AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname StemUp Model Solution

Forename(s) _____

Candidate signature _____

A-level
FURTHER MATHEMATICS

Paper 1

Monday 3 June 2019 Morning Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
TOTAL	

Do not write outside the box

Answer **all** questions in the spaces provided.

1 Which one of these functions has the set $\{x : |x| < 1\}$ as its greatest possible domain?

Circle your answer.

[1 mark]

$\cosh x$ $\cosh^{-1} x$ $\tanh x$ $\tanh^{-1} x$

2 The first two non-zero terms of the Maclaurin series expansion of $f(x)$ are x and $-\frac{1}{2}x^3$

Which one of the following could be $f(x)$?

Circle your answer.

[1 mark]

$xe^{\frac{1}{2}x^2}$ $\frac{1}{2}\sin 2x$ $x\cos x$ $(1+x^3)^{-\frac{1}{2}}$

3 The function $f(x) = x^2 - 1$

Find the mean value of $f(x)$ from $x = -0.5$ to $x = 1.7$
Give your answer to three significant figures.

Circle your answer.

[1 mark]

-0.521 -0.434 -0.237 0.786

Do not write outside the box

4 Solve the equation $2z - 5iz^* = 12$

[4 marks]

Let $z = x + iy$, so $z^* = x - iy$

$2(x + iy) - 5i(x - iy) = 12$

$2x + 2iy - 5ix - 5y = 12$

Real parts: $2x - 5y = 12$

Imaginary parts: $2y - 5x = 0 \Rightarrow y = \frac{5x}{2}$

$2x - \frac{25x}{2} = 12 \Rightarrow -\frac{21x}{2} = 12$

$x = -\frac{8}{7}, \quad y = -\frac{20}{7}$

$z = -\frac{8}{7} - \frac{20}{7}i$

Turn over for the next question

Do not write outside the box

5 A plane has equation $\mathbf{r}.\begin{bmatrix}1\\1\\1\end{bmatrix} = 7$

A line has equation $\mathbf{r} = \begin{bmatrix}2\\0\\1\end{bmatrix} + \mu\begin{bmatrix}1\\0\\1\end{bmatrix}$

Calculate the acute angle between the line and the plane.

Give your answer to the nearest 0.1°

[3 marks]

$n \cdot d = \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} = 1 + 0 + 1 = 2$

$|n| = \sqrt{3}, \quad |d| = \sqrt{2}$

$\cos\alpha = \frac{2}{\sqrt{3} \times \sqrt{2}} = \frac{2}{\sqrt{6}} \Rightarrow \alpha = 35.264...°$ (angle with the normal)

Angle between line and plane = $90° - 35.264...°$

$= 54.7°$

Do not write outside the box

Turn over for the next question

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

6 (a) Show that

$$\cosh^3 x + \sinh^3 x = \frac{1}{4}e^{mx} + \frac{3}{4}e^{nx}$$

where m and n are integers.

[3 marks]

$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh^3 x = \frac{1}{8}(e^x + e^{-x})^3 = \frac{1}{8}(e^{3x} + 3e^x + 3e^{-x} + e^{-3x})$

$\sinh^3 x = \frac{1}{8}(e^x - e^{-x})^3 = \frac{1}{8}(e^{3x} - 3e^x + 3e^{-x} - e^{-3x})$

$\cosh^3 x + \sinh^3 x = \frac{1}{8}(2e^{3x} + 6e^{-x})$

$= \frac{1}{4}e^{3x} + \frac{3}{4}e^{-x}$

so $m = 3$ and $n = -1$

6 (b) Hence find $\cosh^6 x - \sinh^6 x$ in the form

$$\frac{a\cosh(kx) + b}{8}$$

where a, b and k are integers.

[5 marks]

$\cosh^6 x - \sinh^6 x = (\cosh^3 x + \sinh^3 x)(\cosh^3 x - \sinh^3 x)$

$\cosh^3 x - \sinh^3 x = \frac{1}{8}(6e^x + 2e^{-3x}) = \frac{3}{4}e^x + \frac{1}{4}e^{-3x}$

$\cosh^6 x - \sinh^6 x = (\frac{1}{4}e^{3x} + \frac{3}{4}e^{-x})(\frac{3}{4}e^x + \frac{1}{4}e^{-3x})$

$= \frac{3}{16}e^{4x} + \frac{1}{16} + \frac{9}{16} + \frac{3}{16}e^{-4x}$

$= \frac{3}{16}(e^{4x} + e^{-4x}) + \frac{10}{16}$

$= \frac{3}{8} \times \frac{e^{4x} + e^{-4x}}{2} + \frac{5}{8} = \frac{3}{8}\cosh 4x + \frac{5}{8}$

Do not write outside the box

$= \frac{3\cosh(4x) + 5}{8}$

$(a = 3,\ b = 5,\ k = 4)$

Turn over for the next question

Do not write outside the box

7 Three non-singular square matrices, **A**, **B** and **R** are such that

$$\mathbf{AR} = \mathbf{B}$$

The matrix **R** represents a rotation about the z-axis through an angle θ and

$$\mathbf{B} = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7 (a) Show that **A** is independent of the value of θ.

[3 marks]

$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (rotation about the z-axis)

$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (rotation through $-\theta$)

$AR = B \Rightarrow ARR^{-1} = BR^{-1} \Rightarrow A = BR^{-1}$

$A = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} -\cos^2\theta - \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ since $\cos^2\theta + \sin^2\theta = 1$

A contains no θ, so A is independent of the value of θ.

Do not write outside the box

7 (b) Give a full description of the single transformation represented by the matrix **A**.

[1 mark]

Reflection in the plane x = 0 (the y-z plane)

Turn over for the next question

Do not write outside the box

8 (a) If $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to prove that

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

[3 marks]

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (de Moivre)

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ (cos is even, sin is odd)

$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

8 (b) Express $\sin^5\theta$ in terms of $\sin 5\theta$, $\sin 3\theta$ and $\sin\theta$

[4 marks]

$(z - \frac{1}{z})^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$

$= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$

Using part (a) with $z - \frac{1}{z} = 2i\sin\theta$:

$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

Do not write outside the box

8 (c) Hence show that

$$\int_0^{\frac{\pi}{3}} \sin^5\theta \, \mathrm{d}\theta = \frac{53}{480}$$

[3 marks]

$\int_0^{\pi/3} \sin^5\theta \, d\theta = \int_0^{\pi/3} \left(\frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta\right) d\theta$

$= \left[-\frac{1}{80}\cos 5\theta + \frac{5}{48}\cos 3\theta - \frac{5}{8}\cos\theta\right]_0^{\pi/3}$

$= \left(-\frac{1}{80}\cos\frac{5\pi}{3} + \frac{5}{48}\cos\pi - \frac{5}{8}\cos\frac{\pi}{3}\right) - \left(-\frac{1}{80} + \frac{5}{48} - \frac{5}{8}\right)$

$= \left(-\frac{1}{160} - \frac{5}{48} - \frac{5}{16}\right) + \frac{1}{80} - \frac{5}{48} + \frac{5}{8}$

$= \frac{-3 - 50 - 150 + 6 - 50 + 300}{480}$

$= \frac{53}{480}$

Turn over for the next question

Do not write outside the box

9 (a) Solve the equation $z^3 = \sqrt{2} - \sqrt{6}\mathrm{i}$, giving your answers in the form $re^{\mathrm{i}\theta}$ where $r > 0$ and $0 \leq \theta < 2\pi$

[5 marks]

$|z^3| = \sqrt{(2 + 6)} = \sqrt{8} = 2\sqrt{2}$

$\arg(z^3) = -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{2}}\right) = -\tan^{-1}\sqrt{3} = -\frac{\pi}{3}$

$z^3 = 2\sqrt{2}\, e^{i(-\pi/3 + 2k\pi)}$

$r = (2\sqrt{2})^{1/3} = (2^{3/2})^{1/3} = \sqrt{2}$

$\theta = -\frac{\pi}{9} + \frac{2k\pi}{3}$

$k = 1$: $\theta = \frac{5\pi}{9}$, $k = 2$: $\theta = \frac{11\pi}{9}$, $k = 3$: $\theta = \frac{17\pi}{9}$

$z = \sqrt{2}\, e^{5\pi i/9}$, $\sqrt{2}\, e^{11\pi i/9}$, $\sqrt{2}\, e^{17\pi i/9}$

9 (b) The transformation represented by the matrix $\mathbf{M} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$ acts on the points on an Argand Diagram which represent the roots of the equation in part **(a)**.

Find the exact area of the shape formed by joining the transformed points.

[4 marks]

Roots form an equilateral triangle, each vertex $\sqrt{2}$ from 0

Area $= 3 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \times \sin\frac{2\pi}{3} = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$\det \mathbf{M} = 5 \times 3 - 1 \times 1 = 14$ (area scale factor)

Area of transformed shape $= 14 \times \frac{3\sqrt{3}}{2}$

$= 21\sqrt{3}$

Do not write outside the box

Turn over for the next question

Do not write outside the box

10 The points $A(5, -4, 6)$ and $B(6, -6, 8)$ lie on the line L. The point C is $(15, -5, 9)$.

10 (a) D is the point on L that is closest to C.

Find the coordinates of D.

[6 marks]

$AB = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ $\quad L: r = \begin{bmatrix} 5 \\ -4 \\ 6 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

D on L: $CD = \begin{bmatrix} 5 + \mu - 15 \\ -4 - 2\mu + 5 \\ 6 + 2\mu - 9 \end{bmatrix} = \begin{bmatrix} -10 + \mu \\ 1 - 2\mu \\ -3 + 2\mu \end{bmatrix}$

CD is perpendicular to L, so $CD \cdot AB = 0$

$\begin{bmatrix} -10 + \mu \\ 1 - 2\mu \\ -3 + 2\mu \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 0$

$-10 + \mu - 2 + 4\mu - 6 + 4\mu = 0$

$9\mu - 18 = 0 \Rightarrow \mu = 2$

$D = (5 + 2, -4 - 4, 6 + 4)$

$D = (7, -8, 10)$

Do not write outside the box

10 (b) Hence find, in exact form, the shortest distance from C to L.

[2 marks]

$\overrightarrow{CD} = \begin{bmatrix} 7-15 \\ -8+5 \\ 10-9 \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ 1 \end{bmatrix}$

$|\overrightarrow{CD}| = \sqrt{((-8)^2 + (-3)^2 + 1^2)} = \sqrt{(64 + 9 + 1)}$

Shortest distance $= \sqrt{74}$

Turn over for the next question

Do not write outside the box

11 Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \frac{x^3}{\sqrt{4 - 2x - x^2}}$$

where $0 < x < \sqrt{5} - 1$

[7 marks]

Divide by x: $\frac{dy}{dx} - \frac{2}{x}y = \frac{x^2}{\sqrt{(4 - 2x - x^2)}}$

Integrating factor $= e^{\int -2/x\, dx} = e^{-2 \ln x} = x^{-2}$

$x^{-2}\frac{dy}{dx} - 2x^{-3}y = \frac{1}{\sqrt{(4 - 2x - x^2)}}$

$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{\sqrt{(4 - 2x - x^2)}}$

$4 - 2x - x^2 = 5 - (x^2 + 2x + 1) = 5 - (x + 1)^2$

$\frac{y}{x^2} = \int \frac{1}{\sqrt{(5 - (x + 1)^2)}}\, dx$

$\frac{y}{x^2} = \sin^{-1}\left(\frac{x + 1}{\sqrt{5}}\right) + c$

$y = x^2\left(\sin^{-1}\left(\frac{x + 1}{\sqrt{5}}\right) + c\right)$

Do not write outside the box

Turn over for the next question

Do not write outside the box

12 Three planes have equations

$$4x - 5y + z = 8$$
$$3x + 2y - kz = 6$$
$$(k-2)x + ky - 8z = 6$$

where k is a real constant.

The planes do **not** meet at a unique point.

12 (a) Find the possible values of k.

[3 marks]

Not a unique point, so the determinant is zero:

$$\det \begin{bmatrix} 4 & -5 & 1 \\ 3 & 2 & -k \\ k-2 & k & -8 \end{bmatrix} = 0$$

$4(-16 + k^2) + 5(-24 + k^2 - 2k) + 1(3k - 2k + 4) = 0$

$4k^2 - 64 + 5k^2 - 10k - 120 + k + 4 = 0$

$9k^2 - 9k - 180 = 0 \Rightarrow k^2 - k - 20 = 0$

$(k - 5)(k + 4) = 0 \Rightarrow k = 5$ or $k = -4$

Do not write outside the box

12 (b) For each value of k found in part **(a)**, identify the configuration of the given planes.

Fully justify your answer, stating in each case whether or not the equations of the planes form a consistent system.

[5 marks]

$k = 5$: $4x - 5y + z = 8$ (1), $3x + 2y - 5z = 6$ (2), $3x + 5y - 8z = 6$ (3)

$4\times(2) - 3\times(1)$: $23y - 23z = 0 \Rightarrow y = z$

$4\times(3) - 3\times(1)$: $35y - 35z = 0 \Rightarrow y = z$

Same equation, so infinitely many solutions, e.g. $y = z = t$, $x = 2 + t$

The system is consistent. No two normals are parallel, so the three planes form a sheaf: they all meet in one common line.

$k = -4$: $4x - 5y + z = 8$ (1), $3x + 2y + 4z = 6$ (2), $-6x - 4y - 8z = 6$ (3)

(3) $\div (-2)$: $3x + 2y + 4z = -3$, but (2) gives $3x + 2y + 4z = 6$

So planes (2) and (3) are parallel and distinct: the system is inconsistent (no solutions). Plane (1) is not parallel to them, so the configuration is two parallel planes, each crossed by the third plane.

Turn over for the next question

Do not write outside the box

13 The equation $z^3 + kz^2 + 9 = 0$ has roots α, β and γ.

13 (a) (i) Show that

$$\alpha^2 + \beta^2 + \gamma^2 = k^2$$

[3 marks]

$\alpha + \beta + \gamma = -k$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$, $\alpha\beta\gamma = -9$

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$(-k)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(0)$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = k^2$

13 (a) (ii) Show that

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = -18k$$

[4 marks]

$(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2(\alpha\beta^2\gamma + \alpha\beta\gamma^2 + \alpha^2\beta\gamma)$

$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$0^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2(-9)(-k)$

$0 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 18k$

$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = -18k$

Do not write outside the box

13 (b) The equation $9z^3 - 40z^2 + rz + s = 0$ has roots $\alpha\beta + \gamma$, $\beta\gamma + \alpha$ and $\gamma\alpha + \beta$.

13 (b) (i) Show that

$$k = -\frac{40}{9}$$

[1 mark]

Sum of roots: $(\alpha\beta + \gamma) + (\beta\gamma + \alpha) + (\gamma\alpha + \beta) = \frac{40}{9}$

$(\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma) = \frac{40}{9}$

$0 + (-k) = \frac{40}{9} \Rightarrow k = -\frac{40}{9}$

Question 13 continues on the next page

Do not write outside the box

13 (b) (ii) Without calculating the values of α, β and γ, find the value of s.

Show working to justify your answer.

[6 marks]

Product of roots: $(\alpha\beta + \gamma)(\beta\gamma + \alpha)(\gamma\alpha + \beta) = -\frac{s}{9}$

$(\alpha\beta + \gamma)(\beta\gamma + \alpha) = \alpha\beta^2\gamma + \alpha^2\beta + \beta\gamma^2 + \alpha\gamma$

$(\alpha\beta + \gamma)(\beta\gamma + \alpha)(\gamma\alpha + \beta) = \alpha^2\beta^2\gamma^2 + \alpha\beta^3\gamma + \alpha^3\beta\gamma + \alpha\beta\gamma^3$

$+ \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha\beta\gamma$

$= (\alpha\beta\gamma)^2 + \alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2) + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + \alpha\beta\gamma$

$= (-9)^2 + (-9)k^2 + (-18k) + (-9)$

$= 81 - 9k^2 - 18k - 9$, with $k = -\frac{40}{9}$:

$= 81 - 9 \times \frac{1600}{81} + 80 - 9$

$= 152 - \frac{1600}{9} = -\frac{232}{9}$

$-\frac{s}{9} = -\frac{232}{9} \Rightarrow s = 232$

Do not write outside the box

Turn over for the next question

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

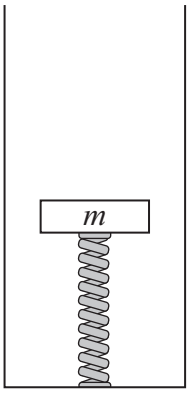
Do not write outside the box

14 **In this question use $g = 10\,\text{m s}^{-2}$**

A light spring is attached to the base of a long tube and has a mass m attached to the other end, as shown in the diagram.

The tube is filled with oil.

When the compression of the spring is ε metres, the thrust in the spring is $9m\varepsilon$ newtons.



The mass is held at rest in a position where the compression of the spring is $\frac{20}{9}$ metres.

The mass is then released from rest. During the subsequent motion the oil causes a resistive force of $6mv$ newtons to act on the mass, where $v\,\text{m s}^{-1}$ is the speed of the mass.

At time t seconds after the mass is released, the displacement of the mass above its starting position is x metres.

14 (a) Find x in terms of t.

[10 marks]

Compression = $\frac{20}{9} - x$, so thrust = $9m(\frac{20}{9} - x)$ upwards

N2L upwards: $9m(\frac{20}{9} - x) - mg - 6m\dot{x} = m\ddot{x}$

$20m - 9mx - 10m - 6m\dot{x} = m\ddot{x}$

$\ddot{x} + 6\dot{x} + 9x = 10$

Auxiliary equation: $\lambda^2 + 6\lambda + 9 = 0 \Rightarrow (\lambda + 3)^2 = 0$

$\lambda = -3$ (repeated root)

CF: $x = Ae^{-3t} + Bte^{-3t}$

PI: $x = c$, $9c = 10 \Rightarrow c = \frac{10}{9}$

General solution: $x = Ae^{-3t} + Bte^{-3t} + \frac{10}{9}$

Do not write outside the box

$t = 0, x = 0$: $0 = A + \frac{10}{9} \Rightarrow A = -\frac{10}{9}$

$\dot{x} = -3Ae^{-3t} + Be^{-3t} - 3Bte^{-3t}$

$t = 0, \dot{x} = 0$ (released from rest): $0 = -3A + B$

$B = 3A = -\frac{10}{3}$

$x = -\frac{10}{9}e^{-3t} - \frac{10}{3}te^{-3t} + \frac{10}{9}$

14 (b) State, giving a reason, the type of damping which occurs.

[1 mark]

Critical damping, because the auxiliary equation has equal (repeated) roots.

Turn over ▶

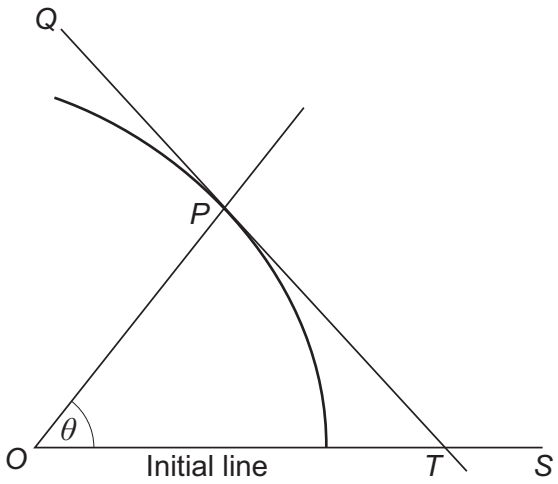
Do not write outside the box

15 The diagram shows part of a spiral curve.

The point P has polar coordinates (r, θ) where $0 \leq \theta \leq \frac{\pi}{2}$

The points T and S lie on the initial line and O is the pole.

TPQ is the tangent to the curve at P.



15 (a) Show that the gradient of TPQ is equal to

$$\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

[4 marks]

$x = r\cos\theta$, $y = r\sin\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$, $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

Gradient of the tangent TPQ at P is $\frac{dy}{dx}$:

Gradient $= \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$ as required

Do not write outside the box

15 (b) The curve has polar equation

$$r = \mathrm{e}^{(\cot b)\theta}$$

where b is a constant such that $0 < b < \frac{\pi}{2}$

Use the result of part **(a)** to show that the angle between the line OP and the tangent TPQ does not depend on θ.

[7 marks]

$r = e^{(\cot b)\theta} \Rightarrow \frac{dr}{d\theta} = (\cot b)e^{(\cot b)\theta}$

Gradient of $TPQ = \frac{(\cot b)e^{(\cot b)\theta}\sin\theta + e^{(\cot b)\theta}\cos\theta}{(\cot b)e^{(\cot b)\theta}\cos\theta - e^{(\cot b)\theta}\sin\theta}$

$= \frac{\cot b \sin\theta + \cos\theta}{\cot b \cos\theta - \sin\theta} = \frac{\cos b \sin\theta + \sin b \cos\theta}{\cos b \cos\theta - \sin b \sin\theta}$ $(\times \sin b)$

$= \frac{\sin(\theta + b)}{\cos(\theta + b)} = \tan(\theta + b)$

So TPQ makes angle $\theta + b$ with the initial line: angle $STP = \theta + b$

In triangle OPT, angle $POT = \theta$ and STP is the exterior angle at T,

so angle STP = angle POT + angle OPT

$\theta + b = \theta +$ angle $OPT \Rightarrow$ angle $OPT = b$

b is a constant, so the angle between OP and the tangent TPQ is b,

which does not depend on θ.

END OF QUESTIONS

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required.** **Write the question numbers in the left-hand margin.**

Copyright information

For confidentiality purposes, from the November 2015 examination series, acknowledgements of third-party copyright material are published in a separate booklet rather than including them on the examination paper or support materials. This booklet is published after each examination series and is available for free download from www.aqa.org.uk after the live examination series.

Permission to reproduce all copyright material has been applied for. In some cases, efforts to contact copyright-holders may have been unsuccessful and AQA will be happy to rectify any omissions of acknowledgements. If you have any queries please contact the Copyright Team, AQA, Stag Hill House, Guildford, GU2 7XJ.

Copyright © 2019 AQA and its licensors. All rights reserved.