT = 28 + 18, so the maximum flow increases by 2 to 46 litres per second.

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8 A motor racing team is undertaking a project to build next season's racing car. The project is broken down into 12 separate activities *A*, *B*, ..., *L*, as shown in the precedence table below. Each activity requires one member of the racing team.

Activity	Duration (days)	Immediate Predecessors
A	7	–
B	6	–
C	15	–
D	9	*A*, *B*
E	8	*D*
F	6	*C*, *D*
G	7	*C*
H	14	*E*
I	17	*F*, *G*
J	9	*H*, *I*
K	8	*I*
L	12	*J*, *K*

8 (a) (i) Complete the activity network for the project on **Figure 3**.

[2 marks]

8 (a) (ii) Find the earliest start time and the latest finish time for each activity and show these values on **Figure 3**.

[3 marks]

8 (b) Write down the critical path(s).

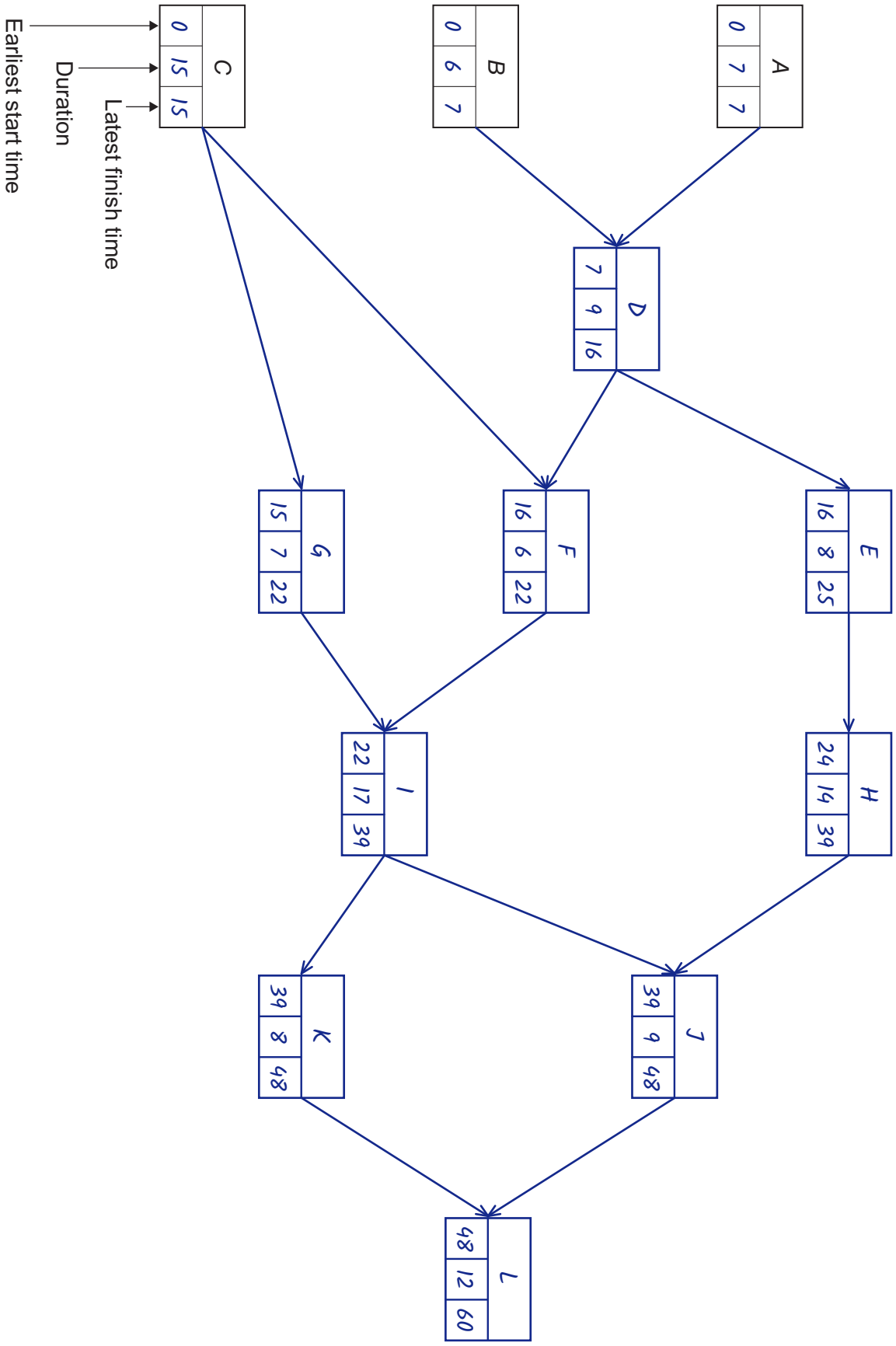
[1 mark]

A – D – F – I – J – L

and C – G – I – J – L

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Figure 3



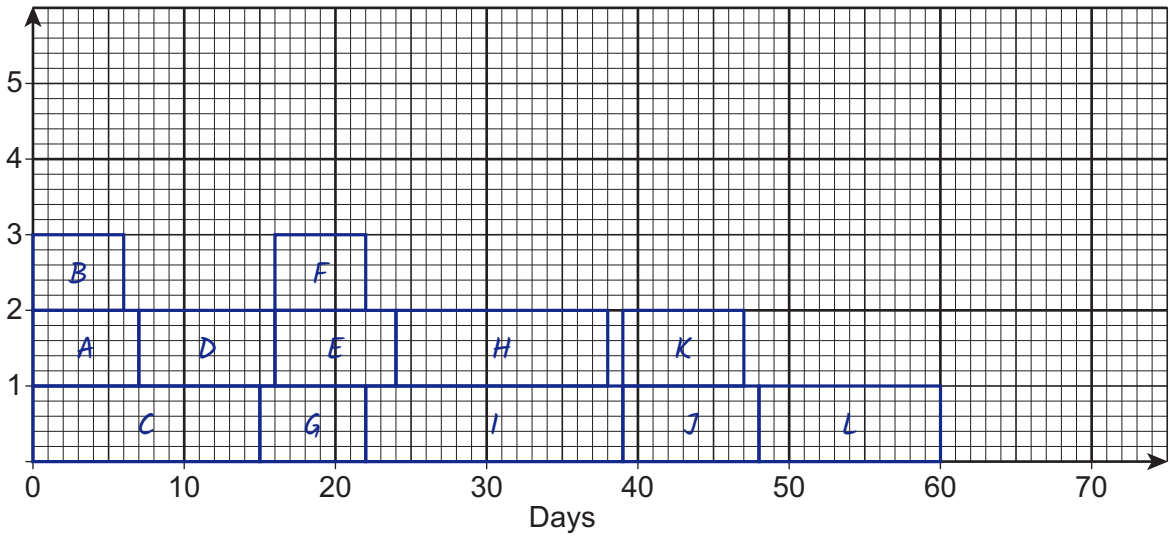
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8 (c) (i) Using **Figure 4**, draw a resource histogram for the project to show how the project can be completed in the shortest possible time. Assume that each activity is to start as early as possible.

[2 marks]

Figure 4



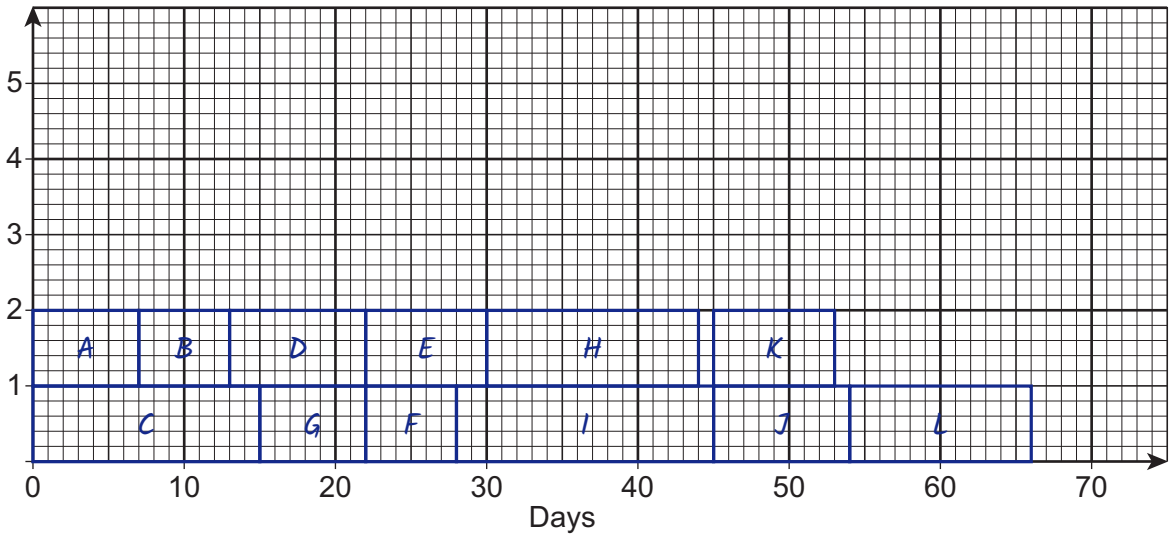
8 (c) (ii) The racing team's boss assigns two members of the racing team to work on the project.

Explain the effect this has on the minimum completion time for the project.

You may use **Figure 5** in your answer.

[2 marks]

Figure 5



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A, B and C cannot all be done at the same time as this needs 3 members, so B must follow A (B finishes on day 13).

This delays the earliest start time for D from 7 to 13 days.

E, F and G then share the two members, and the remaining activities only need two members at a time (Figure 5).

The minimum completion time increases from 60 to 66 days.

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