

Arithmetic Sequences and Series

1 (2 marks)

$$u_n = a + (n - 1)d, \text{ so } u_{20} = 7 + 19 \times 4$$

M1

$$u_{20} = 83$$

A1

Answer: 83

2 marks

2 (2 marks)

$$S_n = \frac{n}{2}(2a + (n - 1)d) \text{ with } a = 3, d = 5, n = 15$$

$$S_{15} = \frac{15}{2}(6 + 70) = 570$$

M1 A1

Answer: 570

2 marks

3 (2 marks)

Common difference 5, so next terms are 19 and 24

B1

$$u_n = 4 + 5(n - 1) = 5n - 1$$

B1

Answer: 19, 24; $u_n = 5n - 1$

2 marks

4 (2 marks)

$$a = 100, d = -6, \text{ so } u_{25} = 100 + 24 \times (-6)$$

M1

$$= -44$$

A1

Answer: -44

2 marks

5 (2 marks)

$$3 + 9d = 48$$

M1

$$d = \frac{45}{9} = 5$$

A1

Answer: 5

2 marks

6 (2 marks)

$$S_{12} = \frac{12}{2}(2 \times 40 + 11 \times (-3))$$

M1

$$= 6 \times 47 = 282$$

A1

Answer: 282

2 marks

7 (5 marks)

(a) $a + 4d = 23$ and $a + 11d = 58$

M1

Subtracting: $7d = 35$, so $d = 5$

A1

$$a = 23 - 4 \times 5 = 3$$

A1

Answer: $a = 3, d = 5$

3 marks

(b) $S_{30} = \frac{30}{2}(2 \times 3 + 29 \times 5)$

M1

$$S_{30} = 15 \times 151 = 2265$$

A1

Answer: 2265

2 marks

8 (4 marks)

(a) $4 + 3(n - 1) = 100$

M1

$$3(n - 1) = 96, \text{ so } n = 33$$

A1

Answer: 33

2 marks

(b) $S = \frac{n}{2}(a + l) = \frac{33}{2}(4 + 100)$

M1

$$S = 1716$$

A1

Answer: 1716

2 marks

9 (5 marks)

(a) $83 - 6n > 0$

M1

$$n < \frac{83}{6} = 13.83\dots$$

A1

so there are 13 positive terms

A1

Answer: 13

3 marks

(b) $u_1 = 77$ and $u_{13} = 83 - 78 = 5$

M1

$$S_{13} = \frac{13}{2}(77 + 5) = 533$$

A1

Answer: 533

2 marks

10 (3 marks)Arithmetic with $a = 11$, $d = 2$, last term 99: $11 + 2(n - 1) = 99$ so $n = 45$

M1 A1

$$S_{45} = \frac{45}{2}(11 + 99) = 2475$$

A1

Answer: 2475

3 marks

11 (4 marks)

$$\frac{n}{2}(4 + 3(n - 1)) = 950$$

M1

$$n(3n + 1) = 1900, \text{ so } 3n^2 + n - 1900 = 0$$

A1

$$(3n + 76)(n - 25) = 0$$

M1

$$n = 25 \text{ (n must be positive)}$$

A1

Answer: 25

4 marks

12 (4 marks)

Equal differences: $(2k + 4) - (k + 3) = (5k - 3) - (2k + 4)$

M1

$$k + 1 = 3k - 7$$

A1

$$k = 4$$

A1

terms 7, 12, 17 so $d = 5$

A1

Answer: $k = 4$, $d = 5$

4 marks

13 (4 marks)

$$u_{n+1} - u_n = 5(n + 1) - 2 - (5n - 2) = 5, \text{ a constant, so arithmetic}$$

M1 A1

$$a = 3, d = 5, S_{20} = 10(6 + 19 \times 5)$$

M1

$$= 1010$$

A1

Answer: 1010

4 marks

14 (8 marks)

(a) Rows form an arithmetic series $9 + 7 + 5 + 3 + 1$ with $a = 9$, $d = -2$, $n = 5$

M1

$$S_5 = \frac{5}{2}(9 + 1) = 25$$

A1

Answer: 25

2 marks

(b) Rows: 15, 13, ..., 1. Number of rows: $15 - 2(n - 1) = 1$, so $n = 8$

M1 A1

$$S_8 = \frac{8}{2}(15 + 1) = 64$$

A1

Answer: 64

3 marks

(c) Rows from the top: 3, 5, 7, ... so $S_k = \frac{k}{2}(6 + 2(k - 1)) = k(k + 2)$

M1

$$k^2 + 2k - 143 = 0, \text{ so } (k - 11)(k + 13) = 0$$

M1

$$k = 11 \text{ (} k > 0 \text{)}$$

A1

Answer: 11

3 marks

15 (7 marks)

(a) $u_{12} = 24\,000 + 11 \times 1250$

M1

$$= £37750$$

A1

Answer: £37750

2 marks

(b) $S_{12} = \frac{12}{2}(24\,000 + 37750)$

M1

$$= £370500$$

Answer: £370500

A1
2 marks

(c) $24\,000 + 1250(n - 1) > 50\,000$

M1

$$n - 1 > 20.8, \text{ so } n > 21.8$$

A1

Year 22 (salary £50250)

A1

Answer: Year 22

3 marks

16 (7 marks)

(a) Equal differences: $(2k - 2) - (k - 3) = (4k - 8) - (2k - 2)$

M1

$$k + 1 = 2k - 6$$

A1

$$k = 7$$

A1

Answer: shown

3 marks

(b) Terms are 4, 12, 20, ... so $a = 4$, $d = 8$; $u_{30} = 4 + 29 \times 8$

M1

$$= 236$$

A1

Answer: 236

2 marks

(c) $S_n = \frac{n}{2}(2 \times 4 + (n - 1) \times 8) = \frac{n}{2} \times 8n$

M1

$$= 4n^2 = (2n)^2$$

A1

Answer: $S_n = (2n)^2$

2 marks

17 (6 marks)

(a) Let the shortest side be a ; the longest is $3a$

B1

$$\text{Perimeter} = \frac{7}{2}(a + 3a)$$

M1

$$14a = 210$$

A1

$$a = 15 \text{ cm}$$

A1

Answer: 15 cm

4 marks

(b) $a + 6d = 3a$, so $6d = 30$

M1

$$d = 5$$

A1

Answer: $d = 5$

2 marks

18 (5 marks)

Let the numbers be $a - d$, a , $a + d$; sum $3a = 36$ so $a = 12$

M1 A1

$$(12 - d)(12)(12 + d) = 1428$$

M1

$$144 - d^2 = 119 \text{ so } d^2 = 25$$

A1

$$d = 5 \text{ (increasing): } 7, 12, 17$$

A1

Answer: 7, 12, 17

5 marks

19 (5 marks)

The last 25 terms are the 126th to the 150th terms

M1

$$u_{126} = 8 + 125 \times 6$$

M1

$$= 758$$

A1

$$S = \frac{25}{2}(758 + 902)$$

M1

$$= 20750$$

A1

Answer: 20750

5 marks

20 (5 marks)

$$\frac{5}{2}(2a + 4d) = 55 \text{ so } a + 2d = 11$$

M1 A1

$$4(2a + 7d) = 124 \text{ so } 2a + 7d = 31$$

M1 A1

$$\text{Subtract } 2(a + 2d) = 22: 3d = 9, \text{ so } d = 3 \text{ and } a = 5$$

A1

Answer: $a = 5$, $d = 3$

5 marks

21 (9 marks)

(a) Write $S_n = a + (a + d) + \dots + (a + (n - 1)d)$ B1

Reverse: $S_n = (a + (n - 1)d) + (a + (n - 2)d) + \dots + a$ M1

Add: $2S_n = n \times (2a + (n - 1)d)$, as each of the n pairs sums to $2a + (n - 1)d$ M1

Hence $S_n = \frac{n}{2}(2a + (n - 1)d)$ A1

Answer: Proof 4 marks

(b) $S_8 = 4(2a + 7d)$ and $S_4 = 2(2a + 3d)$ M1

$4(2a + 7d) = 8(2a + 3d)$, so $8a + 28d = 16a + 24d$ M1

$4d = 8a$, so $d = 2a$ A1

Answer: Proof 3 marks

(c) $u_6 = a + 5d = 11a = 22$, so $a = 2$ and $d = 4$

$S_{10} = 5(4 + 36) = 200$ M1 A1

Answer: 200 2 marks

22 (9 marks)

(a) $5 + 4(n - 1) = 103 - 3(n - 1)$ M1

$7(n - 1) = 98$, so $n = 15$ A1

Common value = $5 + 4 \times 14 = 61$ A1

Answer: $n = 15$, value 61 3 marks

(b) $S_A = \frac{n}{2}(10 + 4(n - 1)) = 2n^2 + 3n$ M1

$S_B = \frac{n}{2}(206 - 3(n - 1)) = \frac{n(209 - 3n)}{2}$ M1

$S_A > S_B$: $4n^2 + 6n > 209n - 3n^2$, so $7n^2 - 203n > 0$ M1

$n(n - 29) > 0$, so $n > 29$ A1

Smallest integer value is $n = 30$ A1

Answer: $n = 30$ 5 marks

(c) When $n = 29$ the two sums are equal ($S_A = S_B = 1769$), so A's sum is not greater E1

Answer: Sums are equal when $n = 29$ 1 mark

23 (9 marks)

(a) $a + 7d = 22$ and $a + 19d = -38$ M1

$12d = -60$, so $d = -5$ A1

$a = 22 + 35 = 57$ A1

Answer: $a = 57$, $d = -5$ 3 marks

(b) $u_n = 57 - 5(n - 1) = 62 - 5n$ M1

$62 - 5n > 0$, so $n < 12.4$ A1

12 positive terms A1

Answer: 12 3 marks

(c) Every term after the 12th is negative, so adding more terms reduces the sum; the maximum is E1

S_{12}

$S_{12} = \frac{12}{2}(2 \times 57 + 11 \times (-5))$ M1

$= 354$ A1

Answer: 354 3 marks

24 (9 marks)

(a) $21\,000 + 1500(N - 1) = 33\,000$ M1

$N - 1 = 8$, so $N = 9$ A1

Answer: $N = 9$ 2 marks

(b) Years 1 to 9 form an arithmetic series: $S_9 = \frac{9}{2}(21\,000 + 33\,000)$ M1

$= 243\,000$ A1

Years 10 to 15: $6 \times 33\,000 = 198\,000$ M1

Total = £441 000 A1

Answer: £441 000**4 marks**

(c) $21\,000 + 1500(n - 1) = 24\,000 + 1000(n - 1)$

M1

$500(n - 1) = 3000$, so $n = 7$

A1

Salary = $21\,000 + 6 \times 1500 = £30\,000$

A1**Answer: year 7, £30 000****3 marks****25 (5 marks)**

$S_{20} = 10(6 + 19 \times 4) = 820$ and $S_{10} = 5(6 + 9 \times 4) = 210$ are correct

B1 $S_{20} - S_{10}$ is the sum of terms 11 to 20, so term 10 is missing**E1**

$S_{20} - S_9$ is needed: $S_9 = \frac{9}{2}(6 + 8 \times 4) = 171$

M1 A1

Correct sum = $820 - 171 = 649$

A1**Answer: 649; the student should subtract S_9 , not S_{10}** **5 marks****26 (7 marks)**

(a) $u_1 = S_1 = 5$

B1

$S_2 = 16$

M1

$u_2 = 16 - 5 = 11$

A1**Answer: 5 and 11****3 marks**

(b) $S_{n-1} = 3(n-1)^2 + 2(n-1) = 3n^2 - 4n + 1$

M1

$u_n = (3n^2 + 2n) - (3n^2 - 4n + 1) = 6n - 1$

A1**Answer: shown****2 marks**

(c) $u_{n+1} - u_n = 6(n+1) - 1 - (6n - 1) = 6$

M1a constant, so arithmetic with $d = 6$ **E1****Answer: $d = 6$** **2 marks****27 (10 marks)**

(a) $a + (p-1)d = q$ and $a + (q-1)d = p$

M1

Subtract: $(p-q)d = q-p$

M1

Since $p \neq q$, divide by $(p-q)$: $d = -1$

A1

$a = q + (p-1)d = p + q - 1$

M1 A1**Answer: $d = -1$, $a = p + q - 1$** **5 marks**

(b) $u_{p+q} = a + (p+q-1)d = (p+q-1) - (p+q-1)$

M1

$= 0$

A1**Answer: Proof****2 marks**

(c) $S_{p+q} = \frac{(p+q)}{2}(\text{first term} + \text{last term})$

M1

$= \frac{(p+q)}{2}(p+q-1+0)$

M1

$= \frac{(p+q)(p+q-1)}{2}$

A1

Answer: $\frac{(p+q)(p+q-1)}{2}$

3 marks**28 (11 marks)**

(a) Multiples of 3: 3, 6, ..., 498. $3n \leq 500$ gives $n = 166$ terms

M1 A1

$S = \frac{166}{2}(3 + 498) = 41583$

A1**Answer: 41583****3 marks**

(b) 5, 10, ..., 500 has 100 terms

M1

$S = \frac{100}{2}(5 + 500) = 25250$

A1**Answer: 25250****2 marks**

(c) Multiples of 15: 15, 30, ..., 495 has 33 terms, sum $\frac{33}{2}(15 + 495) = 8415$

M1 A1

Multiples of 15 are counted in both of the sums in (a) and (b)

M1

Required sum = $41583 + 25250 - 8415 = 58418$

A1

Answer: 58418**4 marks**

$$(d) \text{ Sum of 1 to 500} = \frac{500 \times 501}{2} = 125250$$

M1

$$125250 - 58418 = 66832$$

A1**Answer: 66832****2 marks****29 (10 marks)**

$$(a) S_n = \frac{n}{2}(2a + (n - 1) \times 1) = 100$$

M1

$$\text{so } n(2a + n - 1) = 200$$

A1**Answer: shown****2 marks****(b)** Their sum is $2a + 2n - 1$, which is odd**M1**

so one is even and the other odd

E1**Answer: sum is odd****2 marks****(c)** $200 = 2^3 \times 5^2$; the odd factor of one of n , $2a + n - 1$ is 1, 5 or 25**M1**The smaller of the two factors is n (as $2a + n - 1 > n$): $n < 2a + n - 1$ and $n \geq 2$ **B1**

$$n = 5: 2a + 4 = 40 \text{ so } a = 18: 18 + 19 + 20 + 21 + 22$$

M1 A1

$$n = 8: 2a + 7 = 25 \text{ so } a = 9: 9 + 10 + \dots + 16$$

A1No other factor pair works, e.g. $n = 25$ gives $2a + 24 = 8$, impossible**B1****Answer: 18 to 22 and 9 to 16****6 marks****30 (7 marks)**

$$x = 7p + 2 = 4q + 3. \text{ Numbers satisfying both are } 23, 51, 79, \dots : \text{ common difference } \text{lcm}(7, 4) = 28$$

M1 A1

$$\text{So } x = 28k + 23 \text{ (k a non-negative integer)}$$

A1

$$\text{Three digits: } 100 \leq 28k + 23 \leq 999 \text{ gives } 2.75 \leq k \leq 34.8, \text{ so } k = 3 \text{ to } 34$$

M1

$$\text{First term } 107, \text{ last term } 975, 32 \text{ terms}$$

A1

$$S_{32} = 16(107 + 975)$$

M1

$$= 17312$$

A1**Answer: 17312****7 marks****Geometric Sequences and Series****1 (2 marks)**

$$u_n = ar^{n-1}, \text{ so } u_8 = 3 \times 2^7$$

M1

$$u_8 = 384$$

A1**Answer: 384****2 marks****2 (2 marks)**

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{5(3^6 - 1)}{3 - 1}$$

M1

$$= \frac{5 \times 728}{2} = 1820$$

A1**Answer: 1820****2 marks****3 (2 marks)**

$$r = -\frac{1}{2}, \text{ so } u_6 = 160 \times \left(-\frac{1}{2}\right)^5$$

M1

$$= -5$$

A1**Answer: -5****2 marks****4 (2 marks)**

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{2(2^8 - 1)}{2 - 1}$$

M1

$$= 510$$

A1**Answer: 510****2 marks**

5 (2 marks)

$$2 \times 3^{n-1} = 1458 \text{ so } 3^{n-1} = 729 = 3^6$$

M1

$n = 7$, so 1458 is the 7th term

A1

Answer: 7th term

2 marks

6 (3 marks)

$$r = \frac{18}{12} = \frac{3}{2}$$

B1

$$u_5 = 12 \times \left(\frac{3}{2}\right)^3$$

M1

$$= 40.5$$

A1

Answer: $r = \frac{3}{2}$, $u_5 = 40.5$

3 marks

7 (5 marks)

(a) $ar = 12$ and $ar^4 = 324$

M1

Divide: $r^3 = 27$, so $r = 3$

A1

$a = 12 \div 3 = 4$

A1

Answer: $r = 3$, $a = 4$

3 marks

(b) $S_8 = \frac{4(3^8 - 1)}{3 - 1}$

M1

$$= \frac{4 \times 6560}{2} = 13120$$

A1

Answer: 13120

2 marks

8 (4 marks)

$$S_n = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1$$

M1

$$3^n - 1 > 10\,000, \text{ so } 3^n > 10\,001$$

M1

$$n > \frac{\log 10\,001}{\log 3} = 8.38$$

M1

Smallest n is 9

A1

Answer: $n = 9$

4 marks

9 (4 marks)

$$3 \times 2^{n-1} = 768, \text{ so } 2^{n-1} = 256 = 2^8$$

M1

$$n = 9$$

A1

$$S_9 = \frac{3(2^9 - 1)}{2 - 1}$$

M1

$$= 1533$$

A1

Answer: 9 terms, sum 1533

4 marks

10 (3 marks)

$$256r^4 = 81, \text{ so } r^4 = \frac{81}{256}$$

M1

$$r = \frac{3}{4} \text{ (} r > 0 \text{)}$$

A1

$$u_3 = 256 \times \left(\frac{3}{4}\right)^2 = 144 \text{ tonnes}$$

A1

Answer: 144 tonnes

3 marks

11 (4 marks)

(a) $u_1 = 6000$, $u_2 = 4800$, $u_3 = 3840$

B1

$$\frac{u_{k+1}}{u_k} = 0.8 \text{ for every } k, \text{ so the common ratio is } 0.8$$

B1

Answer: 6000, 4800, 3840; $r = 0.8$

2 marks

(b) $S_{10} = \frac{6000(1 - 0.8^{10})}{1 - 0.8}$

M1

$$= 26779$$

A1

Answer: 26779**2 marks****12 (5 marks)**

(a) 3000×1.025^8
 $= £3655$

M1**A1****Answer: £3655****2 marks**

(b) $1.025^n > 2$

M1

$$n > \frac{\ln 2}{\ln 1.025} = 28.07$$

A1so $n = 29$ years**A1****Answer: 29 years****3 marks****13 (4 marks)**

$$\frac{u_{n+1}}{u_n} = \frac{5 \times 3^{n+1}}{5 \times 3^n} = 3, \text{ a constant, so geometric}$$

M1 A1

$$a = 15, r = 3: S_{10} = \frac{15(3^{10} - 1)}{3 - 1}$$

M1

$$= 442860$$

A1**Answer: 442860****4 marks****14 (6 marks)**

(a) Value $= 2000 \times 1.04^{10}$

M1

$$= £2960.49$$

A1**Answer: £2960.49****2 marks**

(b) The deposit made at the start of year k has had $11 - k$ years of interest: $2000 \times 1.04^{11-k}$

B1

Total $= 2000(1.04^{10} + 1.04^9 + \dots + 1.04)$, a geometric series with $a = 1.04$, $r = 1.04$, $n = 10$

M1

$$S = \frac{2000 \times 1.04(1.04^{10} - 1)}{1.04 - 1}$$

M1

$$= £24973 \text{ (nearest pound)}$$

A1**Answer: £24973****4 marks****15 (7 marks)**

(a) $12^2 = k(2k + 12)$

M1

$$144 = 2k^2 + 12k, \text{ so } k^2 + 6k - 72 = 0$$

A1**Answer: shown****2 marks**

(b) $(k + 12)(k - 6) = 0$

M1

$$k = 6 \text{ or } k = -12$$

A1**Answer: $k = 6$ or $k = -12$** **2 marks**

(c) $k = 6$: terms 6, 12, 24, ... so $r = 2$ and $S_8 = \frac{6(2^8 - 1)}{2 - 1}$

M1

$$= 1530$$

A1

$k = -12$: terms $-12, 12, -12, \dots$ so $r = -1$ and $|r| = 1$, so the series does not converge

E1**Answer: 1530****3 marks****16 (7 marks)**

(a) Each year the value is multiplied by $1 - 0.15 = 0.85$

M1

after n years: $18\,000 \times 0.85^n$

A1**Answer: shown****2 marks**

(b) $18\,000 \times 0.85^5$

M1

$$= £7990$$

A1**Answer: £7990****2 marks**

(c) $0.85^n < \frac{5000}{18\,000}$

M1

$$n > \frac{\ln(\frac{5}{18})}{\ln 0.85} = 7.88$$

A1

$$n = 8 \text{ years}$$

A1

Answer: 8 years**3 marks****17 (5 marks)**Shares a, ar, ar^2 with $ar^2 = 9a$ **M1** $r^2 = 9$, so $r = 3$ ($r > 0$)**A1** $a(1 + 3 + 9) = 2730$, i.e. $13a = 2730$ **M1** $a = 210$ **A1**

Shares £210, £630 and £1890

A1**Answer: £210, £630, £1890****5 marks****18 (6 marks)****(a)** $a + ar + ar^2 = 7a$ **M1** $a \neq 0$, so $1 + r + r^2 = 7$, giving $r^2 + r - 6 = 0$ **A1****Answer: shown****2 marks****(b)** $(r + 3)(r - 2) = 0$ **M1** $r > 0$ so $r = 2$ **A1** $u_8 = 3 \times 2^7$ **M1** $= 384$ **A1****Answer: $r = 2, u_8 = 384$** **4 marks****19 (6 marks)** $S_8 = a(1 + r + \dots + r^3) + r^4 a(1 + r + \dots + r^3) = S_4(1 + r^4)$ **M1 A1** $1 + r^4 = \frac{255}{15} = 17$ **M1** $r^4 = 16, r = 2$ ($r > 0$)**A1** $S_4 = a(2^4 - 1) = 15a = 15$ **M1** $a = 1$ **A1****Answer: $r = 2, a = 1$** **6 marks****20 (5 marks)** $ar + ar^3 = 78$ and $ar^2 + ar^4 = 117$ **M1**Second equation = $r \times$ (first equation): $117 = 78r$ **M1** $r = \frac{3}{2}$ **A1** $a \times \frac{3}{2} \times (1 + \frac{9}{4}) = 78$, so $a \times \frac{39}{8} = 78$ **M1** $a = 16$ **A1****Answer: $a = 16, r = \frac{3}{2}$** **5 marks****21 (9 marks)****(a)** $S_n = a + ar + ar^2 + \dots + ar^{n-1}$ **B1** $rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$ **M1**Subtract: $rS_n - S_n = ar^n - a$ **M1** $S_n(r - 1) = a(r^n - 1)$, so $S_n = \frac{a(r^n - 1)}{r - 1}$ since $r \neq 1$ **A1****Answer: Proof****4 marks****(b)** $a + ar + ar^2 = 7a$, so $1 + r + r^2 = 7$ **M1** $r^2 + r - 6 = 0$, so $(r + 3)(r - 2) = 0$ **M1** $r > 0$, so $r = 2$ **A1****Answer: $r = 2$** **3 marks****(c)** $ar^3 = 8a = 40$, so $a = 5$; $S_{10} = \frac{5(2^{10} - 1)}{2 - 1}$ **M1** $= 5 \times 1023 = 5115$ **A1****Answer: 5115****2 marks**

22 (8 marks)

(a) $u_{10} = 5 \times 1.08^9$ M1

$= 10.0 \text{ km (3 s.f.)}$ A1

Answer: 10.0 km 2 marks

(b) $S_{10} = \frac{5(1.08^{10} - 1)}{1.08 - 1}$ M1

$= 72.4 \text{ km}$ A1

Answer: 72.4 km 2 marks

(c) $5 \times 1.08^{n-1} > 15$, so $1.08^{n-1} > 3$ M1

$n - 1 > \frac{\log 3}{\log 1.08} = 14.27$ M1

$n = 16$ A1

Answer: Week 16 3 marks

(d) The weekly distance grows without limit (for example it would exceed the number of kilometres in a week), which is not realistic E1

Answer: Distance would grow without bound 1 mark**23 (6 marks)**

$ar^2 = 12$ and $ar + ar^3 = 40$ B1 B1

Divide: $\frac{ar(1+r^2)}{ar^2} = \frac{40}{12}$, so $\frac{1+r^2}{r} = \frac{10}{3}$ M1

$3r^2 - 10r + 3 = 0$, $(3r - 1)(r - 3) = 0$ M1

$r = 3$: $a = \frac{12}{9} = \frac{4}{3}$; $r = \frac{1}{3}$: $a = 12 \times 9 = 108$ A1 A1

Answer: $r = 3$, $a = \frac{4}{3}$ or $r = \frac{1}{3}$, $a = 108$ 6 marks**24 (10 marks)**

(a) $B_1 = 10\,000 \times 1.01 - 400 = 9700$ B1

$B_2 = 9700 \times 1.01 - 400 = 9397$ B1

$B_{n+1} = 1.01B_n - 400$ B1

Answer: 9700, 9397; $B_{n+1} = 1.01B_n - 400$ 3 marks

(b) $B_n = 10\,000 \times 1.01^n - 400(1.01^{n-1} + \dots + 1.01 + 1)$ M1 A1

The bracket is geometric with sum $\frac{1.01^n - 1}{0.01} = 100(1.01^n - 1)$ M1

$B_n = 10\,000 \times 1.01^n - 40\,000(1.01^n - 1) = 40\,000 - 30\,000 \times 1.01^n$ A1

Answer: shown 4 marks

(c) $B_n \leq 0$: $1.01^n \geq \frac{4}{3}$ M1

$n \geq \frac{\ln(\frac{4}{3})}{\ln 1.01} = 28.91$ A1

so 29 repayments (the last one smaller than £400) A1

Answer: 29 repayments 3 marks**25 (8 marks)**

(a) $2^{n-1} > 10^6$ M1

$(n - 1) > \frac{\lg 10^6}{\lg 2} = 19.93$ A1

$n = 21$ A1

Answer: square 21 3 marks

(b) $S_{64} = \frac{1 \times (2^{64} - 1)}{2 - 1} = 2^{64} - 1$ M1 A1

$= 1.84 \times 10^{19}$ A1

Answer: 1.84×10^{19} 3 marks

(c) $25 \times 1.84 \times 10^{19} \div 10^9$ M1

$$= 4.6 \times 10^{11} \text{ tonnes}$$

A1

Answer: 4.6×10^{11} tonnes

2 marks

26 (8 marks)

(a) Output of A in month n is $6000 \times 0.88^{n-1}$

M1

$$6000 \times 0.88^4 = 3598.2 = 3598$$

A1

Answer: shown

2 marks

(b) $S_{12} = \frac{6000(1 - 0.88^{12})}{1 - 0.88}$

M1

$$= 39216$$

A1

Answer: 39216

2 marks

(c) $800 \times 1.09^{k-1} > 6000 \times 0.88^{k-1}$

M1

$$\left(\frac{1.09}{0.88}\right)^{k-1} > 7.5$$

A1

$$k - 1 > \frac{\ln 7.5}{\ln\left(\frac{1.09}{0.88}\right)} = 9.41$$

M1

$$k = 11$$

A1

Answer: k = 11

4 marks

27 (10 marks)

(a) The terms are a + d, a + 2d, a + 5d

B1

Geometric: $(a + 2d)^2 = (a + d)(a + 5d)$

M1

$$a^2 + 4ad + 4d^2 = a^2 + 6ad + 5d^2, \text{ so } d^2 + 2ad = 0$$

A1

$$d \neq 0, \text{ so } d + 2a = 0 \text{ and } d = -2a$$

A1

Answer: Proof

4 marks

(b) Terms are -a, -3a, -9a so r = 3

B1

Answer: r = 3

1 mark

(c) $S_{10} = 5(2a + 9d) = 5(2a - 18a) = -80a$

M1 A1

$$-80a = -400, \text{ so } a = 5 \text{ and } d = -10$$

A1

Answer: a = 5, d = -10

3 marks

(d) First term -5, ratio 3: $S_8 = \frac{-5(3^8 - 1)}{3 - 1}$

M1

$$= -16400$$

A1

Answer: -16400

2 marks

28 (11 marks)

(a) $\ln u_n = \ln 4 + (n - 1) \ln 1.5$

M1

Consecutive terms differ by $\ln 1.5$ (constant), so arithmetic

A1

First term $\ln 4$, common difference $\ln 1.5$

A1

Answer: a = $\ln 4$, d = $\ln 1.5$

3 marks

(b) $4 \times 1.5^{n-1} > 10^6$, so $(n - 1) \ln 1.5 > \ln 250\,000$

M1

$$n - 1 > 30.65$$

M1

$$n = 32$$

A1

Answer: n = 32

3 marks

(c) $S_{20} = \frac{20}{2}(2 \ln 4 + 19 \ln 1.5)$

M1

$$= 20 \ln 4 + 190 \ln 1.5$$

A1

$$= 104.8$$

A1

Answer: $20 \ln 4 + 190 \ln 1.5 = 104.8$

3 marks

(d) $u_1 u_2 \dots u_{20} = 4^{20} \times (1.5)^{190} = 2^{40} \times 3^{190} \times 2^{-190}$

M1

$$= 2^{-150} \times 3^{190}, \text{ so } p = -150, q = 190$$

A1

Answer: p = -150, q = 190

2 marks

29 (9 marks)

(a) $S_{2n} - S_n$ is the sum of terms $n + 1$ to $2n$

M1

each is r^n times the corresponding term of the first n terms, so $S_{2n} - S_n = r^n S_n$

A1

Terms $2n + 1$ to $3n$ are r^{2n} times the first n terms

M1

$$S_{3n} - S_{2n} = r^{2n} S_n$$

A1

Answer: $r^n S_n$ and $r^{2n} S_n$

4 marks

(b) $S_n, r^n S_n, r^{2n} S_n$ each term is r^n times the previous

M1

geometric with common ratio r^n

A1

Answer: ratio r^n

2 marks

(c) $S_{10} - S_5 = 22 = r^5 \times 11$, so $r^5 = 2$

M1 A1

$$S_{15} - S_{10} = r^{10} \times 11 = 44, \text{ so } S_{15} = 33 + 44 = 77$$

A1

Answer: 77

3 marks

30 (9 marks)

(a) The first payment is multiplied by 1.004 sixty times, the last by 1.004 once

M1

$$\text{Total} = x(1.004 + 1.004^2 + \dots + 1.004^{60})$$

A1

Geometric, first term 1.004x, ratio 1.004

M1

$$S_{60} = \frac{1.004x(1.004^{60} - 1)}{1.004 - 1} = \frac{1.004x(1.004^{60} - 1)}{0.004}$$

A1

Answer: shown

4 marks

(b) $20\,000 = 251x(1.004^{60} - 1)$

M1

$$1.004^{60} = 1.2706, \text{ so } x = 294.417$$

A1

$x = \text{£}294.42$ (rounded up)

A1

Answer: £294.42

3 marks

(c) 12 compounding periods: $1.004^{12} = 1.0491$

M1

annual rate 4.91% > 4.8% because interest earns interest

A1

Answer: 4.91%

2 marks

Sum to Infinity

1 (2 marks)

$$S_{\infty} = \frac{a}{1-r} = \frac{12}{1-\frac{1}{3}}$$

M1

$$= \frac{12}{\frac{2}{3}} = 18$$

A1

Answer: 18

2 marks

2 (3 marks)

(i) $r = \frac{1}{2}$, $|r| < 1$, so converges: $S_{\infty} = \frac{6}{1-\frac{1}{2}} = 12$

B1

(ii) $r = -3$, $|r| > 1$, so diverges (no sum to infinity)

B1

(iii) $r = -\frac{1}{2}$, converges: $S_{\infty} = \frac{10}{1+\frac{1}{2}} = \frac{20}{3}$

B1

Answer: (i) 12; (ii) diverges; (iii) $\frac{20}{3}$

3 marks

3 (2 marks)

$$r = \frac{1}{2}, |r| < 1 \text{ so } S_{\infty} = \frac{a}{1-r} = \frac{20}{1-\frac{1}{2}}$$

M1

$$= 40$$

A1

Answer: 40

2 marks

4 (2 marks)A geometric series converges when $|r| < 1$

B1

(i) no, $|1.2| > 1$; (ii) yes, $|-0.5| < 1$; (iii) no, $|1|$ is not less than 1

B1

Answer: only (ii) converges

2 marks

5 (2 marks)

$$\frac{8}{1-r} = 10$$

M1

$$1-r = 0.8 \text{ so } r = 0.2$$

A1

$$\text{Answer: } r = \frac{1}{5}$$

2 marks

6 (2 marks)

$$a = S_{\infty}(1-r) = 14 \times 0.7$$

M1

$$= 9.8$$

A1

Answer: 9.8

2 marks

7 (4 marks)(a) Geometric, $a = \frac{1}{3}$, $r = \frac{1}{3}$: width $= (\frac{1}{3})^n$

B1

$$\text{Answer: } (\frac{1}{3})^n$$

1 mark

$$(b) S_{\infty} = \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$

M1

$$= \frac{1}{2}$$

A1

$$\text{Uncovered width} = 1 - \frac{1}{2} = \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2}; \text{ uncovered width } \frac{1}{2}$$

3 marks

8 (4 marks)(a) $0.36 + 0.0036 + 0.000036 + \dots$

M1

$$a = 0.36, r = 0.01$$

A1

Answer: a = 0.36, r = 0.01

2 marks

$$(b) S_{\infty} = \frac{0.36}{1 - 0.01} = \frac{0.36}{0.99}$$

M1

$$= \frac{36}{99} = \frac{4}{11}$$

A1

$$\text{Answer: } \frac{4}{11}$$

2 marks

9 (4 marks)

$$S_3 = \frac{a(1 - (\frac{1}{4})^3)}{1 - \frac{1}{4}} = \frac{21}{16}a$$

M1

$$\frac{21}{16}a = 21, \text{ so } a = 16$$

A1

$$S_{\infty} = \frac{16}{1 - \frac{1}{4}}$$

M1

$$= \frac{64}{3}$$

A1

$$\text{Answer: } a = 16, S_{\infty} = \frac{64}{3}$$

4 marks

10 (4 marks)

$$0.1666... = 0.1 + (0.06 + 0.006 + 0.0006 + ...)$$

M1

geometric part: $a = 0.06$, $r = 0.1$

B1

$$S_{\infty} = \frac{0.06}{0.9} = \frac{1}{15}$$

M1

$$0.1 + \frac{1}{15} = \frac{1}{6}$$

A1

Answer: $\frac{1}{6}$

4 marks

11 (4 marks)

$$r = 2x$$

B1

$$|2x| < 1 \text{ so } -\frac{1}{2} < x < \frac{1}{2}$$

M1 A1

$$x = \frac{1}{4}: S_{\infty} = \frac{1}{1 - \frac{1}{2}} = 2$$

A1

Answer: $-\frac{1}{2} < x < \frac{1}{2}$; sum 2

4 marks

12 (4 marks)

$$\frac{a}{1 - a^2} = \frac{2}{3}$$

M1

$$3a = 2 - 2a^2, \text{ so } 2a^2 + 3a - 2 = 0$$

A1

$$(2a - 1)(a + 2) = 0$$

M1

$$a = \frac{1}{2}. \text{ If } a = -2 \text{ then } r = 4, \text{ so } |r| > 1 \text{ and the series diverges}$$

A1

Answer: $a = \frac{1}{2}$

4 marks

13 (3 marks)

$$\text{Fourth term} = 64 \times \left(\frac{1}{2}\right)^3 = 8, r = \frac{1}{2}$$

B1

$$S_{\infty} = \frac{8}{1 - \frac{1}{2}}$$

M1

$$= 16$$

A1

Answer: 16

3 marks

14 (6 marks)

$$\frac{a}{1 - r} = 27 \text{ so } a = 27(1 - r)$$

M1

$$a(1 + r) = 24, \text{ so } 27(1 - r)(1 + r) = 24$$

M1

$$1 - r^2 = \frac{8}{9}, \text{ so } r^2 = \frac{1}{9}$$

M1

$$r = \frac{1}{3} \text{ or } r = -\frac{1}{3}$$

A1

$$r = \frac{1}{3}: a = 27 \times \frac{2}{3} = 18$$

A1

$$r = -\frac{1}{3}: a = 27 \times \frac{4}{3} = 36$$

A1

Answer: $a = 18, r = \frac{1}{3}$ or $a = 36, r = -\frac{1}{3}$

6 marks

15 (6 marks)

(a) Heights after bounces form a geometric sequence with first term 1.2 and ratio 0.6

M1

$$\text{Height} = 2 \times 0.6^3 = 0.432 \text{ m}$$

A1

Answer: 0.432 m

2 marks

(b) Each rebound height h is travelled twice (up and down)

B1

$$\text{Total rebounds distance} = 2(1.2 + 0.72 + ...) = 2 \times \frac{1.2}{1 - 0.6}$$

M1 A1

$$= 6, \text{ so total} = 2 + 6 = 8 \text{ m}$$

Answer: 8 m

A1

4 marks

16 (6 marks)

(a) $\frac{800}{1-r} = 1000$

M1

$$1 - r = 0.8, \text{ so } r = 0.2$$

A1

Answer: 0.2

2 marks

(b) $S_5 = \frac{800(1 - 0.2^5)}{1 - 0.2}$

M1

$$= 999.68$$

A1

Answer: 999.68

2 marks

(c) The sum from the 6th term onwards = $S_\infty - S_5$

M1

$$= 1000 - 999.68 = 0.32$$

A1

Answer: 0.32

2 marks

17 (6 marks)

(a) $r = 0.95$

B1

$$S_\infty = \frac{40}{1 - 0.95}$$

M1

$$= 800 \text{ cm}$$

A1

Answer: 800 cm

3 marks

(b) $40 \times 0.95^{n-1} < 1$

M1

$$n - 1 > \frac{\ln(\frac{1}{40})}{\ln 0.95} = 71.92$$

A1

$$n = 73$$

A1

Answer: 73rd swing

3 marks

18 (6 marks)

(a) $ar = 48$ and $ar^4 = -6$, so $r^3 = -\frac{1}{8}$

M1

$$r = -\frac{1}{2}$$

A1

$$a = \frac{48}{-\frac{1}{2}} = -96$$

A1

Answer: $r = -\frac{1}{2}$, $a = -96$

3 marks

(b) $S_\infty = \frac{-96}{1 + \frac{1}{2}} = -64$

M1

$$S_n = \frac{-96(1 - (-\frac{1}{2})^n)}{1 + \frac{1}{2}} = -64(1 - (-\frac{1}{2})^n)$$

M1

$$S_\infty - S_n = -64 + 64(1 - (-\frac{1}{2})^n) = -64(-\frac{1}{2})^n$$

A1

Answer: shown

3 marks

19 (4 marks)

$$S_3 = \frac{a(1 - r^3)}{1 - r}, S_\infty = \frac{a}{1 - r}$$

B1

$$\frac{S_3}{S_\infty} = 1 - r^3 = \frac{19}{27}$$

M1

$$r^3 = \frac{8}{27}$$

A1

$$r = \frac{2}{3}$$

A1

Answer: $r = \frac{2}{3}$

4 marks

20 (7 marks)

$$(a) \frac{a}{1-r} = 4ar \quad \text{M1}$$

$$1 = 4r(1-r), \text{ so } 4r^2 - 4r + 1 = 0 \quad \text{A1}$$

$$(2r-1)^2 = 0, \text{ so } r = \frac{1}{2} \quad \text{E1}$$

Answer: shown 3 marks

$$(b) S_5 = \frac{a(1 - (\frac{1}{2})^5)}{1 - \frac{1}{2}} = \frac{31}{16}a = 1550 \quad \text{M1}$$

$$a = 800 \quad \text{A1}$$

Answer: 800 2 marks

$$(c) S_\infty = 1600, S_n = 1600(1 - (\frac{1}{2})^n) > 1599, \text{ so } (\frac{1}{2})^n < \frac{1}{1600}, n > \frac{\ln 1600}{\ln 2} = 10.64 \quad \text{M1}$$

$$n = 11 \quad \text{A1}$$

Answer: 11 2 marks

21 (8 marks)

$$(a) \text{ Side of } S_2: \sqrt{4^2 + 4^2} = 4\sqrt{2}, \text{ so area} = 32 = \frac{1}{2} \times 64 \quad \text{M1 A1}$$

$$\text{Each square has half the area of the previous one (same construction), so } r = \frac{1}{2} \quad \text{E1}$$

$$\text{Answer: } r = \frac{1}{2} \quad \text{3 marks}$$

$$(b) S_\infty = \frac{64}{1 - \frac{1}{2}} \quad \text{M1}$$

$$= 128 \text{ cm}^2 \quad \text{A1}$$

Answer: 128 cm² 2 marks

$$(c) \text{ Side lengths } 8, 4\sqrt{2}, 4, \dots \text{ have ratio } \frac{1}{\sqrt{2}}, \text{ so perimeters } 32, 16\sqrt{2}, \dots: a = 32, r = \frac{1}{\sqrt{2}} \quad \text{M1}$$

$$S_\infty = \frac{32}{1 - \frac{1}{\sqrt{2}}} = \frac{32\sqrt{2}}{\sqrt{2} - 1} = \frac{32\sqrt{2}(\sqrt{2} + 1)}{1} \quad \text{M1}$$

$$= 64 + 32\sqrt{2} \text{ cm} \quad \text{A1}$$

Answer: 64 + 32√2 cm 3 marks

22 (8 marks)

$$(a) \text{ Convergent when } |r| < 1: \left| \frac{2x}{x+3} \right| < 1 \quad \text{M1}$$

$$\text{Square both sides: } 4x^2 < (x+3)^2 \quad \text{M1}$$

$$(2x - (x+3))(2x + (x+3)) < 0, \text{ i.e. } (x-3)(3x+3) < 0 \quad \text{M1}$$

$$\text{Critical values } -1 \text{ and } 3, \text{ so } -1 < x < 3 \quad \text{A1}$$

Answer: Proof 4 marks

$$(b) S_\infty = \frac{1}{1 - \frac{2x}{x+3}} = \frac{x+3}{(x+3) - 2x} \quad \text{M1}$$

$$= \frac{x+3}{3-x} \quad \text{A1}$$

Answer: Proof 2 marks

$$(c) x+3 = 7(3-x), \text{ so } 8x = 18 \text{ and } x = \frac{9}{4} \quad \text{M1}$$

$$-1 < \frac{9}{4} < 3 \text{ so the value is valid} \quad \text{A1}$$

Answer: x = 9/4 2 marks

23 (6 marks)

(a) $r = \cos \theta$

B1

For $0 < \theta < \pi$, $-1 < \cos \theta < 1$, so $|r| < 1$

E1

Answer: $|\cos \theta| < 1$

2 marks

(b) $S_{\infty} = \frac{1}{1 - \cos \theta}$

B1

Answer: $\frac{1}{1 - \cos \theta}$

1 mark

(c) $\frac{1}{1 - \cos \theta} = 2$ so $\cos \theta = \frac{1}{2}$

M1 A1

$\theta = \frac{\pi}{3}$

A1

Answer: $\theta = \frac{\pi}{3}$

3 marks

24 (5 marks)

(a) $\frac{2^r + 3^r}{5^r} = \left(\frac{2}{5}\right)^r + \left(\frac{3}{5}\right)^r$

M1

so the sum is $\sum_{r=1}^{\infty} \left(\frac{2}{5}\right)^r + \sum_{r=1}^{\infty} \left(\frac{3}{5}\right)^r$, each geometric

A1

Answer: shown

2 marks

(b) First: $a = \frac{2}{5}$, $r = \frac{2}{5}$: $\text{sum} = \frac{2}{5/3/5} = \frac{2}{3}$

M1 A1

Second: $a = r = \frac{3}{5}$: $\text{sum} = \frac{3}{2}$; total $\frac{2}{3} + \frac{3}{2} = \frac{13}{6}$

A1

Answer: $\frac{13}{6}$

3 marks

25 (6 marks)

(a) The terms are $5^x, (5^x)^2, (5^x)^3, \dots$ so $r = 5^x$

B1

Converges when $|5^x| < 1$, i.e. $5^x < 1$, so $x < 0$

B1

Answer: $x < 0$

2 marks

(b) $\frac{5^x}{1 - 5^x} = \frac{3}{2}$

M1

$2 \times 5^x = 3 - 3 \times 5^x$, so $5^x = \frac{3}{5}$

A1

$x = \log_5 \frac{3}{5} = \frac{\ln 0.6}{\ln 5}$

M1

$x = -0.317$

A1

Answer: $x = \log_5 \frac{3}{5} = -0.317$

4 marks

26 (9 marks)

(a) Midpoint triangle is similar with scale factor $\frac{1}{2}$, area $\frac{1}{4}A$

B1

Stage 2 removes 3 triangles, each $\frac{1}{4}$ of a corner triangle of area $\frac{1}{4}A$

M1

$3 \times \frac{1}{4} \times \frac{1}{4}A = \frac{3}{16}A$

A1

Answer: shown

3 marks

(b) Ratio $\frac{3}{16}A \div \frac{1}{4}A = \frac{3}{4}$; each stage removes 3 triangles of quarter area of those of the stage before

M1

$S_{\infty} = \frac{\frac{1}{4}A}{1 - \frac{3}{4}}$

M1

$= A$

A1

Answer: A

3 marks

(c) Stage k removes 3^{k-1} triangles of side $16 \times (\frac{1}{2})^k$, perimeter $3 \times 16 \times (\frac{1}{2})^k$ each M1

Stage k total = $48 \times (\frac{3}{2})^{k-1} \times \frac{1}{2} = 24(\frac{3}{2})^{k-1}$; sum = $48((\frac{3}{2})^n - 1)$ A1

$r = \frac{3}{2} > 1$ so the sum diverges E1

Answer: $48((\frac{3}{2})^n - 1)$; $r > 1$ 3 marks

27 (10 marks)

(a) Terms of H: $a^2, (ar^2)^2, (ar^2)^2, \dots = a^2, a^2r^2, a^2r^4, \dots$ M1

Geometric with first term a^2 and ratio r^2 A1

Answer: First term a^2 , ratio r^2 2 marks

(b) $\frac{a}{1-r} = 12$ so $a = 12(1-r)$ M1

$\frac{a^2}{1-r^2} = 72$ M1

$\frac{144(1-r)^2}{(1-r)(1+r)} = \frac{144(1-r)}{1+r} = 72$ M1 A1

$2(1-r) = 1+r$, so $r = \frac{1}{3}$ and $a = 8$ A1

Answer: $r = \frac{1}{3}$, $a = 8$ 5 marks

(c) Cubes form a geometric series with first term $a^3 = 512$ and ratio $r^3 = \frac{1}{27}$ B1 M1

$S_\infty = \frac{512}{1 - \frac{1}{27}} = \frac{6912}{13}$ A1

Answer: $\frac{6912}{13}$ (≈ 531.7) 3 marks

28 (12 marks)

(a) $A_1 = 200$, $A_2 = 200 \times 0.7 + 200 = 340$ M1

$A_3 = 340 \times 0.7 + 200 = 438$ mg A1

Answer: 438 mg 2 marks

(b) $A_n = 200 + 200(0.7) + 200(0.7)^2 + \dots + 200(0.7)^{n-1}$ M1

Geometric series with $a = 200$, $r = 0.7$: $A_n = \frac{200(1 - 0.7^n)}{1 - 0.7}$ M1

$= \frac{2000}{3}(1 - 0.7^n)$ A1

Answer: Proof 3 marks

(c) $0.7^n > 0$, so $A_n < \frac{2000}{3}$ M1

$\frac{2000}{3} = 666.7 < 700$ A1

Answer: $A_n < 666.7 < 700$ 2 marks

(d) $\frac{2000}{3}(1 - 0.7^n) > 600$ M1

$1 - 0.7^n > 0.9$, so $0.7^n < 0.1$ A1

$n > \frac{\log 0.1}{\log 0.7} = 6.46$ M1

$n = 7$ ($A_6 = 588$, $A_7 = 612$) A1

Answer: After the 7th tablet 4 marks

(e) For example: the same proportion is removed every day, whatever the amount present, and tablets are taken at exactly the same time each day E1

Answer: e.g. a constant proportion removed each day 1 mark

29 (9 marks)

(a) $100 \times 0.8^3 = 51.2$ and $100 \times 0.8^4 = 40.96$

B1**Answer: 51.2 cm, 40.96 cm****1 mark**

(b) East-west sections are 1st, 3rd, 5th ...: 100, -64, 40.96, ...

M1 $a = 100, r = -0.64$ ($|r| < 1$)**A1**

$$x = \frac{100}{1 + 0.64} = \frac{100}{1.64}$$

M1

$$= \frac{2500}{41} \approx 61.0$$

A1**Answer: $\frac{2500}{41}$ cm****4 marks**(c) North-south sections: 80, -51.2, ...: $a = 80, r = -0.64$ **M1**

$$y = \frac{80}{1.64} = 48.78 \text{ cm}$$

A1

$$\text{distance} = \sqrt{x^2 + y^2}$$

M1

$$= 78.1 \text{ cm (3 s.f.)}$$

A1**Answer: 78.1 cm****4 marks****30 (9 marks)**

(a) The decimal is $\frac{N}{10^k} + \frac{N}{10^{2k}} + \frac{N}{10^{3k}} + \dots$

M1

geometric with $a = \frac{N}{10^k}, r = \frac{1}{10^k}$

A1

$$S_{\infty} = \frac{N}{10^k/1 - \frac{1}{10^k}}$$

M1

$$= \frac{N}{10^k - 1}$$

A1**Answer: shown****4 marks**

(b) $k = 6, N = 142857: \frac{142857}{999999}$

M1

$$999999 = 7 \times 142857, \text{ so the fraction is } \frac{1}{7}$$

A1

Answer: $\frac{1}{7}$

2 marks

(c) $0.999... = \frac{9}{10 - 1} = \frac{9}{9} = 1$

M1 A1The sum to infinity is the limit of $S_n = 1 - 10^{-n}$, which has no gap from 1 in the limit, so the claim is wrong**E1****Answer: 0.999... = 1****3 marks****Sigma Notation and Standard Series****1 (2 marks)**Substitute $r = 1, 2, 3, 4, 5$: $4 + 7 + 10 + 13 + 16$ **M1**

$$= 50$$

A1**Answer: 50****2 marks****2 (2 marks)**The r th term is $3r$ **B1**

$$60 = 3 \times 20, \text{ so the series is } \sum_{r=1}^{20} 3r$$

B1

Answer: $\sum_{r=1}^{20} 3r$

2 marks

3 (2 marks)

$$(1 + 1) + (4 + 1) + (9 + 1) + (16 + 1)$$

$$= 2 + 5 + 10 + 17 = 34$$

Answer: 34

M1

A1

2 marks

4 (2 marks)

$$2 + 4 + 8 + 16 + 32 + 64$$

$$= 126$$

Answer: 126

M1

A1

2 marks

5 (2 marks)rth term is $3r - 1$ ($a = 2$, $d = 3$)

M1

$$29 = 3 \times 10 - 1, \text{ so the series is } \sum_{r=1}^{10} (3r - 1)$$

A1

$$\text{Answer: } \sum_{r=1}^{10} (3r - 1)$$

2 marks

6 (2 marks)

$$\frac{100}{2}(1 + 100)$$

$$= 5050$$

Answer: 5050

M1

A1

2 marks

7 (4 marks)

(a) Arithmetic series: first term 3, last term 58, 12 terms

M1

$$S = \frac{12}{2}(3 + 58) = 366$$

A1

Answer: 366

2 marks

(b) Geometric series: first term 6, ratio 2, 10 terms

M1

$$S = \frac{6(2^{10} - 1)}{2 - 1} = 6138$$

A1

Answer: 6138

2 marks

8 (5 marks)(a) Total = $\sum_{k=1}^n (2k + 1)$, an arithmetic series with first term 3 and last term $2n + 1$

M1

$$= \frac{n}{2}(3 + 2n + 1)$$

M1

$$= \frac{n}{2}(2n + 4) = n^2 + 2n$$

A1

Answer: Proof

3 marks

(b) $n^2 + 2n - 224 = 0$, so $(n - 14)(n + 16) = 0$

M1

$$n = 14$$

A1

Answer: 14

2 marks

9 (4 marks)(a) $a = 5$, $b = 3$

B1

$$5 \times 3^{N-1} = 3645, \text{ so } 3^{N-1} = 729 = 3^6, N = 7$$

B1

Answer: a = 5, b = 3, N = 7

2 marks

$$(b) S_7 = \frac{5(3^7 - 1)}{3 - 1}$$

M1

$$= 5465$$

A1

Answer: 5465

2 marks

10 (3 marks)

$r = 5$ to 15 gives 11 terms; first term 11 , last term 41

B1

$$S = \frac{11}{2}(11 + 41)$$

M1

$$= 286$$

A1

Answer: 286

3 marks

11 (3 marks)

Arithmetic series with $a = 1$, $d = 4$

B1

$$S_n = \frac{n}{2}(2 + 4(n - 1))$$

M1

$$= \frac{n}{2}(4n - 2) = n(2n - 1)$$

A1

Answer: shown

3 marks

12 (3 marks)

Geometric: $a = -6$, $r = -2$, $n = 10$

B1

$$S = \frac{-6(1 - (-2)^{10})}{1 - (-2)}$$

M1

$$= \frac{-6 \times (-1023)}{3} = 2046$$

A1

Answer: 2046

3 marks

13 (4 marks)

$$\sum_{r=1}^{12} r^2 = \frac{12 \times 13 \times 25}{6} = 650$$

M1 A1

$$\sum_{r=1}^{12} 2r = 12 \times 13 = 156$$

M1

$$\text{Total} = 806$$

A1

Answer: 806

4 marks

14 (6 marks)

$$(a) \sum_{r=1}^{10} 3r = 3(1 + 2 + \dots + 10) = 3 \times 55 = 165$$

M1 A1

$$165 + 10k = 215, \text{ so } k = 5$$

A1

Answer: $k = 5$

3 marks

(b) Sum from 11 to $30 = (\text{sum to } 30) - (\text{sum to } 10)$

M1

$$\text{Sum to } 30: \frac{30}{2}(8 + 95) = 1545$$

A1

$$1545 - 215 = 1330$$

A1

Answer: 1330

3 marks

15 (6 marks)

$$(a) \text{ Geometric, } a = \frac{3}{2}, r = \frac{1}{2}: S_8 = \frac{\frac{3}{2}(1 - (\frac{1}{2})^8)}{1 - \frac{1}{2}}$$

M1

$$= 3(1 - \frac{1}{256}) = \frac{765}{256}$$

A1

$$\text{Answer: } \frac{765}{256}$$

2 marks

$$(b) S_{\infty} = \frac{\frac{3}{2}}{1 - \frac{1}{2}} = 3$$

B1

Answer: 3

1 mark

$$(c) S_n = 3(1 - (\frac{1}{2})^n) > 2.99, \text{ so } (\frac{1}{2})^n < \frac{1}{300}$$

M1

$$2^n > 300, \text{ so } n > \frac{\log 300}{\log 2} = 8.23$$

M1

$$n = 9$$

Answer: $n = 9$

A1

3 marks

16 (4 marks)

$$\sum_{r=1}^n r(r+1) = \sum_{r=1}^n r^2 + \sum_{r=1}^n r$$

M1

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

A1

$$= \frac{n(n+1)}{6} [(2n+1) + 3]$$

M1

$$= \frac{n(n+1)(2n+4)}{6} = \frac{n(n+1)(n+2)}{3}$$

A1

Answer: shown

4 marks

17 (4 marks)

$$\sum_{r=11}^{20} r^3 = \sum_{r=1}^{20} r^3 - \sum_{r=1}^{10} r^3$$

M1

$$\sum_{r=1}^{20} r^3 = 210^2 = 44\,100$$

A1

$$\sum_{r=1}^{10} r^3 = 55^2 = 3025$$

A1

$$\text{Difference} = 41075$$

A1

Answer: 41075

4 marks

18 (6 marks)

(a) With $r = s + 1$: $4r - 3 = 4(s + 1) - 3 = 4s + 1$

M1

$r = 2$ gives $s = 1$ and $r = n + 1$ gives $s = n$, so the two sums are equal

A1

Answer: shown

2 marks

(b) Use (a): $\sum_{r=1}^n (4r + 1) = 4 \times \frac{n(n+1)}{2} + n$

M1

$$= 2n^2 + 3n$$

A1

$$2n^2 + 3n - 1325 = 0, \text{ so } (2n + 53)(n - 25) = 0$$

M1

$$n = 25 \text{ (} n > 0 \text{)}$$

A1

Answer: $n = 25$

4 marks

19 (6 marks)

(a) Arithmetic with $a = 4$, $d = 5$

B1

$$S_n = \frac{n}{2}(8 + 5(n - 1))$$

M1

$$= \frac{n(5n + 3)}{2}$$

A1

Answer: shown

3 marks

(b) $S_{2n} - S_n = \frac{2n(10n + 3)}{2} - \frac{n(5n + 3)}{2}$

M1 A1

$$= \frac{n(15n + 3)}{2}$$

A1

Answer: $\frac{n(15n + 3)}{2}$

3 marks

20 (5 marks)

$$(r + 1)(r + 3) = r^2 + 4r + 3$$

B1

$$\text{Sum} = \frac{n(n+1)(2n+1)}{6} + 4 \times \frac{n(n+1)}{2} + 3n$$

M1 A1

$$= \frac{n}{6}[(n+1)(2n+1) + 12(n+1) + 18]$$

M1

$$= \frac{n}{6}(2n^2 + 3n + 1 + 12n + 12 + 18) = \frac{n(2n^2 + 15n + 31)}{6}$$

A1

Answer: shown

5 marks

21 (7 marks)

$$(a) u_1 = S_1 = 2^3 + 1 - 4 = 5$$

B1

Answer: 5

1 mark

$$(b) S_{n-1} = 2^{n+1} + (n-1)^2 - 4$$

M1

$$u_n = 2^{n+2} - 2^{n+1} + n^2 - (n-1)^2$$

M1 A1

$$= 2^{n+1} + 2n - 1$$

A1

Answer: shown

4 marks

$$(c) \text{Sum} = S_8 - S_3 = (2^{10} + 64 - 4) - (2^5 + 9 - 4)$$

M1

$$= 1084 - 37 = 1047$$

A1

Answer: 1047

2 marks

22 (8 marks)

$$(a) \text{Split the sum: } \sum_{r=1}^n 2^r + \sum_{r=1}^n 3r$$

M1

$$\text{First is geometric: } a = 2, r = 2: \frac{2(2^n - 1)}{2 - 1} = 2^{n+1} - 2$$

M1 A1

$$\text{Second is arithmetic: } 3 \times \frac{n(n+1)}{2} = \frac{3n(n+1)}{2}$$

A1

Answer: Proof

4 marks

$$(b) S_{10} - S_5 = 2211 - 107$$

M1

$$= 2104$$

A1

Answer: 2104

2 marks

$$(c) \text{Left side, } n = 2: 1 \times 2 + 2 \times 4 = 10$$

B1

$$\text{Right side: } (1+2)(2+4) = 18 \neq 10, \text{ so sum of a product is not the product of the sums}$$

E1

Answer: 10 $\neq$ 18

2 marks

23 (9 marks)

$$(a) r(r+1)(r+2) = r^3 + 3r^2 + 2r$$

M1

$$\text{Sum} = \frac{n^2(n+1)^2}{4} + 3 \times \frac{n(n+1)(2n+1)}{6} + 2 \times \frac{n(n+1)}{2}$$

M1 A1

$$\text{Common factor } \frac{n(n+1)}{4}: = \frac{n(n+1)}{4}[n(n+1) + 2(2n+1) + 4]$$

M1

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

A1

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

A1

Answer: shown

6 marks

$$(b) S_{10} = \frac{10 \times 11 \times 12 \times 13}{4} = 4290$$

M1 A1

$$S_4 = \frac{4 \times 5 \times 6 \times 7}{4} = 210, \text{ so answer } 4290 - 210 = 4080$$

A1

Answer: 4080

3 marks

24 (7 marks)

$$(a) \frac{1}{r} - \frac{1}{r+2} = \frac{(r+2) - r}{r(r+2)} \quad \text{M1}$$

$$= \frac{2}{r(r+2)}, \text{ so the result follows} \quad \text{A1}$$

Answer: shown 2 marks

$$(b) \text{ Write out: } \frac{1}{2}[(1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \dots + (\frac{1}{n} - \frac{1}{n+2})] \quad \text{M1}$$

Terms cancel in pairs (telescoping), leaving the first two positives and last two negatives A1

$$\frac{1}{2}[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}] \quad \text{A1}$$

$$= \frac{1}{2}(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}) \quad \text{A1}$$

Answer: shown 4 marks

$$(c) \text{ As } n \rightarrow \infty \text{ the last two terms } \rightarrow 0, \text{ so the sum is } \frac{1}{2} \times \frac{3}{2} = \frac{3}{4} \quad \text{B1}$$

$$\text{Answer: } \frac{3}{4} \quad \text{1 mark}$$

25 (7 marks)

$$(a) \text{ The } r\text{th term is } 2 \times (1 + 10 + 10^2 + \dots + 10^{r-1}) = 2 \times \frac{10^r - 1}{10 - 1} \quad \text{M1}$$

$$= \frac{2}{9}(10^r - 1) \quad \text{A1}$$

Answer: shown 2 marks

$$(b) \text{ Sum} = \frac{2}{9} \sum_{r=1}^n (10^r - 1) = \frac{2}{9}(\sum_{r=1}^n 10^r - n) \quad \text{M1}$$

$$\text{Geometric: } \sum_{r=1}^n 10^r = \frac{10(10^n - 1)}{10 - 1} \quad \text{M1 A1}$$

$$\text{Sum} = \frac{2}{9}(\frac{10^{n+1} - 10}{9} - n) = \frac{2}{81}(10^{n+1} - 10 - 9n) \quad \text{A1}$$

Answer: shown 4 marks

$$(c) \frac{2}{81}(10^7 - 10 - 54) = 246\,912 \quad \text{B1}$$

$$\text{Answer: } 246\,912 \quad \text{1 mark}$$

26 (7 marks)

$$(a) \sum_{r=1}^n (ar + b) = a \times \frac{n(n+1)}{2} + bn \quad \text{M1 A1}$$

$$\text{Compare: } \frac{a}{2}n^2 + (\frac{a}{2} + b)n = 3n^2 + 2n \quad \text{M1}$$

$$a = 6, b = -1 \quad \text{A1}$$

Answer: a = 6, b = -1 4 marks

$$(b) S_{20} - S_{10} \quad \text{M1}$$

$$20 \times 62 - 10 \times 32 = 1240 - 320 \quad \text{A1}$$

$$= 920 \quad \text{A1}$$

Answer: 920 3 marks

27 (9 marks)

$$(a) \sum_{r=1}^{12} f(r) = \sum_{r=1}^{12} (f(r) - 5) + \sum_{r=1}^{12} 5 \quad \text{M1}$$

$$= 60 + 60 = 120 \quad \text{A1}$$

Answer: 120 2 marks

$$(b) (f(r) - 5)^2 = (f(r))^2 - 10f(r) + 25 \quad \text{M1}$$

$$\sum_{r=1}^{12} (f(r))^2 = \sum_{r=1}^{12} (f(r) - 5)^2 + 10 \sum_{r=1}^{12} f(r) - \sum_{r=1}^{12} 25 \quad \text{M1 M1}$$

$$= 700 + 1200 - 300 = 1600$$

Answer: 1600

A1
4 marks

$$(c) \text{ Mean} = \frac{120}{12} = 10$$

B1

$$\text{Variance} = \frac{1600}{12} - 10^2$$

M1

$$= \frac{100}{3} = 33.3 \text{ (3 s.f.)}$$

A1

$$\text{Answer: } \frac{100}{3} = 33.3$$

3 marks

28 (9 marks)

$$(a) T_3 = -6 + 11 - 16 + 21 - 26 + 31$$

M1

$$= 15$$

A1

Answer: 15

2 marks

$$(b) \text{ The } (2k - 1)\text{th and } (2k)\text{th terms together give } -(5(2k - 1) + 1) + (5(2k) + 1)$$

M1 A1

$$= -10k + 4 + 10k + 1 = 5$$

A1

There are n such pairs, so $T_n = 5n$

E1

Answer: $T_n = 5n$

4 marks

$$(c) 2n = 40, \text{ so } n = 20 \text{ and the sum is } 5 \times 20 = 100$$

B1

Answer: 100

1 mark

$$(d) = T_n + (-1)^{2n+1}(5(2n + 1) + 1)$$

M1

$$= 5n - (10n + 6) = -5n - 6$$

A1

Answer: $-5n - 6$

2 marks

29 (9 marks)

$$(a) (r + 1)! = (r + 1) \times r!$$

M1

$$(r + 1)! - r! = r!(r + 1 - 1) = r \times r!$$

A1

Answer: shown

2 marks

$$(b) \text{ Sum} = (2! - 1!) + (3! - 2!) + \dots + ((n + 1)! - n!)$$

M1

All middle terms cancel

A1

$$= (n + 1)! - 1! = (n + 1)! - 1$$

A1

Answer: shown

3 marks

$$(c) 7! - 1 = 5040 - 1 = 5039$$

B1

Answer: 5039

1 mark

$$(d) (n + 1)! > 1\,000\,000\,001$$

M1

$$12! = 479\,001\,600 \text{ and } 13! = 6\,227\,020\,800$$

A1

$$n + 1 = 13 \text{ so } n = 12$$

A1

Answer: 12

3 marks

30 (11 marks)

$$(a) \frac{1}{r(r+1)(r+2)} = \frac{A(r+2) - Br}{r(r+1)(r+2)}, \text{ so } 1 = A(r+2) - Br$$

M1

$$r^0: 2A = 1; r: A - B = 0$$

A1

$$A = B = \frac{1}{2}$$

A1

$$\text{Answer: } A = B = \frac{1}{2}$$

3 marks

$$(b) \text{ Write } \frac{1}{2} \left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \right) \text{ and sum from } r = 1 \text{ to } n$$

M1

$$\text{Consecutive terms cancel, leaving } \frac{1}{2} \left(\frac{1}{1 \times 2} - \frac{1}{(n+1)(n+2)} \right)$$

M1 A1

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

A1

Answer: shown

4 marks

$$(c) \text{ As } n \rightarrow \infty, \frac{1}{2(n+1)(n+2)} \rightarrow 0, \text{ so the sum is } \frac{1}{4}$$

B1

Answer: $\frac{1}{4}$

1 mark

(d) $\frac{1}{2(n+1)(n+2)} < 0.001$, so $(n+1)(n+2) > 500$

M1

 $n = 20: 21 \times 22 = 462 < 500; n = 21: 22 \times 23 = 506 > 500$

A1

 $n = 21$

A1

Answer: 21

3 marks

Recurrence Relations

1 (2 marks)

$$u_2 = 2 \times 1 + 3 = 5, u_3 = 2 \times 5 + 3 = 13$$

M1

$$u_4 = 2 \times 13 + 3 = 29$$

A1

Answer: 5, 13, 29

2 marks

2 (2 marks)

$$a_2 = 3, a_3 = 8, a_4 = 63$$

M1

$$a_5 = 63^2 - 1 = 3968$$

A1

Answer: 3968

2 marks

3 (2 marks)

$$u_2 = 3 \times 2 - 2 \times 1 + 1 = 5, u_3 = 3 \times 5 - 2 \times 2 + 1 = 12$$

M1

$$u_4 = 3 \times 12 - 2 \times 3 + 1 = 31$$

A1

Answer: 5, 12, 31

2 marks

4 (2 marks)

$$u_2 = 1 + 1 = 2, u_3 = 2 + 2 = 4$$

M1

$$u_4 = 4 + 3 = 7, u_5 = 7 + 4 = 11$$

A1

Answer: 1, 2, 4, 7, 11

2 marks

5 (2 marks)

$$u_2 = \frac{1}{2}, u_3 = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

M1

$$u_4 = \frac{1}{1 + \frac{2}{3}} = \frac{3}{5}$$

A1

Answer: $\frac{3}{5}$

2 marks

6 (3 marks)

(i) 2, 5, 8, ... increasing

B1

(ii) 5, -5, 5, -5, ... periodic (order 2)

B1

(iii) 8, 4, 2, ... decreasing

B1

Answer: (i) increasing (ii) periodic (iii) decreasing

3 marks

7 (4 marks)

(a) $u_2 = 3k + 2 = 11$

M1

$k = 3$

A1

Answer: k = 3

2 marks

(b) $u_3 = 3 \times 11 + 2 = 35$

M1

$u_1 + u_2 + u_3 = 3 + 11 + 35 = 49$

A1

Answer: $u_3 = 35$, sum = 49

2 marks

8 (4 marks)

(a) Each term is 3 times the previous one, so it is geometric with $a = 2$, $r = 3$

B1

$$u_n = 2 \times 3^{n-1}$$

B1

Answer: Geometric; $u_n = 2 \times 3^{n-1}$

2 marks

(b) $S_8 = \frac{2(3^8 - 1)}{3 - 1}$

M1

$$= 3^8 - 1 = 6560$$

A1

Answer: 6560

2 marks

9 (5 marks)

(a) $u_2 = 2k + 1$

M1

$$u_3 = k(2k + 1) + 1 = 2k^2 + k + 1$$

A1

Answer: shown

2 marks

(b) $2 + (2k + 1) + (2k^2 + k + 1) = 18$, so $2k^2 + 3k - 14 = 0$

M1

$$(2k + 7)(k - 2) = 0$$

M1

$$k = 2 \text{ or } k = -\frac{7}{2}$$

A1

Answer: $k = 2$ or $k = -\frac{7}{2}$

3 marks

10 (7 marks)

(a) $u_2 = \frac{1}{3}$, $u_3 = -\frac{1}{2}$, $u_4 = -3$, $u_5 = 2$

B1 B1

The sequence is periodic with order 4

B1

Answer: periodic, order 4

3 marks

(b) $2025 = 4 \times 506 + 1$

M1

$$\text{so } u_{2025} = u_1 = 2$$

A1

Answer: 2

2 marks

(c) One cycle sums to $2 + \frac{1}{3} - \frac{1}{2} - 3 = -\frac{7}{6}$, and $100 = 25$ cycles

M1

$$25 \times \left(-\frac{7}{6}\right) = -\frac{175}{6}$$

A1

Answer: $-\frac{175}{6}$

2 marks

11 (3 marks)

Of the form $u_{n+1} = ru_n + c$ with $|r| = 0.6 < 1$, so it converges

E1

$$L = 0.6L + 8$$

M1

$$0.4L = 8, L = 20$$

A1

Answer: 20

3 marks

12 (4 marks)

(a) $u_{n+1} = u_n + 3$, $u_1 = 5$

B1

$$u_{n+1} = 3u_n, u_1 = 2$$

B1

Answer: $u_{n+1} = u_n + 3$; $u_{n+1} = 3u_n$

2 marks

(b) $u_{n+1} = 2(n+1)^2 - 3 = 2n^2 + 4n - 1$

M1

$$u_n + 4n + 2 = 2n^2 - 3 + 4n + 2 = 2n^2 + 4n - 1$$

A1

Answer: shown

2 marks

13 (4 marks)

$$u_3 = 2u_2 - 3 = 5, \text{ so } u_2 = 4$$

M1 A1

$$u_2 = 2u_1 - 3 = 4, \text{ so } u_1 = 3.5$$

A1

$$u_4 = 2 \times 5 - 3 = 7$$

A1

Answer: $u_2 = 4$, $u_1 = 3.5$, $u_4 = 7$

4 marks

14 (6 marks)

$$(a) u_2 = \frac{1}{2} \times 4 + 6 = 8$$

M1

$$u_3 = \frac{1}{2} \times 8 + 6 = 10$$

A1

Answer: $u_2 = 8, u_3 = 10$

2 marks

$$(b) \text{ As } n \rightarrow \infty, u_{n+1} \text{ and } u_n \text{ both tend to } L, \text{ so } L = \frac{1}{2}L + 6$$

M1

$$L = 12$$

A1

Answer: $L = 12$

2 marks

$$(c) 12 - u_{n+1} = 12 - \frac{1}{2}u_n - 6 = 6 - \frac{1}{2}u_n = \frac{1}{2}(12 - u_n)$$

M1

The gap from 12 halves each time, so the terms increase towards 12 but never reach it

E1

Answer: Proof

2 marks

15 (6 marks)

$$(a) u_2 = \frac{1}{1-3} = -\frac{1}{2}, u_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}, u_4 = \frac{1}{1-\frac{2}{3}} = 3$$

M1

 $u_4 = u_1$, so the sequence repeats with period 3

A1

Answer: $-\frac{1}{2}, \frac{2}{3}, 3$; period 3

2 marks

$$(b) 100 = 3 \times 33 + 1, \text{ so } u_{100} = u_1$$

M1

$$= 3$$

A1

Answer: 3

2 marks

$$(c) \text{ Each complete cycle sums to } 3 - \frac{1}{2} + \frac{2}{3} = \frac{19}{6}, \text{ and there are 33 complete cycles plus } u_{100} = 3$$

M1

$$33 \times \frac{19}{6} + 3 = 107.5$$

A1

Answer: 107.5

2 marks

16 (6 marks)

$$(a) P_2 = 1.1 \times 3000 - 300$$

M1

$$= 3000 = P_1, \text{ so every term is } 3000$$

A1

Answer: shown

2 marks

$$(b) P_2 = 1.1 \times 2800 - 300 = 2780$$

M1

$$P_3 = 2758, P_4 = 2733$$

A1

Decreasing, moving further below 3000, so the fish will die out

E1

Answer: 2780, 2758, 2733; population falls

3 marks

$$(c) \text{ The relation has } r = 1.1 > 1, \text{ so the fixed point } 3000 \text{ is unstable}$$

E1

Answer: $r = 1.1 > 1$

1 mark

17 (6 marks)

$$(a) u_2 = \frac{1}{1+2} = \frac{1}{3}, u_3 = \frac{1}{3/1 + \frac{2}{3}} = \frac{1}{5}$$

M1

$$u_4 = \frac{1}{5/1 + \frac{2}{5}} = \frac{1}{7}$$

A1

Answer: $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$

2 marks

$$(b) \text{ The denominators are the odd numbers: } u_n = \frac{1}{2n-1}$$

B1

Answer: $u_n = \frac{1}{2n-1}$

1 mark

$$(c) \frac{u_n}{1+2u_n} = \frac{\frac{1}{2n-1}}{1+\frac{2}{2n-1}}$$

M1

$$= \frac{1}{2n-1+2}$$

M1

$$= \frac{1}{2n+1} = \frac{1}{2(n+1)-1} = u_{n+1}$$

Answer: shown

A1

3 marks

18 (5 marks)

(a) $u_3 = 7, u_4 = 12, u_5 = 19$

M1

$u_6 = 31, u_7 = 50, u_8 = 81$

A1

Answer: 2, 5, 7, 12, 19, 31, 50, 81

2 marks

(b) Running sums: 2, 7, 14, 26, 45, 76, 126, 207

M1 A1

sum of first 8 terms = 207 > 200 and of first 7 terms = 126 < 200, so $n = 8$

A1

Answer: $n = 8$

3 marks

19 (5 marks)

$u_2 = 3 + 3 = 6, u_3 = 6 + 5 = 11, u_4 = 11 + 7 = 18$

B1

$$u_n = u_1 + \sum_{r=1}^{n-1} (2r + 1)$$

M1

$$\sum_{r=1}^{n-1} (2r + 1) = (n - 1)^2 + 2(n - 1) = n^2 - 1$$

M1 A1

$u_n = 3 + n^2 - 1 = n^2 + 2$

A1

Answer: $n^2 + 2$

5 marks

20 (4 marks)

Limit: $L = 0.4L + c$

M1

$6 = 2.4 + c$ so $c = 3.6$

A1

$u_2 = 0.4 \times 10 + 3.6 = 7.6$

M1

$u_3 = 0.4 \times 7.6 + 3.6 = 6.64$

A1

Answer: $c = 3.6, u_3 = 6.64$

4 marks

21 (7 marks)

(a) $u_2 = 2k + 4$

B1

$u_3 = k(2k + 4) + 4 = 2k^2 + 4k + 4 = 34$, so $2k^2 + 4k - 30 = 0$

M1

$k^2 + 2k - 15 = 0$, so $(k + 5)(k - 3) = 0$

M1

$k = 3$ or $k = -5$

A1

Answer: $k = 3$ or $k = -5$

4 marks

(b) Increasing requires $k = 3$ ($k = -5$ gives $u_2 = -6 < u_1$)

M1

$u_4 = 3 \times 34 + 4 = 106$; sum = $2 + 10 + 34 + 106 = 152$

A1

Answer: $u_4 = 106$, sum = 152

2 marks

(c) The terms grow without bound ($|k| > 1$ so the terms diverge)

E1

Answer: Diverges since $|k| > 1$

1 mark

22 (8 marks)

(a) $P_{n+1} = 0.85P_n + 2000$

B1

Answer: $P_{n+1} = 0.85P_n + 2000, P_1 = 10\,000$

1 mark

(b) $P_2 = 0.85 \times 10\,000 + 2000 = 10500$

M1

$P_3 = 0.85 \times 10500 + 2000 = 10925$

A1

Answer: 10500, 10925

2 marks

(c) $L = 0.85L + 2000$

M1

$0.15L = 2000$, so $L = 13333$

A1

Answer: 13333

2 marks

(d) $13\,333.3 - 3333.3 \times 0.85^{n-1} > 13\,000$, so $0.85^{n-1} < 0.1$

M1

$n - 1 > \frac{\log 0.1}{\log 0.85} = 14.17$

M1

$n = 16$, so year 16

A1

Answer: Year 16

3 marks

23 (8 marks)**(a)** Terms: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89

M1

$$u_{10} = 55, u_{11} = 89$$

A1

$$\frac{89}{55} = 1.61818$$

A1

Answer: 55, 89, 1.61818

3 marks

(b)
$$r_{n+1} = \frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1} + u_n}{u_{n+1}}$$

M1

$$= 1 + \frac{u_n}{u_{n+1}} = 1 + \frac{1}{r_n}$$

A1

Answer: shown

2 marks

(c)
$$\varphi = 1 + \frac{1}{\varphi}$$

M1

$$\varphi^2 - \varphi - 1 = 0$$

A1

$$\varphi = \frac{1 + \sqrt{5}}{2} \text{ (the positive root)}$$

A1

Answer: $\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.6180$

3 marks

24 (7 marks)

(a) $u_2 = -5 + k, u_3 = (-5 + k) + k = 2k - 5$

M1

$u_4 = -(2k - 5) + k = 5 - k \text{ and } u_5 = (5 - k) + k = 5$

A1

Answer: shown

2 marks

(b) The sequence repeats every 4 terms and $26 = 4 \times 6 + 2$, so $u_{26} = u_2 = k - 5$

B1

Answer: $k - 5$

1 mark

(c) $5 + (k - 5) + (2k - 5) + (5 - k) = 2k$

M1

$2k = 14, \text{ so } k = 7$

A1

Answer: $k = 7$

2 marks

(d) 6 complete cycles of sum 14, plus $u_{25} + u_{26} = 5 + 2$

M1

$6 \times 14 + 7 = 91$

A1

Answer: 91

2 marks

25 (7 marks)

(a) $20 = 8p + q \text{ and } 44 = 20p + q$

M1 A1

$12p = 24 \text{ so } p = 2, q = 4$

A1

$8 = 2u_1 + 4 \text{ so } u_1 = 2$

A1

Answer: $p = 2, q = 4, u_1 = 2$

4 marks

(b) $u_{n+1} + 4 = 2u_n + 8 = 2(u_n + 4)$

M1

ratio 2, first term $u_1 + 4 = 6$

A1

$u_n + 4 = 6 \times 2^{n-1} = 3 \times 2^n, \text{ so } u_n = 3 \times 2^n - 4$

A1

Answer: shown

3 marks

26 (7 marks)

(a) Constant: $k = k^2 - 2k + 2$

M1

$k^2 - 3k + 2 = 0, (k - 1)(k - 2) = 0$

A1

$k = 1 \text{ or } k = 2$

A1

Answer: $k = 1 \text{ or } 2$

3 marks

(b) 3, 5, 17, 257

M1

The terms increase rapidly without limit

A1

Answer: 3, 5, 17, 257; diverges

2 marks

(c) 1.5, 1.25, 1.0625, 1.00391 (5 d.p.)

M1

It appears to converge to 1

A1

Answer: approaches 1

2 marks

27 (10 marks)

(a) $u_2 = 2 \times 5 - 3 = 7, u_3 = 11$

M1

$u_4 = 2 \times 11 - 3 = 19$

A1

Answer: 7, 11, 19

2 marks

(b) $v_{n+1} = u_{n+1} - 3 = 2u_n - 6 = 2(u_n - 3) = 2v_n$

M1

So v_n is geometric with $v_1 = 2$ and ratio 2, giving $v_n = 2 \times 2^{n-1} = 2^n$

M1

$u_n = v_n + 3 = 3 + 2^n$

A1

Answer: Proof

3 marks

(c) $\sum_{r=1}^n u_r = \sum_{r=1}^n 3 + \sum_{r=1}^n 2^r$

M1

$= 3n + \frac{2(2^n - 1)}{2 - 1}$

M1 A1

Answer: Proof

3 marks

(d) $2^{20} = 1\,048\,576$, and $n = 19$ gives $57 + 2^{20} - 2 = 1\,048\,631$

M1

$n = 18$ gives $524\,340 < 1\,000\,000$, so $n = 19$

A1

Answer: $n = 19$

2 marks

28 (11 marks)

(a) $u_2 = \frac{1}{2}(2 + 3) = 2.5, u_3 = \frac{1}{2}(2.5 + 2.4) = 2.45$

M1 A1

$u_4 = \frac{1}{2}(2.45 + \frac{6}{2.45}) = 2.44949$

A1

Answer: 2.5, 2.45, 2.44949

3 marks

(b) $L = \frac{1}{2}(L + \frac{6}{L})$

M1

$2L^2 = L^2 + 6$, so $L^2 = 6$

A1

The terms are positive, so $L = \sqrt{6}$

A1

Answer: $L = \sqrt{6}$

3 marks

(c) $u_{n+1} - \sqrt{6} = \frac{u_n^2 + 6 - 2\sqrt{6}u_n}{2u_n}$
 $= \frac{(u_n - \sqrt{6})^2}{2u_n}$

M1 A1

A1

A square divided by a positive number is not negative, so $u_{n+1} \geq \sqrt{6}$

E1

Answer: Proof

4 marks

(d) $u_{n+1} \geq \sqrt{6}$ holds for every $n \geq 1$, so $u_2, u_3, \dots$ are all at least $\sqrt{6}$, which is the limit

E1

Answer: Each term is at least the limit

1 mark

29 (12 marks)

(a) $u_2 = 2 + 2 = 4, u_3 = 8 + 4 = 12$

M1

$u_4 = 24 + 8 = 32$

A1

Answer: 4, 12, 32

2 marks

(b) $v_{n+1} = \frac{u_{n+1}}{2^{n+1}} = \frac{2u_n + 2^n}{2^{n+1}}$

M1

$= \frac{u_n}{2^n} + \frac{1}{2}$

A1

$= v_n + \frac{1}{2}$

A1

Answer: shown

3 marks

(c) $v_1 = \frac{1}{2}$, so $v_n = \frac{1}{2} + (n - 1) \times \frac{1}{2} = \frac{n}{2}$

M1 A1

$u_n = 2^n v_n = n \times 2^{n-1}$

A1

Answer: $u_n = n \times 2^{n-1}$

3 marks

(d) $n \times 2^{n-1} > 1\,000\,000$; $u_{16} = 16 \times 2^{15} = 524\,288$

M1 A1

$$u_{17} = 17 \times 2^{16} = 1114112$$

M1

u_n is increasing, so the least n is 17

A1

Answer: 17

4 marks

30 (11 marks)

(a) $F_{n+1} = 1.03F_n - W$

B1

Answer: $F_{n+1} = 1.03F_n - W$

1 mark

(b) Fixed point: $L = 1.03L - 9000$, so $L = 300\,000$

M1 A1

$F_n - 300\,000$ satisfies $G_{n+1} = 1.03G_n$: geometric, ratio 1.03

M1

$G_0 = -50\,000$, so $F_n = 300\,000 - 50\,000 \times 1.03^n$

A1

Answer: shown

4 marks

(c) $F_n < 0$: $1.03^n > 6$

M1

$$n > \frac{\ln 6}{\ln 1.03} = 60.62$$

A1

$n = 61$

A1

Answer: after the 61st withdrawal

3 marks

(d) Pot never decreases if $F_1 \geq F_0$: $1.03 \times 250\,000 - W \geq 250\,000$

M1

$W \leq 7500$

A1

Largest is £7500: it withdraws exactly the interest each year, so the pot stays at £250 000

E1

Answer: £7500

3 marks

Binomial Expansion (Positive Integer Powers)

1 (2 marks)

$${}^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

B1

$${}^{10}C_2 = \frac{10 \times 9}{2 \times 1} = 45$$

B1

Answer: (a) 35 (b) 45

2 marks

2 (3 marks)

Row $n = 4$ is 1, 4, 6, 4, 1

B1

$$(2 + x)^4 = 2^4 + 4(2^3)x + 6(2^2)x^2 + 4(2)x^3 + x^4$$

M1

$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

A1

Answer: $16 + 32x + 24x^2 + 8x^3 + x^4$

3 marks

3 (2 marks)

$$(1 + x)^5 = 1 + {}^5C_1x + {}^5C_2x^2 + {}^5C_3x^3 + {}^5C_4x^4 + x^5$$

M1

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

A1

Answer: $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

2 marks

4 (2 marks)

$$x^3 + 3(x^2)(2) + 3(x)(2^2) + 2^3$$

M1

$$= x^3 + 6x^2 + 12x + 8$$

A1

Answer: $x^3 + 6x^2 + 12x + 8$

2 marks

5 (2 marks)

$${}^5C_2(3x)^2 = 10 \times 9x^2$$

M1

coefficient is 90

A1

Answer: 90

2 marks

6 (2 marks)

$${}^5C_3 \times 2^2 = 10 \times 4$$

M1

$$= 40$$

A1

Answer: 40

2 marks

7 (4 marks)

$$(1 + 2x)^7 = 1 + {}^7C_1(2x) + {}^7C_2(2x)^2 + {}^7C_3(2x)^3 + \dots$$

M1

$${}^7C_1 \times 2 = 14, \text{ so second term } 14x$$

A1

$${}^7C_2 \times 4 = 84, \text{ so third term } 84x^2$$

A1

$${}^7C_3 \times 8 = 280, \text{ so fourth term } 280x^3$$

A1

$$\text{Answer: } 1 + 14x + 84x^2 + 280x^3 + \dots$$

4 marks

8 (4 marks)

$$(a) {}^7C_4 \times 2^3 \times (-1)^4$$

M1

$$= 35 \times 8 = 280$$

A1

Answer: 280

2 marks

$$(b) {}^6C_3 \times 3^3 = 20 \times 27$$

M1

$$= 540$$

A1

Answer: 540

2 marks

9 (4 marks)

$$(3 - 2x)^4 = 3^4 + {}^4C_1(3^3)(-2x) + {}^4C_2(3^2)(-2x)^2 + {}^4C_3(3)(-2x)^3 + (-2x)^4$$

M1

$$\text{1st two terms: } 81 - 216x$$

A1

$$216x^2 - 96x^3$$

A1

$$+ 16x^4$$

A1

$$\text{Answer: } 81 - 216x + 216x^2 - 96x^3 + 16x^4$$

4 marks

10 (3 marks)

$${}^8C_5 \left(-\frac{1}{2}x\right)^5$$

M1

$$= 56 \times \left(-\frac{1}{32}\right)$$

A1

$$= -\frac{7}{4}$$

A1

$$\text{Answer: } -\frac{7}{4}$$

3 marks

11 (3 marks)

$${}^nC_2 = \frac{n(n-1)}{2} = 45$$

M1

$$n^2 - n - 90 = 0, (n - 10)(n + 9) = 0$$

M1

$$n = 10$$

A1

Answer: 10

3 marks

12 (3 marks)

$${}^5C_3(2x)^3(3)^2$$

M1

$$= 10 \times 8x^3 \times 9$$

A1

$$= 720x^3$$

A1

$$\text{Answer: } 720x^3$$

3 marks

13 (4 marks)

$$\begin{aligned}(2x - \frac{1}{x})^4 &= (2x)^4 + 4(2x)^3(-\frac{1}{x}) + 6(2x)^2(-\frac{1}{x})^2 + 4(2x)(-\frac{1}{x})^3 + (-\frac{1}{x})^4 \\ &= 16x^4 - 32x^2 \\ &\quad + 24 - \frac{8}{x^2} + \frac{1}{x^4}\end{aligned}$$

M1 A1

A1

A1

Answer: $16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$

4 marks

14 (6 marks)

(a) $(1 - 3x)^5 = 1 + {}^5C_1(-3x) + {}^5C_2(-3x)^2 + \dots$
 $= 1 - 15x$
 $+ 10 \times 9x^2 = + 90x^2$

M1

A1

A1

Answer: $1 - 15x + 90x^2$

3 marks

(b) $(2 + x)(1 - 15x + 90x^2 + \dots)$
 Constant 2; $x: 2(-15) + 1 = -29$
 $x^2: 2(90) + 1(-15) = 165$

M1

A1

A1

Answer: $2 - 29x + 165x^2$

3 marks

15 (6 marks)

(a) $x^2: {}^6C_2 \times 2^4 \times a^2 = 240a^2$; $x^3: {}^6C_3 \times 2^3 \times a^3 = 160a^3$

M1 A1

Answer: $240a^2$ and $160a^3$

2 marks

(b) $240a^2 = 160a^3$

M1

$a \neq 0$, so $a = \frac{240}{160} = \frac{3}{2}$

A1

Answer: $a = \frac{3}{2}$

2 marks

(c) ${}^6C_4 \times 2^2 \times (\frac{3}{2})^4 = 15 \times 4 \times \frac{81}{16}$
 $= \frac{1215}{4}$

M1

A1

Answer: $\frac{1215}{4}$

2 marks

16 (6 marks)

(a) $(2x)^4 + 4(2x)^3(3) + 6(2x)^2(3)^2 + 4(2x)(3)^3 + 3^4$
 $= 16x^4 + 96x^3 + 216x^2 + 216x + 81$

M1 A1

A1

Answer: $16x^4 + 96x^3 + 216x^2 + 216x + 81$

3 marks

(b) x^2 terms: $x \times 216x$ and $-2 \times 216x^2$
 $216 - 432 = -216$

M1 M1

A1

Answer: -216

3 marks

17 (5 marks)

Term in x^3 : ${}^5C_3 k^2(2x)^3 = 10 \times 8k^2x^3 = 80k^2x^3$

M1 A1

$80k^2 = 720$, so $k^2 = 9$

M1

$k = 3$ ($k > 0$)

A1

Term in x^2 : ${}^5C_2 k^3(2x)^2 = 10 \times 27 \times 4 = 1080$

A1

Answer: $k = 3$, coefficient 1080

5 marks

18 (5 marks)

General term: ${}^6C_r(2x)^{6-r}\left(\frac{1}{x^2}\right)^r$

M1

Power of x: $(6 - r) - 2r = 6 - 3r$

A1

Independent of x: $r = 2$

M1

${}^6C_2 \times 2^4 = 15 \times 16$

A1

$= 240$

A1

Answer: 240

5 marks

19 (5 marks)

(a) $2^4 + 4(2^3)(5x) + 6(2^2)(5x)^2 + 4(2)(5x)^3 + (5x)^4$

M1 A1

$= 16 + 160x + 600x^2 + 1000x^3 + 625x^4$

A1

Answer: $16 + 160x + 600x^2 + 1000x^3 + 625x^4$

3 marks

(b) $2 + 5x = 102$ when $x = 20$

M1

$16 + 3200 + 240\,000 + 8\,000\,000 + 100\,000\,000 = 108\,243\,216$

A1

Answer: 108 243 216

2 marks

20 (6 marks)

(a) $3^4 + 4(3^3)x + 6(3^2)x^2 + 4(3)x^3 + x^4$

M1

$= 81 + 108x + 54x^2 + 12x^3 + x^4$

A1

Answer: $81 + 108x + 54x^2 + 12x^3 + x^4$

2 marks

(b) Replace x by $-4x^2$: $81 + 108(-4x^2) + 54(-4x^2)^2 + 12(-4x^2)^3 + (-4x^2)^4$

M1

$= 81 - 432x^2 + 864x^4 - 768x^6 + 256x^8$

A1

Answer: $81 - 432x^2 + 864x^4 - 768x^6 + 256x^8$

2 marks

(c) $3 - 4(0.05)^2 = 2.99$, so $x = 0.05$

M1

$81 - 1.08 + 0.0054 = 79.9254$

A1

Answer: 79.9254

2 marks

21 (8 marks)

(a) ${}^8C_1 = 8$, ${}^8C_2 = 28$, ${}^8C_3 = 56$

M1

$1 + 8x + 28x^2 + 56x^3 + \dots$

A1

Answer: $1 + 8x + 28x^2 + 56x^3 + \dots$

2 marks

(b) $1 + 0.16 + 28(0.0004) + 56(0.000008) = 1 + 0.16 + 0.0112 + 0.000448$

M1

$= 1.171648$

A1

Error = $1.1716594 - 1.171648 = 0.0000114$, percentage error = 0.00097%

A1

Answer: 1.171648, error 0.00097%

3 marks

(c) $x = -0.02$: $1 - 0.16 + 0.0112 - 0.000448$

M1

$= 0.850752$

A1

Answer: 0.850752

2 marks

(d) All the terms for $x = 0.02$ are positive, so each extra term increases the total

E1

Answer: All terms are positive

1 mark

22 (8 marks)

(a) General term: ${}^8C_r x^{8-r} \left(\frac{2}{x}\right)^r = {}^8C_r 2^r x^{8-2r}$

M1 A1

Independent of x when $8 - 2r = 0$, so $r = 4$

M1

${}^8C_4 \times 2^4 = 70 \times 16 = 1120$

A1

Answer: 1120

4 marks

(b) $8 - 2r = 4$, so $r = 2$

M1

${}^8C_2 \times 2^2 = 28 \times 4 = 112$

A1

Answer: 112

2 marks

(c) Put $x = 1$: $(1 + 2)^8$

M1

$= 3^8 = 6561$

A1

Answer: 6561

2 marks

23 (7 marks)

(a) $(1 + x)^n = {}^nC_0 + {}^nC_1x + \dots + {}^nC_nx^n$

M1

Put $x = 1$: $2^n = \text{sum of the coefficients}$

A1

Answer: shown

2 marks

(b) Put $x = -1$: $(1 - 1)^n = 0$

M1

so the alternating sum is 0

A1

Answer: 0

2 marks

(c) Even sum $-$ odd sum $= 0$ and even $+ \text{odd} = 2^n$

M1

so even sum $= 2^{n-1}$

A1

$n = 8$: $2^7 = 128$

A1

Answer: 128

3 marks

24 (7 marks)

(a) $T = {}^7C_r(2x)^{7-r}\left(-\frac{3}{x}\right)^r$

M1

$x^{7-r} \times x^{-r} = x^{7-2r}$, giving the stated term

A1

Answer: shown

2 marks

(b) x : $7 - 2r = 1$ so $r = 3$

M1

${}^7C_3 \times 2^4 \times (-27) = 35 \times 16 \times (-27) = -15\,120$

A1

$\frac{1}{x}$: $7 - 2r = -1$ so $r = 4$

M1

${}^7C_4 \times 2^3 \times 81 = 35 \times 8 \times 81 = 22\,680$

A1

Answer: $-15\,120$ and $22\,680$

4 marks

(c) $7 - 2r = 0$ has no integer solution

E1

Answer: $r = 3.5$ is not an integer

1 mark

25 (7 marks)

(a) $(1 + \sqrt{3})^5 = 1 + 5\sqrt{3} + 10(\sqrt{3})^2 + 10(\sqrt{3})^3 + 5(\sqrt{3})^4 + (\sqrt{3})^5$

M1 A1

$= 1 + 5\sqrt{3} + 30 + 30\sqrt{3} + 45 + 9\sqrt{3}$

M1

$= 76 + 44\sqrt{3}$

A1

Answer: shown

4 marks

(b) Change sign of every odd power of $\sqrt{3}$: $76 - 44\sqrt{3}$

B1

Answer: $76 - 44\sqrt{3}$

1 mark

(c) $(1 + \sqrt{3})^5(1 - \sqrt{3})^5 = (1 - 3)^5 = -32$

M1

$(76 + 44\sqrt{3})(76 - 44\sqrt{3}) = 76^2 - 3 \times 44^2$, so this equals -32

A1

Answer: shown

2 marks

26 (7 marks)

(a) $(1 + ax)^5 = 1 + 5ax + 10a^2x^2 + 10a^3x^3 + \dots$

B1

$(2 - x)^4 = 16 - 32x + 24x^2 - 8x^3 + \dots$

B1

Coefficient of x : $16 \times 5a + (-32) = 0$

M1

$a = \frac{2}{5}$

A1

Answer: $a = \frac{2}{5}$

4 marks

(b) x^2 : $16 \times 10a^2 + 5a \times (-32) + 24$

M1

$= 160 \times \frac{4}{25} - 64 + 24$

A1

$= -\frac{72}{5}$

A1

Answer: $-\frac{72}{5}$

3 marks

27 (10 marks)

(a) Terms: $2^6 + 6(2^5)\sqrt{3} + 15(2^4)(3) + 20(2^3)(3\sqrt{3}) + 15(2^2)(9) + 6(2)(9\sqrt{3}) + 27$

M1 A1

Rational terms: $64 + 720 + 540 + 27 = 1351$

M1

Surd terms: $(192 + 480 + 108)\sqrt{3} = 780\sqrt{3}$

A1

Answer: Proof

4 marks

(b) Signs of the odd powers of $\sqrt{3}$ change: $1351 - 780\sqrt{3}$

B1

Answer: $1351 - 780\sqrt{3}$

1 mark

(c) Add the results of (a) and (b)

M1

$1351 + 1351 + (780 - 780)\sqrt{3} = 2702$

A1

Answer: 2702

2 marks

(d) $(2 + \sqrt{3})^6 = 2702 - (2 - \sqrt{3})^6$, and $(2 - \sqrt{3})^6 > 0$, so $(2 + \sqrt{3})^6 < 2702$

M1

$(2 - \sqrt{3})^6 < 0.3^6 = 0.000729$

M1

$0.000729 < 0.001$, so $2702 - 0.001 < (2 + \sqrt{3})^6 < 2702$

A1

Answer: Proof

3 marks

28 (10 marks)

(a) Coefficient of x : $na = 12$; coefficient of x^2 : $\frac{n(n-1)}{2}a^2 = 60$

M1 A1

$\frac{n(n-1)}{2}a^2 = \frac{n-1}{2n}(na)^2 = \frac{n-1}{2n} \times 144 = 60$

M1

$72(n-1) = 60n$, so $12n = 72$, $n = 6$

A1

$a = \frac{12}{6} = 2$

A1

Answer: $n = 6$, $a = 2$

5 marks

(b) ${}^6C_3 \times 2^3 = 20 \times 8 = 160$

B1

Answer: 160

1 mark

(c) ${}^{n-1}C_r + {}^{n-1}C_{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!}$

M1

Common denominator $r!(n-r)!$: $\frac{(n-1)!(n-r)}{r!(n-r)!} + \frac{(n-1)! \times r}{r!(n-r)!}$

M1 A1

$= \frac{(n-1)!(n-r+r)}{r!(n-r)!} = \frac{n!}{r!(n-r)!} = {}^nC_r$

A1

Answer: Proof

4 marks

29 (8 marks)

(a) Choosing r items to take is the same as choosing $n-r$ items to leave: $\frac{n!}{r!(n-r)!}$ is unchanged

E1

when r is replaced by $n-r$

Answer: symmetry

1 mark

(b) Coefficient of x^n in $(1+x)^{2n}$ is ${}^{2n}C_n$

B1

In the product, x^n arises from ${}^nC_r x^r \times {}^nC_{n-r} x^{n-r}$ for $r = 0, \dots, n$

M1 A1

Coefficient $= \sum_{r=0}^n {}^nC_r {}^nC_{n-r}$

A1

$= \sum_{r=0}^n ({}^nC_r)^2$ by part (a), so this equals ${}^{2n}C_n$

A1

Answer: shown

5 marks

(c) $1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70$

M1

${}^8C_4 = 70$

A1

Answer: $70 = 70$

2 marks

30 (8 marks)Let the coefficients be ${}^nC_r$, ${}^nC_{r+1}$, ${}^nC_{r+2}$ **B1**

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1} = 2$$

M1 A1

$$n - r = 2r + 2, \text{ so } n = 3r + 2$$

A1

$$\frac{{}^nC_{r+2}}{{}^nC_{r+1}} = \frac{n-r-1}{r+2} = \frac{3}{2}$$

M1 A1

$$2n - 2r - 2 = 3r + 6, \text{ so } 2n = 5r + 8$$

A1Solve: $6r + 4 = 5r + 8$, $r = 4$, $n = 14$; coefficients 1001, 2002, 3003**A1****Answer: $n = 14$; 1001, 2002, 3003****8 marks****Binomial Expansion (Fractional and Negative Powers)****1 (2 marks)**

$$(1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \dots$$

M1

$$= 1 - 2x + 3x^2 + \dots$$

A1**Answer: $1 - 2x + 3x^2$** **2 marks****2 (3 marks)**

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \dots$$

M1

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

A1Valid for $|x| < 1$ **B1****Answer: $1 + \frac{1}{2}x - \frac{1}{8}x^2$; $|x| < 1$** **3 marks****3 (2 marks)**Valid when $|4x| < 1$ **M1**

$$-\frac{1}{4} < x < \frac{1}{4}$$

A1**Answer: $|x| < \frac{1}{4}$** **2 marks****4 (2 marks)**

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2$$

M1

$$= 1 + x + x^2$$

A1**Answer: $1 + x + x^2$** **2 marks****5 (2 marks)**

$$\frac{(-3)(-4)}{2!}x^2$$

M1

coefficient = 6

A1**Answer: 6****2 marks****6 (3 marks)**

$$1 + \frac{1}{2}(2x)$$

M1

$$+ \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(2x)^2$$

M1

$$= 1 + x - \frac{1}{2}x^2$$

A1**Answer: $1 + x - \frac{1}{2}x^2$** **3 marks**

7 (4 marks)

$$(1 - 3x)^{-2} = 1 + (-2)(-3x) + \frac{(-2)(-3)}{2!}(-3x)^2 + \dots$$

M1

$$= 1 + 6x$$

A1

$$+ 3 \times 9x^2 = + 27x^2$$

A1

$$\text{Valid when } |3x| < 1, \text{ i.e. } |x| < \frac{1}{3}$$

B1

$$\text{Answer: } 1 + 6x + 27x^2, |x| < \frac{1}{3}$$

4 marks

8 (5 marks)

$$\sqrt{4+x} = 4^{1/2} \left(1 + \frac{x}{4}\right)^{1/2} = 2 \left(1 + \frac{x}{4}\right)^{1/2}$$

M1

$$\left(1 + \frac{x}{4}\right)^{1/2} = 1 + \frac{1}{2} \left(\frac{x}{4}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2} \left(\frac{x}{4}\right)^2 + \dots$$

M1

$$= 1 + \frac{x}{8} - \frac{x^2}{128} + \dots$$

A1

$$\sqrt{4+x} = 2 + \frac{x}{4} - \frac{x^2}{64} + \dots$$

A1

$$\text{Valid for } \left|\frac{x}{4}\right| < 1, \text{ i.e. } |x| < 4$$

B1

$$\text{Answer: } 2 + \frac{x}{4} - \frac{x^2}{64}, |x| < 4$$

5 marks

9 (4 marks)

$$(-3)\left(-\frac{1}{2}x\right) = \frac{3}{2}x$$

M1 A1

$$\frac{(-3)(-4)}{2} \left(-\frac{1}{2}x\right)^2 = 6 \times \frac{1}{4}x^2 = \frac{3}{2}x^2$$

A1

$$\text{Valid for } |x/2| < 1, \text{ i.e. } |x| < 2$$

B1

$$\text{Answer: } 1 + \frac{3}{2}x + \frac{3}{2}x^2; |x| < 2$$

4 marks

10 (4 marks)

$$\frac{1}{\sqrt{1+2x}} = (1+2x)^{-1/2}$$

B1

$$1 + \left(-\frac{1}{2}\right)(2x) = 1 - x$$

M1 A1

$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})}{2} (2x)^2 = \frac{3}{2}x^2$$

A1

$$\text{Answer: } 1 - x + \frac{3}{2}x^2$$

4 marks

11 (4 marks)

$$1 + \frac{1}{3}(3x) = 1 + x$$

M1 A1

$$\frac{(\frac{1}{3})(-\frac{2}{3})}{2} (3x)^2 = -\frac{1}{9} \times 9x^2 = -x^2$$

A1

$$\text{Valid for } |3x| < 1, \text{ i.e. } |x| < \frac{1}{3}$$

B1

$$\text{Answer: } 1 + x - x^2; |x| < \frac{1}{3}$$

4 marks

12 (4 marks)

$$\begin{aligned}\sqrt{9 - 4x} &= 3\left(1 - \frac{4x}{9}\right)^{1/2} & \text{B1} \\ &= 3\left[1 + \frac{1}{2}\left(-\frac{4x}{9}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(-\frac{4x}{9}\right)^2\right] & \text{M1 A1} \\ &= 3 - \frac{2}{3}x - \frac{2}{27}x^2 & \text{A1} \\ \text{Answer: } &3 - \frac{2}{3}x - \frac{2}{27}x^2 & \text{4 marks}\end{aligned}$$

13 (3 marks)

$$\begin{aligned}(1 - 2x)^{-2}: \text{ term in } x^4 \text{ is } &\frac{(-2)(-3)(-4)(-5)}{4!}(-2x)^4 & \text{M1 A1} \\ &= 5 \times 16 = 80 & \text{A1} \\ \text{Answer: } &80 & \text{3 marks}\end{aligned}$$

14 (6 marks)

$$\begin{aligned}(\text{a}) (1 + x)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \dots & \text{M1 A1} \\ &= 1 - \frac{x}{2} + \frac{3x^2}{8} & \text{A1} \\ \text{Answer: } &1 - \frac{x}{2} + \frac{3x^2}{8} & \text{3 marks} \\ (\text{b}) &1 - 0.01 + \frac{3}{8}(0.0004) & \text{M1} \\ &= 0.99015 & \text{A1} \\ \text{Answer: } &0.99015 & \text{2 marks} \\ (\text{c}) &\text{The expansion is only valid for } |x| < 1, \text{ and the diagram shows the curve and the quadratic are far apart at } x = 2 & \text{E1} \\ \text{Answer: } &\text{Outside } |x| < 1 \text{ the approximation is poor} & \text{1 mark}\end{aligned}$$

15 (6 marks)

$$\begin{aligned}(\text{a}) &\frac{1}{(2 + x)^2} = \frac{1}{4}\left(1 + \frac{x}{2}\right)^{-2} & \text{M1} \\ \left(1 + \frac{x}{2}\right)^{-2} &= 1 - 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + \dots = 1 - x + \frac{3x^2}{4} + \dots & \text{M1 A1} \\ \text{Hence } &\frac{1}{4} - \frac{x}{4} + \frac{3x^2}{16} + \dots, \text{ valid for } |x| < 2 & \text{A1} \\ \text{Answer: } &\frac{1}{4} - \frac{x}{4} + \frac{3x^2}{16}, |x| < 2 & \text{4 marks} \\ (\text{b}) \text{ Term in } (1 + u)^{-2}: &\frac{(-2)(-3)(-4)}{3!}u^3 = -4u^3 \text{ with } u = \frac{x}{2}, \text{ giving } -\frac{x^3}{2} & \text{M1} \\ \text{Multiply by } &\frac{1}{4}: \text{ coefficient} = -\frac{1}{8} & \text{A1} \\ \text{Answer: } &-\frac{1}{8} & \text{2 marks}\end{aligned}$$

16 (6 marks)

$$\begin{aligned}(\text{a}) (1 - 2x)^{-1} &= 1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2 + \dots & \text{M1} \\ &= 1 + 2x + 4x^2 + \dots & \text{A1} \\ \text{Answer: } &1 + 2x + 4x^2 & \text{2 marks} \\ (\text{b}) (2 + 3x)^2 &= 4 + 12x + 9x^2 & \text{B1} \\ (4 + 12x + 9x^2)(1 + 2x + 4x^2): &x: 8 + 12 = 20; x^2: 16 + 24 + 9 = 49 & \text{M1 A1} \\ \text{Answer: } &4 + 20x + 49x^2 & \text{3 marks} \\ (\text{c}) &|2x| < 1 & \text{B1} \\ \text{Answer: } &|x| < \frac{1}{2} & \text{1 mark}\end{aligned}$$

17 (5 marks)

$$(a) (1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}x^2 + \dots$$

M1

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \dots$$

A1

$$\text{Answer: } 1 + \frac{1}{3}x - \frac{1}{9}x^2$$

2 marks

$$(b) \sqrt[3]{65} = \sqrt[3]{64(1 + \frac{1}{64})} = 4(1 + \frac{1}{64})^{1/3}$$

B1

$$4(1 + \frac{1}{192} - \frac{1}{36864})$$

M1

$$= 4.0207$$

A1

$$\text{Answer: } 4.0207$$

3 marks

18 (6 marks)

$$(a) \frac{(-2)(-3)}{2!}(ax)^2 = 3a^2x^2$$

M1

$$3a^2 = 12$$

M1

$$a = 2 \text{ or } a = -2$$

A1

$$\text{Answer: } a = \pm 2$$

3 marks

$$(b) \frac{(-2)(-3)(-4)}{3!}(2x)^3 = -4 \times 8x^3 = -32x^3$$

M1 A1

$$\text{Valid for } |2x| < 1, \text{ so } |x| < \frac{1}{2}$$

B1

$$\text{Answer: } -32; |x| < \frac{1}{2}$$

3 marks

19 (7 marks)

$$(a) \frac{1}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-2x}, 1 = A(1-2x) + B(1+x)$$

M1

$$x = -1: A = \frac{1}{3}; x = \frac{1}{2}: B = \frac{2}{3}$$

M1

$$f(x) = \frac{1}{3(1+x)} + \frac{2}{3(1-2x)}$$

A1

$$\text{Answer: } \frac{1}{3(1+x)} + \frac{2}{3(1-2x)}$$

3 marks

$$(b) \frac{1}{3}(1-x+x^2) + \frac{2}{3}(1+2x+4x^2)$$

M1 A1

$$= 1 + x + 3x^2$$

A1

$$\text{Answer: } 1 + x + 3x^2$$

3 marks

$$(c) \text{ Need } |x| < 1 \text{ and } |2x| < 1, \text{ so } |x| < \frac{1}{2}$$

B1

$$\text{Answer: } |x| < \frac{1}{2}$$

1 mark

20 (5 marks)

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{1/2}(1-x)^{-1/2}$$

B1

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2; (1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

M1 A1

$$\text{Product: } 1 + (\frac{1}{2} + \frac{1}{2})x + (\frac{3}{8} - \frac{1}{8} + \frac{1}{4})x^2$$

M1

$$= 1 + x + \frac{1}{2}x^2$$

A1

$$\text{Answer: shown}$$

5 marks

21 (8 marks)

$$(a) (1 - 2x)^{-1} = 1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2 + \frac{(-1)(-2)(-3)}{3!}(-2x)^3 \quad \text{M1}$$

$$= 1 + 2x + 4x^2 + 8x^3 \quad \text{A1}$$

$$\text{Answer: } 1 + 2x + 4x^2 + 8x^3 \quad \text{3 marks}$$

$$(b) \frac{d}{dx}(1 - 2x)^{-1} = 2(1 - 2x)^{-2} \quad \text{M1}$$

$$\frac{d}{dx}(1 + 2x + 4x^2 + 8x^3) = 2 + 8x + 24x^2 + \dots \quad \text{M1}$$

$$\text{Dividing by 2: } (1 - 2x)^{-2} = 1 + 4x + 12x^2 + \dots \quad \text{A1}$$

$$\text{Answer: shown} \quad \text{3 marks}$$

$$(c) (3 - x)(1 + 4x + 12x^2 + \dots) \quad \text{M1}$$

$$x: 12 - 1 = 11; x^2: 36 - 4 = 32 \quad \text{A1}$$

$$3 + 11x + 32x^2; \text{ valid for } |x| < \frac{1}{2} \quad \text{A1}$$

$$\text{Answer: } 3 + 11x + 32x^2; |x| < \frac{1}{2} \quad \text{3 marks}$$

22 (8 marks)

$$(a) \frac{5+x}{(1-x)(2+x)} = \frac{A}{1-x} + \frac{B}{2+x}, \text{ so } 5+x = A(2+x) + B(1-x) \quad \text{M1}$$

$$x = 1: 6 = 3A, \text{ so } A = 2; x = -2: 3 = 3B, \text{ so } B = 1 \quad \text{M1}$$

$$f(x) = \frac{2}{1-x} + \frac{1}{2+x} \quad \text{A1}$$

$$\text{Answer: } \frac{2}{1-x} + \frac{1}{2+x} \quad \text{3 marks}$$

$$(b) 2(1-x)^{-1} = 2(1+x+x^2+\dots) = 2+2x+2x^2 \quad \text{M1 A1}$$

$$(2+x)^{-1} = \frac{1}{2}(1+\frac{x}{2})^{-1} = \frac{1}{2}(1-\frac{x}{2}+\frac{x^2}{4}) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{5}{2} + \frac{7x}{4} + \frac{17x^2}{8} \quad \text{4 marks}$$

$$(c) |x| < 1 \text{ for } (1-x)^{-1} \text{ and } |x| < 2 \text{ for } (2+x)^{-1}, \text{ so both hold when } |x| < 1 \quad \text{B1}$$

$$\text{Answer: } |x| < 1 \quad \text{1 mark}$$

23 (7 marks)

$$(a) (1+4x)^{-1/2} = 1 + (-\frac{1}{2})(4x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(4x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(4x)^3 \quad \text{M1 A1}$$

$$= 1 - 2x + \frac{3}{8} \times 16x^2 - \frac{5}{16} \times 64x^3 = 1 - 2x + 6x^2 - 20x^3 \quad \text{A1}$$

$$\text{Answer: shown} \quad \text{3 marks}$$

$$(b) (1+ax)(1+bx^2) = 1+ax+bx^2+abx^3 \quad \text{B1}$$

$$x: a+2=0, \text{ so } a=-2; x^2: b-6=0, \text{ so } b=6 \quad \text{M1 A1}$$

$$x^3: ab+20=-12+20=8 \quad \text{A1}$$

$$\text{Answer: } a = -2, b = 6, \text{ coefficient } 8 \quad \text{4 marks}$$

24 (7 marks)

$$\text{Coefficient of } x: qp = -4 \quad \text{M1 A1}$$

$$\text{Coefficient of } x^2: \frac{q(q-1)}{2}p^2 = 24 \quad \text{M1}$$

$$\text{Substitute } p = -\frac{4}{q}: \frac{q(q-1)}{2} \times \frac{16}{q^2} = 24, \text{ so } \frac{8(q-1)}{q} = 24 \quad \text{A1}$$

$$q-1 = 3q \text{ so } q = -\frac{1}{2} \quad \text{A1}$$

$$p = 8 \quad \text{A1}$$

$$\text{Valid for } |8x| < 1, \text{ i.e. } |x| < \frac{1}{8} \quad \text{B1}$$

$$\text{Answer: } p = 8, q = -\frac{1}{2}; |x| < \frac{1}{8} \quad \text{7 marks}$$

25 (7 marks)

$$(a) \left(1 + \frac{1}{49}\right)^{1/2} = \sqrt{\frac{50}{49}} = \frac{5\sqrt{2}}{7}$$

M1

$$\text{so } \frac{7}{5}\left(1 + \frac{1}{49}\right)^{1/2} = \sqrt{2}$$

A1

Answer: shown

2 marks

$$(b) (1 + x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 \text{ with } x = \frac{1}{49}$$

M1

$$\frac{7}{5}\left(1 + \frac{1}{98} - \frac{1}{19208}\right) = 1.4 \times 1.010152$$

M1

$$= 1.414213$$

A1

Answer: 1.414213

3 marks

(c) $x = 1$ is on the boundary of the validity range and the terms do not decrease quickly: $1 + 0.5 - 0.125 = 1.375$

M1

$x = \frac{1}{49}$ is tiny so the terms decrease very rapidly

E1

Answer: x is small

2 marks

26 (7 marks)

$$(a) (1 - 4x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-4x)^2$$

M1

$$= 1 + 2x + \frac{3}{8} \times 16x^2 = 1 + 2x + 6x^2$$

A1

Answer: shown

2 marks

(b) The expansion needs $|4x| < 1$, i.e. $|x| < \frac{1}{4}$, not $|x| < 1$

E1

$x = 0.3$ is outside this range (and $(1 - 1.2) < 0$ has no real square root), so the series is meaningless

E1

Answer: range is $|x| < \frac{1}{4}$; $x = 0.3$ invalid

2 marks

(c) $1 - 4x = 0.96$ gives $x = 0.01$

M1

$$1 + 2(0.01) + 6(0.0001) = 1.0206$$

A1

Calculator value 1.0206: agrees

A1

Answer: 1.0206

3 marks

27 (10 marks)

$$(a) \frac{12}{(1 - 2x)(1 + x)^2} = \frac{A}{1 - 2x} + \frac{B}{1 + x} + \frac{C}{(1 + x)^2}, \text{ so } 12 = A(1 + x)^2 + B(1 - 2x)(1 + x) + C(1 - 2x)$$

M1

$$x = \frac{1}{2}: 12 = \frac{9}{4}A, \text{ so } A = \frac{16}{3}$$

M1 A1

$$x = -1: 12 = 3C, \text{ so } C = 4$$

A1

$$x = 0: 12 = A + B + C, \text{ so } B = \frac{8}{3}$$

A1

$$\text{Answer: } \frac{16}{3}/(1 - 2x) + \frac{8}{3}/(1 + x) + 4/(1 + x)^2$$

5 marks

$$(b) (1 - 2x)^{-1} = 1 + 2x + 4x^2, (1 + x)^{-1} = 1 - x + x^2, (1 + x)^{-2} = 1 - 2x + 3x^2$$

B1 B1

$$f(x) \approx \frac{16}{3}(1 + 2x + 4x^2) + \frac{8}{3}(1 - x + x^2) + 4(1 - 2x + 3x^2)$$

M1

$$\text{constant: } \frac{16}{3} + \frac{8}{3} + 4 = 12; x: \frac{32}{3} - \frac{8}{3} - 8 = 0; x^2: \frac{64}{3} + \frac{8}{3} + 12 = 36$$

A1

Answer: $12 + 36x^2$

4 marks

(c) $(1 - 2x)^{-1}$ needs $|2x| < 1$; $(1 + x)^{-1}$ and $(1 + x)^{-2}$ need $|x| < 1$; both hold only for $|x| < \frac{1}{2}$

E1

Answer: the most restrictive condition is $|2x| < 1$

1 mark

28 (11 marks)

(a) $na = -6$ and $\frac{n(n-1)}{2}a^2 = 27$

M1 A1

$$\frac{n(n-1)}{2}a^2 = \frac{n-1}{2n}(na)^2 = 18 \times \frac{n-1}{n} = 27$$

M1

$$\frac{n-1}{n} = \frac{3}{2}, \text{ so } 2n - 2 = 3n, n = -2$$

A1

$$a = \frac{-6}{-2} = 3$$

A1

Answer: $n = -2, a = 3$

5 marks

(b) $|3x| < 1$, so $|x| < \frac{1}{3}$

B1

Answer: $|x| < \frac{1}{3}$

1 mark

(c) $\frac{1}{1-x} = 1 + x + x^2 + \dots$

B1

$$(1 - 6x + 27x^2)(1 + x + x^2): x: 1 - 6 = -5$$

M1

$$x^2: 1 - 6 + 27 = 22, \text{ so } 1 - 5x + 22x^2$$

A1

Answer: $1 - 5x + 22x^2$

3 marks

(d) $1 - 0.05 + 22(0.0001)$

M1

$$= 0.9522$$

A1

Answer: 0.9522

2 marks

29 (8 marks)

(a) $(1 - 4x)^{-1/2} = 1 + (-\frac{1}{2})(-4x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-4x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{6}(-4x)^3$

M1

$$= 1 + 2x + \frac{3}{8} \times 16x^2 + \frac{5}{16} \times 64x^3$$

A1

$$= 1 + 2x + 6x^2 + 20x^3$$

A1

Answer: shown

3 marks

(b) Coefficient = $\frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-1}{2})}{n!}(-4)^n$

M1

$$= \frac{1 \times 3 \times \dots \times (2n-1)}{2^n n!} \times 4^n$$

M1

$$= \frac{2^n \times 1 \times 3 \times \dots \times (2n-1)}{n!}$$

A1

Answer: shown

3 marks

(c) Coefficient = $\frac{2^n}{n!} \times \frac{(2n)!}{2^n n!}$

M1

$$= \frac{(2n)!}{(n!)^2} = {}^{2n}C_n$$

A1

Answer: ${}^{2n}C_n$

2 marks

30 (11 marks)

(a) $y = (1 - x)^{-1/2}$

B1

$$= 1 + (-\frac{1}{2})(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-x)^2$$

M1

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

A1

Answer: shown

3 marks

(b) $E \approx mc^2(\frac{1}{2}x + \frac{3}{8}x^2)$

M1

$$= \frac{1}{2}mc^2x(1 + \frac{3}{4}x)$$

A1

$$= \frac{1}{2}mv^2(1 + \frac{3v^2}{4c^2}) \text{ since } c^2x = v^2$$

A1

Answer: shown**3 marks****(c)** Exact: $(\gamma - 1)mc^2$ with $\gamma = \frac{1}{\sqrt{0.99}} = 1.005038$, so $E = 0.005038mc^2$ **M1 A1**Classical: $\frac{1}{2}(0.01)mc^2 = 0.005mc^2$ **B1**Percentage = $\frac{0.005038 - 0.005}{0.005038} \times 100 = 0.75\%$ **M1****Answer: 0.75%****4 marks****(d)** Corrected: $(0.005 + 0.0000375)mc^2 = 0.0050375mc^2$; error $\frac{0.0050378 - 0.0050375}{0.005038} \times 100 = 0.0063\%$ **B1****Answer: 0.0063%****1 mark**