AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

A-level
FURTHER MATHEMATICS

Paper 2

Thursday 6 June 2019 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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TOTAL	

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Answer **all** questions in the spaces provided.

1 Given that z is a complex number, and that z^* is the complex conjugate of z, which of the following statements is **not** always true?

Circle your answer.

[1 mark]

$(z^*)^* = z$ $\quad$ $zz^* = |z|^2$ $\quad$ $(-z)^* = -(z^*)$ $\quad$ $z - z^* = z^* - z$

2 Which of the straight lines given below is an asymptote to the curve

$$y = \frac{ax^2}{x - 1}$$

where a is a non-zero constant?

Circle your answer.

[1 mark]

$y = ax + a$ $\quad$ $y = ax$ $\quad$ $y = ax - a$ $\quad$ $y = a$

3 The set $\mathcal{A}$ is defined by $\mathcal{A} = \{x : -\sqrt{2} < x < 0\} \cup \{x : 0 < x < \sqrt{2}\}$

Which of the inequalities given below has $\mathcal{A}$ as its solution?

Circle your answer.

[1 mark]

$|x^2 - 1| > 1$ $\quad$ $|x^2 - 1| \geq 1$ $\quad$ $|x^2 - 1| < 1$ $\quad$ $|x^2 - 1| \leq 1$

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4 The positive integer k is such that

$$\sum_{r=1}^{k}(3r-k)=90$$

Find the value of k.

[3 marks]

$\sum_{r=1}^{k}(3r-k) = 3\sum_{r=1}^{k} r - k \times k = \frac{3}{2}k(k+1) - k^2$

$\frac{3}{2}k^2 + \frac{3}{2}k - k^2 = 90$

$\frac{1}{2}k^2 + \frac{3}{2}k = 90$

$k^2 + 3k - 180 = 0$

$(k-12)(k+15) = 0$

k is a positive integer, so $k = 12$

Turn over for the next question

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5 A curve has equation $y = \cosh x$

Show that the arc length of the curve from $x = a$ to $x = b$, where $0 < a < b$, is equal to

$$\sinh b - \sinh a$$

[4 marks]

Arc length $s = \int_a^b \sqrt{(1 + (dy/dx)^2)}\, dx$

$y = \cosh x \Rightarrow dy/dx = \sinh x$

$1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$

$s = \int_a^b \sqrt{(\cosh^2 x)}\, dx = \int_a^b \cosh x\, dx$ (since $\cosh x > 0$)

$= [\sinh x]_a^b$

$= \sinh b - \sinh a$ as required

Do not write outside the box

6 A circle C in the complex plane has equation $|z - 2 - 5i| = a$

The point z_1 on C has the least argument of any point on C, and $\arg(z_1) = \frac{\pi}{4}$

Prove that $a = \frac{3\sqrt{2}}{2}$

[6 marks]

C has centre 2 + 5i, i.e. the point (2, 5), and radius a.

z_1 has the least argument of any point on C, so the half-line

arg z = π/4 (the line y = x, x > 0) is a tangent to C at z_1.

arg z_1 = π/4, so Re z_1 = Im z_1: let $z_1 = k + ki$

The radius to z_1 is perpendicular to the tangent (gradient 1), so

gradient from (2, 5) to (k, k) is −1: $\frac{5 - k}{2 - k} = -1$

$5 - k = k - 2 \Rightarrow 2k = 7 \Rightarrow k = 3.5$

$a^2 = (5 - 3.5)^2 + (2 - 3.5)^2 = 2.25 + 2.25 = 4.5$

$a = \sqrt{4.5} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ (a > 0)

Check: distance from (2, 5) to x − y = 0 is $\frac{|2 - 5|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ ✓

$\therefore a = \frac{3\sqrt{2}}{2}$ as required

Turn over ▶

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7 The points A, B and C have coordinates $A(4, 5, 2)$, $B(-3, 2, -4)$ and $C(2, 6, 1)$

7 (a) Use a vector product to show that the area of triangle ABC is $\frac{5\sqrt{11}}{2}$

[4 marks]

$\overrightarrow{AB} = b - a = -7i - 3j - 6k$

$\overrightarrow{AC} = c - a = -2i + j - k$

$\overrightarrow{AB} \times \overrightarrow{AC} = ((-3)(-1) - (-6)(1))i - ((-7)(-1) - (-6)(-2))j$

$+ ((-7)(1) - (-3)(-2))k$

$= (3 + 6)i - (7 - 12)j + (-7 - 6)k = 9i + 5j - 13k$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(9^2 + 5^2 + (-13)^2)} = \sqrt{275} = 5\sqrt{11}$

Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \times 5\sqrt{11}$

Area $= \frac{5\sqrt{11}}{2}$ as required

7 (b) The points A, B and C lie in a plane.

Find a vector equation of the plane in the form $\mathbf{r.n} = k$

[1 mark]

Normal $n = 9i + 5j - 13k$; $a.n = 4(9) + 5(5) + 2(-13) = 35$

$r.(9i + 5j - 13k) = 35$

Do not write outside the box

7 (c) Hence find the exact distance of the plane from the origin.

[1 mark]

Distance $= \frac{35}{|n|} = \frac{35}{5\sqrt{11}} = \frac{7}{\sqrt{11}} = \frac{7\sqrt{11}}{11}$

Turn over for the next question

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8 A parabola P_1 has equation $y^2 = 4ax$ where $a > 0$

P_1 is translated by the vector $\begin{bmatrix} b \\ 0 \end{bmatrix}$, where $b > 0$, to give the parabola P_2

8 (a) The line $y = mx$ is a tangent to P_2

Prove that $m = \pm\sqrt{\frac{a}{b}}$

Solutions using differentiation will be given no marks.

[4 marks]

P_2: $y^2 = 4a(x - b)$

Sub $y = mx$: $m^2x^2 = 4a(x - b)$

$m^2x^2 - 4ax + 4ab = 0$

$y = mx$ is a tangent $\Rightarrow$ equal roots $\Rightarrow$ discriminant $= 0$

$(-4a)^2 - 4(m^2)(4ab) = 0$

$16a^2 = 16m^2ab \quad \Rightarrow \quad m^2 = \frac{a}{b} \quad (a \neq 0)$

$m = \pm\sqrt{\frac{a}{b}}$ as required

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8 (b) The line $y = \sqrt{\frac{a}{b}}\,x$ meets P_2 at the point D.

The finite region R is bounded by the x-axis, P_2 and a line through D perpendicular to the x-axis.

The region R is rotated through 2π radians about the x-axis to form a solid.

Find, in terms of a and b, the volume of this solid.

Fully justify your answer.

[5 marks]

Sub $y = \sqrt{\frac{a}{b}}\,x$ into P_2: $\frac{a}{b}x^2 = 4a(x - b)$

$ax^2 = 4abx - 4ab^2 \Rightarrow x^2 - 4bx + 4b^2 = 0 \quad (a \neq 0)$

$(x - 2b)^2 = 0 \Rightarrow x = 2b$, so D is the point where $x = 2b$

P_2 meets the x-axis at its vertex $(b, 0)$, so R lies between $x = b$ and $x = 2b$

$V = \pi\int_b^{2b} y^2\,dx = \pi\int_b^{2b} 4a(x - b)\,dx$

$= 4a\pi\left[\frac{x^2}{2} - bx\right]_b^{2b}$

$= 4a\pi\left\{\left(\frac{4b^2}{2} - 2b^2\right) - \left(\frac{b^2}{2} - b^2\right)\right\}$

$= 4a\pi\left\{0 + \frac{b^2}{2}\right\}$

$V = 2\pi ab^2$

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9 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{-3}{5} & \frac{13}{10} \end{bmatrix}$$

[5 marks]

$\det(M - \lambda I) = (\frac{1}{5} - \lambda)(\frac{13}{10} - \lambda) - (\frac{2}{5})(\frac{-3}{5}) = 0$

$\lambda^2 - 1.5\lambda + 0.26 + 0.24 = 0 \Rightarrow \lambda^2 - 1.5\lambda + 0.5 = 0$

$(\lambda - 1)(\lambda - 0.5) = 0 \Rightarrow$ eigenvalues $\lambda = 1$ and $\lambda = 0.5$

$\lambda = 1$: $-0.8x + 0.4y = 0$ and $-0.6x + 0.3y = 0 \Rightarrow y = 2x$

eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (or any non-zero multiple)

$\lambda = 0.5$: $-0.3x + 0.4y = 0$ and $-0.6x + 0.8y = 0 \Rightarrow 3x = 4y$

eigenvector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (or any non-zero multiple)

9 (b) Find matrices **U** and **D** such that **D** is a diagonal matrix and $\mathbf{M} = \mathbf{UDU}^{-1}$

[2 marks]

$U = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$

(columns of U are the eigenvectors, in the same order as D)

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9 (c) Given that $\mathbf{M}^n \to \mathbf{L}$ as $n \to \infty$, find the matrix $\mathbf{L}$.

[4 marks]

$U^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

$M^n = UD^nU^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0.5^n & 0 \\ 0 & 1^n \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

As $n \to \infty$, $0.5^n \to 0$ and $1^n = 1$ for all n

$L = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ -6 & 8 \end{bmatrix}$

$L = \begin{bmatrix} -0.6 & 0.8 \\ -1.2 & 1.6 \end{bmatrix}$

9 (d) The transformation represented by **L** maps all points onto a line.

Find the equation of this line.

[2 marks]

$\begin{bmatrix} x' \\ y' \end{bmatrix} = L\begin{bmatrix} x \\ y \end{bmatrix}$: $x' = -0.6x + 0.8y$, $y' = -1.2x + 1.6y$

$y' = 2(-0.6x + 0.8y) = 2x'$ for every point (x, y)

Line: $y = 2x$

Turn over ▶

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10 Prove by induction that $f(n) = n^3 + 3n^2 + 8n$ is divisible by 6 for all integers $n \geq 1$ **[7 marks]**

$n = 1$: $f(1) = 1 + 3 + 8 = 12 = 6 \times 2$, divisible by 6, so true for $n = 1$

Assume true for $n = k$: $f(k) = k^3 + 3k^2 + 8k = 6m$ for some integer m

$f(k + 1) = (k + 1)^3 + 3(k + 1)^2 + 8(k + 1)$

$= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 8k + 8$

$= k^3 + 6k^2 + 17k + 12$

$f(k + 1) - f(k) = 3k^2 + 9k + 12 = 3(k^2 + 3k + 4)$

$k^2 + 3k = k(k + 3)$: one of k and $k + 3$ is even, so $k(k + 3)$ is even

$\therefore k^2 + 3k + 4$ is even, so $3(k^2 + 3k + 4)$ is a multiple of 6

$f(k + 1) = f(k) + 3(k^2 + 3k + 4) = 6m + 6q$ for some integer q

$= 6(m + q)$, so $f(k + 1)$ is divisible by 6

So if the result is true for $n = k$, it is true for $n = k + 1$.

It is true for $n = 1$, so by induction $f(n)$ is divisible by 6

for all integers $n \geq 1$.

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Turn over for the next question

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ANSWER IN THE SPACES PROVIDED

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11 The line L_1 has equation

$$\frac{x-2}{3}=\frac{y+4}{8}=\frac{4z-5}{5}$$

The line L_2 has equation

$$\left(\mathbf{r}-\begin{bmatrix}-2\\0\\3\end{bmatrix}\right)\times\begin{bmatrix}2\\1\\3\end{bmatrix}=\mathbf{0}$$

Find the shortest distance between the two lines, giving your answer to three significant figures.

[8 marks]

L_1: $\frac{x-2}{3}=\frac{y+4}{8}=\frac{z-1.25}{1.25}$

Point on L_1: $(2, -4, 1.25)$; direction $3i + 8j + 1.25k$, so $d_1 = 12i + 32j + 5k$

L_2: $r = (-2i + 3k) + \lambda(2i + j + 3k)$; $d_2 = 2i + j + 3k$

Between points: $(2i - 4j + 1.25k) - (-2i + 3k) = 4i - 4j - 1.75k$

Common perpendicular $n = d_1 \times d_2$

$= (32\times3 - 5\times1)i - (12\times3 - 5\times2)j + (12\times1 - 32\times2)k$

$= 91i - 26j - 52k = 13(7i - 2j - 4k)$

Use $7i - 2j - 4k$: $|7i - 2j - 4k| = \sqrt{(49 + 4 + 16)} = \sqrt{69}$

Shortest distance $= \frac{|(4i - 4j - 1.75k).(7i - 2j - 4k)|}{\sqrt{69}}$

$= \frac{|28 + 8 + 7|}{\sqrt{69}} = \frac{43}{\sqrt{69}}$

$= 5.176591\ldots = 5.18$ (3 s.f.)

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Turn over for the next question

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12 Abel and Bonnie are trying to solve this mathematical problem:

$z = 2 - 3\mathrm{i}$ is a root of the equation
$2z^3 + mz^2 + pz + 91 = 0$

Find the value of m and the value of p.

Abel says he has solved the problem.

Bonnie says there is not enough information to solve the problem.

12 (a) Abel's solution begins as follows:

Since $z = 2 - 3\mathrm{i}$ is a root of the equation,
$z = 2 + 3\mathrm{i}$ is another root.

State **one extra** piece of information about m and p which could be added to the problem to make the beginning of Abel's solution correct.

[1 mark]

m and p are both real numbers.

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12 (b) Prove that Bonnie is right.

[4 marks]

Without m and p real, $2 + 3i$ need not be a root. For any m and p:

product of roots $= -\frac{91}{2}$

Case 1: roots $2 - 3i$, $2 + 3i$, γ: $13\gamma = -\frac{91}{2} \Rightarrow \gamma = -\frac{7}{2}$

sum of roots $= 4 - \frac{7}{2} = \frac{1}{2} = -\frac{m}{2} \Rightarrow m = -1$

sum of pairs $= 13 + 4(-\frac{7}{2}) = -1 = \frac{p}{2} \Rightarrow p = -2$

Case 2: roots $2 - 3i$, $4 + 6i$, δ: $(2 - 3i)(4 + 6i) = 8 + 18 = 26$

$26\delta = -\frac{91}{2} \Rightarrow \delta = -\frac{7}{4}$

sum of roots $= 2 - 3i + 4 + 6i - \frac{7}{4} = \frac{17}{4} + 3i = -\frac{m}{2}$

$\Rightarrow m = -\frac{17}{2} - 6i$

sum of pairs $= 26 - \frac{7}{4}(6 + 3i) = 15.5 - 5.25i = \frac{p}{2} \Rightarrow p = 31 - 10.5i$

Both cubics have leading coefficient 2, constant term $-2 \times (-91/2) = 91$

and the root $2 - 3i$, but they give different values of m and p

($m = -1$, $p = -2$ or $m = -8.5 - 6i$, $p = 31 - 10.5i$).

There is more than one possible pair of values of m and p, so there

is not enough information to solve the problem: Bonnie is right.

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13 (a) Explain why $\int_3^\infty x^2 e^{-2x}\,dx$ is an improper integral.

[1 mark]

The upper limit of integration is infinite (∞), so the interval of integration is unbounded: it is an improper integral.

13 (b) Evaluate $\int_3^\infty x^2 e^{-2x}\,dx$

Show the limiting process.

[9 marks]

By parts: $u = x^2$, $v' = e^{-2x} \Rightarrow u' = 2x$, $v = -\frac{1}{2}e^{-2x}$

$\int x^2 e^{-2x}\,dx = -\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\,dx$

Again: $u = x$, $v' = e^{-2x} \Rightarrow u' = 1$, $v = -\frac{1}{2}e^{-2x}$

$\int x e^{-2x}\,dx = -\frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx = -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}$

$\therefore \int x^2 e^{-2x}\,dx = -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + c$

$\int_3^\infty x^2 e^{-2x}\,dx = \lim_{n\to\infty} \int_3^n x^2 e^{-2x}\,dx$

$= \lim_{n\to\infty} \left[-\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right]_3^n$

$= \lim_{n\to\infty} \left(-\frac{n^2}{2}e^{-2n} - \frac{n}{2}e^{-2n} - \frac{1}{4}e^{-2n}\right)$

$+ \left(\frac{9}{2}e^{-6} + \frac{3}{2}e^{-6} + \frac{1}{4}e^{-6}\right)$

As $n \to \infty$: $n^2 e^{-2n} \to 0$, $n e^{-2n} \to 0$, $e^{-2n} \to 0$

(the exponential decays faster than any power of n grows)

$\int_3^\infty x^2 e^{-2x}\,dx = 0 + \frac{25}{4}e^{-6}$

$= \frac{25}{4}e^{-6}$ $(= 0.01549\ldots)$

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Turn over for the next question

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14 Let

$$S_n = \sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$$

where $n \geq 1$

14 (a) Use the method of differences to show that

$$S_n = \frac{5n^2 + an}{12(n+b)(n+c)}$$

where a, b and c are integers.

[6 marks]

$\frac{1}{(r+1)(r+3)} \equiv \frac{A}{r+1} + \frac{B}{r+3}$: $1 \equiv A(r+3) + B(r+1)$

$r = -1$: $A = \frac{1}{2}$; $r = -3$: $B = -\frac{1}{2}$

$S_n = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$

$r = 1$: $\frac{1}{2} - \frac{1}{4}$

$r = 2$: $\frac{1}{3} - \frac{1}{5}$

$r = 3$: $\frac{1}{4} - \frac{1}{6}$

⋮ ⋮

$r = n-1$: $\frac{1}{n} - \frac{1}{n+2}$

$r = n$: $\frac{1}{n+1} - \frac{1}{n+3}$

All terms cancel except the first two positive and last two negative:

$S_n = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right)$

$= \frac{1}{2}\left(\frac{5}{6} - \frac{2n+5}{(n+2)(n+3)}\right)$

$= \frac{5(n+2)(n+3) - 6(2n+5)}{12(n+2)(n+3)}$

$= \frac{5n^2 + 25n + 30 - 12n - 30}{12(n+2)(n+3)}$

$S_n = \frac{5n^2 + 13n}{12(n+2)(n+3)}$ $a = 13$, $b = 2$, $c = 3$

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Question 14 continues on the next page

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14 (b) Show that, for any number k greater than $\frac{12}{5}$, if the difference between $\frac{5}{12}$ and S_n is less than $\frac{1}{k}$, then

$$n > \frac{k - 5 + \sqrt{k^2 + 1}}{2}$$

[6 marks]

$S_n < \frac{5}{12}$ for all n, so the difference is $\frac{5}{12} - S_n < \frac{1}{k}$

$\frac{5}{12} - \frac{5n^2 + 13n}{12(n+2)(n+3)} = \frac{5(n^2 + 5n + 6) - 5n^2 - 13n}{12(n+2)(n+3)}$

$= \frac{12n + 30}{12(n+2)(n+3)} < \frac{1}{k}$

Both denominators are positive ($n \geq 1$, $k > 0$), so

$k(12n + 30) < 12(n+2)(n+3) = 12n^2 + 60n + 72$

$12n^2 + (60 - 12k)n + (72 - 30k) > 0$

$\div 6$: $2n^2 + (10 - 2k)n + (12 - 5k) > 0$

Roots of $2n^2 + (10 - 2k)n + (12 - 5k) = 0$:

$n = \frac{2k - 10 \pm \sqrt{((10 - 2k)^2 - 8(12 - 5k))}}{4}$

$= \frac{2k - 10 \pm \sqrt{(4k^2 + 4)}}{4} = \frac{k - 5 \pm \sqrt{(k^2 + 1)}}{2}$

$k > \frac{12}{5} \Rightarrow 12 - 5k < 0$, so the product of the roots $\frac{12 - 5k}{2} < 0$:

one root is negative and the other is positive.

The quadratic (positive n^2 coefficient) is positive only outside the

roots, and $n > 0$, so n must be greater than the positive root:

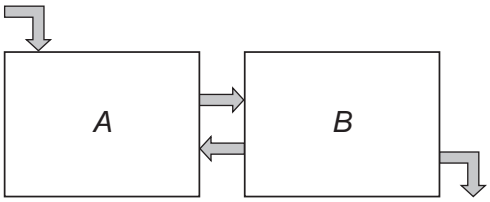
$n > \frac{k - 5 + \sqrt{(k^2 + 1)}}{2}$ as required

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Turn over for the next question

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15



Two tanks, *A* and *B*, each have a capacity of 800 litres.

At time $t = 0$ both tanks are full of pure water.

When $t > 0$, water flows in the following ways:

- Water with a salt concentration of μ grams per litre flows into tank *A* at a constant rate
- Water flows from tank *A* to tank *B* at a rate of 16 litres per minute
- Water flows from tank *B* to tank *A* at a rate of r litres per minute
- Water flows out of tank *B* through a waste pipe
- The amount of water in each tank remains at 800 litres.

At time t minutes $(t \geq 0)$ there are x grams of salt in tank *A* and y grams of salt in tank *B*.

This system is represented by the coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 36 - 0.02x + 0.005y \qquad (1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.02x - 0.02y \qquad (2)$$

15 (a) Find the value of r.

[2 marks]

Water from B to A carries r/800 of the salt in B each minute:

$\frac{r}{800} y = 0.005y \quad \Rightarrow \quad r = 0.005 \times 800$

$r = 4$

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15 (b) Show that $\mu = 3$

[3 marks]

Volume of A stays at 800 litres, so flow into A = flow out of A:

inflow from pipe + 4 = 16 ⇒ 12 litres per minute enter A through the pipe

Salt entering through the pipe per minute = $12\mu = 36$ (constant in (1))

$\mu = 36 \div 12 = 3$ as required

15 (c) Solve the coupled differential equations to find both x and y in terms of t.

[9 marks]

From (2): $0.02x = \frac{dy}{dt} + 0.02y \Rightarrow x = 50\frac{dy}{dt} + y$

$\frac{dx}{dt} = 50\frac{d^2y}{dt^2} + \frac{dy}{dt}$

Sub in (1): $50\frac{d^2y}{dt^2} + \frac{dy}{dt} = 36 - 0.02(50\frac{dy}{dt} + y) + 0.005y$

$50\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 0.015y = 36$

Auxiliary equation: $50m^2 + 2m + 0.015 = 0$

$m = \frac{-2 \pm \sqrt{(4 - 3)}}{100} \Rightarrow m = -0.01$ or $m = -0.03$

CF: $y = Ae^{-0.03t} + Be^{-0.01t}$

PI: try $y = \lambda$: $0.015\lambda = 36 \Rightarrow \lambda = 2400$

$y = Ae^{-0.03t} + Be^{-0.01t} + 2400$

$\frac{dy}{dt} = -0.03Ae^{-0.03t} - 0.01Be^{-0.01t}$

$x = 50\frac{dy}{dt} + y = -1.5Ae^{-0.03t} - 0.5Be^{-0.01t} + Ae^{-0.03t} + Be^{-0.01t} + 2400$

$x = -0.5Ae^{-0.03t} + 0.5Be^{-0.01t} + 2400$

$t = 0$: $x = 0$ and $y = 0$ (both tanks hold pure water)

$A + B + 2400 = 0$ and $-0.5A + 0.5B + 2400 = 0$

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$\Rightarrow \quad A + B = -2400 \quad \text{and} \quad B - A = -4800$

$\Rightarrow \quad B = -3600, \quad A = 1200$

$x = -600e^{-0.03t} - 1800e^{-0.01t} + 2400$

$y = 1200e^{-0.03t} - 3600e^{-0.01t} + 2400$

END OF QUESTIONS

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