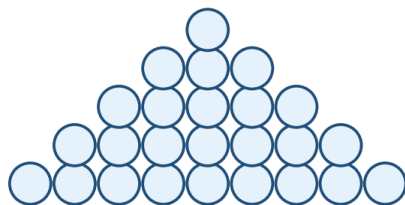


Arithmetic Sequences and Series

- 1 An arithmetic sequence has first term 7 and common difference 4.
Find the 20th term of the sequence. [2]
- 2 Find the sum of the first 15 terms of the arithmetic series
 $3 + 8 + 13 + \dots$ [2]
- 3 The first three terms of an arithmetic sequence are 4, 9, 14.
Write down the next two terms and a formula for the n th term. [2]
- 4 Find the 25th term of the arithmetic sequence 100, 94, 88, ... [2]
- 5 An arithmetic sequence has first term 3 and tenth term 48. Find the common difference. [2]
- 6 Find the sum of the first 12 terms of the arithmetic series $40 + 37 + 34 + \dots$ [2]
- 7 The 5th term of an arithmetic sequence is 23 and the 12th term is 58.
(a) Find the first term and the common difference. [3]
(b) Find the sum of the first 30 terms. [2]
- 8 The series $4 + 7 + 10 + \dots + 100$ is arithmetic.
(a) Find the number of terms in the series. [2]
(b) Find the sum of the series. [2]
- 9 The n th term of an arithmetic sequence is $u_n = 83 - 6n$.
(a) Find the number of positive terms in the sequence. [3]
(b) Find the sum of the positive terms. [2]
- 10 Find the sum of all the odd numbers from 11 to 99 inclusive. [3]
- 11 The first term of an arithmetic series is 2 and the common difference is 3.
The sum of the first n terms is 950. Find the value of n . [4]
- 12 The first three terms of an arithmetic sequence are $k + 3$, $2k + 4$ and $5k - 3$.
Find the value of k and the common difference. [4]
- 13 The n th term of a sequence is $u_n = 5n - 2$.
Show that the sequence is arithmetic and find the sum of the first 20 terms. [4]
- 14 Logs are stacked in rows. The diagram shows stack A, which has 5 rows. The bottom row has 9 logs and each row above it has 2 fewer logs than the row below, so the top row has 1 log.



Stack A

- (a) Show that stack A contains 25 logs. [2]
- (b) Stack B is built in the same way, but its bottom row has 15 logs and its top row has 1 log. Find the total number of logs in stack B. [3]
- (c) Stack C is built in the same way, but its top row has 3 logs and it contains 143 logs in total. Find the number of rows in stack C. [3]

- 15** Priya starts a job with an annual salary of £24 000. Her salary increases by £1250 at the start of each subsequent year, so that her salaries form an arithmetic sequence.
- Find her salary in her 12th year. [2]
 - Find the total she earns in her first 12 years. [2]
 - In which year does her salary first exceed £50 000? [3]
-
- 16** The first three terms of an arithmetic sequence are $(k - 3)$, $(2k - 2)$ and $(4k - 8)$, where k is a constant.
- Show that $k = 7$. [3]
 - Find the 30th term of the sequence. [2]
 - Show that the sum of the first n terms is a perfect square for every positive integer n . [2]
-
- 17** A non-regular heptagon (a polygon with 7 sides) has side lengths, in cm, that form an arithmetic sequence with common difference d . The longest side is three times as long as the shortest side and the perimeter of the heptagon is 210 cm.
- Find the length of the shortest side. [4]
 - Find the value of d . [2]
-
- 18** Three numbers are in arithmetic progression. Their sum is 36 and their product is 1428. Find the three numbers, given that they are increasing. [5]
-
- 19** The arithmetic series $8 + 14 + 20 + \dots + 902$ has 150 terms. Find the sum of the last 25 terms of the series. [5]
-
- 20** The sum of the first 5 terms of an arithmetic series is 55 and the sum of the first 8 terms is 124. Find the first term and the common difference. [5]
-
- 21** The first term of an arithmetic series is a and the common difference is d .
- Prove that the sum of the first n terms is $S_n = \frac{n}{2}(2a + (n - 1)d)$. [4]
 - Given that $S_8 = 4S_4$, show that $d = 2a$. [3]
 - Given also that the 6th term is 22, find the sum of the first 10 terms. [2]
-
- 22** Sequence A is arithmetic with first term 5 and common difference 4. Sequence B is arithmetic with first term 103 and common difference -3 .
- Find the value of n for which the n th term of A equals the n th term of B, and state this common value. [3]
 - Find the smallest value of n for which the sum of the first n terms of A is greater than the sum of the first n terms of B. [5]
 - A student gives the answer to (b) as $n = 29$. Explain why this is wrong. [1]
-
- 23** The 8th term of an arithmetic series is 22 and the 20th term is -38 .
- Find the first term and the common difference of the series. [3]
 - Find the number of positive terms in the series. [3]
 - Find the greatest value of S_n , the sum of the first n terms of the series, giving a reason for your answer. [3]
-
- 24** Dev starts a job on an annual salary of £21 000. Each year his salary rises by £1500 until it reaches £33 000, after which it stays at £33 000. The year in which Dev's salary first reaches £33 000 is year N .
- Find the value of N . [2]
 - Find Dev's total earnings in his first 15 years. [4]
 - Elena starts at the same time on £24 000 with rises of £1000 a year (she is not yet at her maximum). In which year do Dev and Elena first have the same salary, and what is it? [3]

- 25** A student is asked to find the sum of the 10th to the 20th terms inclusive of the arithmetic series $3 + 7 + 11 + \dots$
Her working is: $S_{20} - S_{10} = 820 - 210 = 610$.
Explain the error and find the correct answer. [5]
- 26** The sum of the first n terms of a sequence is $S_n = 3n^2 + 2n$.
(a) Find the first term u_1 and the second term u_2 . [3]
(b) Use $u_n = S_n - S_{n-1}$ to show that $u_n = 6n - 1$. [2]
(c) Hence show that the sequence is arithmetic and state its common difference. [2]
- 27** An arithmetic sequence has first term a and common difference d . The p th term of the sequence is q and the q th term is p , where p and q are different positive integers.
(a) Show that $d = -1$ and find an expression for a in terms of p and q . [5]
(b) Hence show that the $(p + q)$ th term is zero. [2]
(c) Find, in terms of p and q , the sum of the first $p + q$ terms. [3]
- 28** In this question, all integers considered are between 1 and 500 inclusive.
(a) Find the sum of all the multiples of 3. [3]
(b) Find the sum of all the multiples of 5. [2]
(c) Find the sum of all the integers that are divisible by 3 or by 5 (or both). [4]
(d) Hence find the sum of the integers that are divisible by neither 3 nor 5. [2]
- 29** A positive integer can sometimes be written as the sum of two or more consecutive positive integers, e.g. $9 = 4 + 5 = 2 + 3 + 4$.
(a) Show that if 100 is the sum of n consecutive positive integers beginning with a , then $n(2a + n - 1) = 200$. [2]
(b) Explain why n and $2a + n - 1$ must have opposite parity (one even, one odd). [2]
(c) Find all the ways of writing 100 as a sum of two or more consecutive positive integers. [6]
- 30** Find the sum of all the three-digit positive integers which leave remainder 2 when divided by 7 and remainder 3 when divided by 4. [7]

Geometric Sequences and Series

- 1** A geometric sequence has first term 3 and common ratio 2.
Find the 8th term. [2]
- 2** Find the sum of the first 6 terms of the geometric series with first term 5 and common ratio 3. [2]
- 3** Find the 6th term of the geometric sequence 160, -80 , 40, ... [2]
- 4** Find the sum of the first 8 terms of the geometric series $2 + 4 + 8 + \dots$ [2]
- 5** Show that 1458 is a term of the geometric sequence 2, 6, 18, ... and state which term it is. [2]
- 6** The second and third terms of a geometric sequence are 12 and 18.
Find the common ratio and the fifth term. [3]
- 7** The 2nd term of a geometric sequence is 12 and the 5th term is 324.
(a) Find the common ratio and the first term. [3]
(b) Find the sum of the first 8 terms. [2]
- 8** The geometric series $2 + 6 + 18 + \dots$ has first term 2 and common ratio 3.
Find the smallest number of terms for which the sum of the series exceeds 10 000. [4]

- 9** The geometric series $3 + 6 + 12 + \dots + 768$ has a last term of 768.
Find the number of terms and the sum of the series. [4]
-
- 10** The masses, in tonnes, of the five stages of a rocket form a geometric sequence with a positive common ratio. The first stage has mass 256 tonnes and the fifth stage has mass 81 tonnes.
Find the mass of the third stage. [3]
-
- 11** The k th term of a geometric sequence is given by $u_k = 7500 \times 0.8^k$.
(a) Write down the first three terms of the sequence and state the common ratio. [2]
(b) Find the sum of the first 10 terms, to the nearest whole number. [2]
-
- 12** Ruth invests £3000 in an account paying 2.5% compound interest per year.
(a) Find the value of her investment after 8 years, to the nearest pound. [2]
(b) Find the number of complete years after which the investment first more than doubles. [3]
-
- 13** A sequence has n th term $u_n = 5 \times 3^n$.
Show that the sequence is geometric and find the sum of the first 10 terms. [4]
-
- 14** Sam invests £2000 at the start of each year in an account that pays 4% compound interest per year. Interest is added at the end of each year.
(a) Find the value, at the end of the tenth year, of the first £2000 invested at the start of year 1. Give your answer to the nearest penny. [2]
(b) Show that the total value of all ten investments at the end of the tenth year is $2000(1.04 + 1.04^2 + \dots + 1.04^{10})$, and hence find this total to the nearest pound. [4]
-
- 15** The first three terms of a geometric sequence are k , 12 and $(2k + 12)$, where k is a non-zero constant.
(a) Show that $k^2 + 6k - 72 = 0$. [2]
(b) Find the two possible values of k . [2]
(c) For the positive value of k , find the sum of the first 8 terms. Explain why the series with the negative value of k has no sum to infinity. [3]
-
- 16** A car is bought for £18 000. Its value falls by 15% at the end of each year.
(a) Show that the value of the car after n years is $£18\,000 \times 0.85^n$. [2]
(b) Find the value of the car after 5 years, to the nearest £10. [2]
(c) Find the number of years after which the car is first worth less than £5000. [3]
-
- 17** A prize fund of £2730 is shared between three winners so that the shares, from smallest to largest, form a geometric sequence. The largest share is nine times the smallest share.
Find the value of each share. [5]
-
- 18** The sum of the first three terms of a geometric series is 7 times its first term. The common ratio r is positive.
(a) Show that $r^2 + r - 6 = 0$. [2]
(b) Find r and, given that the first term is 3, find the eighth term. [4]
-
- 19** A geometric series with positive common ratio has $S_4 = 15$ and $S_8 = 255$.
Show that $S_8 = S_4(1 + r^4)$ and hence find the common ratio and the first term. [6]
-
- 20** The sum of the second and fourth terms of a geometric series is 78 and the sum of the third and fifth terms is 117.
Find the first term and the common ratio of the series. [5]
-

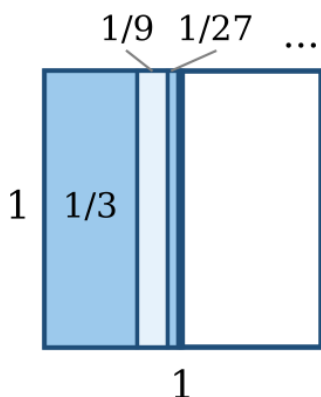
- 21** A geometric series has first term a and common ratio r , where $r \neq 1$.
- (a) Prove that the sum of the first n terms is $S_n = \frac{a(r^n - 1)}{r - 1}$. [4]
- (b) Given that $r > 0$ and that the sum of the first three terms is 7 times the first term, find r . [3]
- (c) Given also that the fourth term is 40, find the sum of the first 10 terms. [2]
-
- 22** A runner follows a training plan. In week 1 she runs 5 km. In each later week she runs 8% further than in the previous week.
- (a) Find the distance she runs in week 10, to 3 significant figures. [2]
- (b) Find the total distance she runs in the first 10 weeks, to 3 significant figures. [2]
- (c) Find the first week in which she runs more than 15 km. [3]
- (d) Give one reason why this model is unlikely to be valid for a long period of time. [1]
-
- 23** A geometric sequence has third term 12 and the sum of its second and fourth terms equal to 40.
- Find the two possible values of the common ratio and the corresponding first terms. [6]
-
- 24** Jo borrows £10 000. Interest of 1% of the outstanding balance is added at the end of each month, and then Jo immediately repays £400. Let B_n be the balance after n repayments.
- (a) Write down B_1 and B_2 and state a recurrence relation for B_{n+1} in terms of B_n . [3]
- (b) Show that $B_n = 40\,000 - 30\,000 \times 1.01^n$. [4]
- (c) Find the number of repayments needed to clear the debt. [3]
-
- 25** According to a legend, a reward is given in grains of rice on a board of 64 squares: 1 grain on the first square, 2 on the second, 4 on the third, and so on, doubling each time.
- (a) Find the number of the first square that holds more than 1 million grains. [3]
- (b) Show that the total number of grains on the board is $2^{64} - 1$ and give this to 3 significant figures. [3]
- (c) One grain has mass 25 mg. Find the mass of rice in tonnes, to 2 significant figures (1 tonne = 10^9 mg). [2]
-
- 26** A factory has two machines. In month 1 machine A makes 6000 components and its output then falls by 12% each month. In month 1 machine B makes 800 components and its output then rises by 9% each month.
- (a) Show that machine A makes 3598 components in month 5, to the nearest whole number. [2]
- (b) Find the total number of components made by machine A in the first 12 months, to the nearest whole number. [2]
- (c) In month k machine B makes more components than machine A for the first time. Show that $(\frac{1.09}{0.88})^{k-1} > 7.5$ and find the value of k . [4]
-
- 27** An arithmetic sequence has first term a and non-zero common difference d . The 2nd, 3rd and 6th terms of this arithmetic sequence are the first three terms of a geometric sequence.
- (a) Show that $d = -2a$. [4]
- (b) Find the common ratio of the geometric sequence. [1]
- (c) The sum of the first 10 terms of the arithmetic sequence is -400 . Find a and d . [3]
- (d) Find the sum of the first 8 terms of the geometric sequence. [2]
-
- 28** A geometric sequence has n th term $u_n = 4 \times (1.5)^{n-1}$.
- (a) Show that the sequence $\ln u_1, \ln u_2, \ln u_3, \dots$ is arithmetic, stating its first term and common difference. [3]
- (b) Find the least value of n for which $u_n > 1\,000\,000$. [3]
- (c) Find the exact value of $\ln u_1 + \ln u_2 + \dots + \ln u_{20}$, and give it to 3 significant figures. [3]
- (d) Hence show that $u_1 u_2 \dots u_{20} = 2^p \times 3^q$, where p and q are integers to be found. [2]

- 29** A geometric series has first term a , common ratio r ($r \neq 1$) and sum to n terms S_n .
- (a) Show that $S_{2n} - S_n = r^n S_n$ and find a similar expression for $S_{3n} - S_{2n}$. [4]
- (b) Hence show that S_n , $S_{2n} - S_n$ and $S_{3n} - S_{2n}$ are consecutive terms of a geometric sequence and state its common ratio. [2]
- (c) For a certain geometric series $S_5 = 11$ and $S_{10} = 33$. Find S_{15} . [3]
-
- 30** Sara pays £ x into a savings account at the start of each month for 60 months. The account pays interest of 0.4% per month, added at the end of each month.
- (a) Show that the amount in the account at the end of the 60th month is $\frac{1.004x(1.004^{60} - 1)}{0.004}$. [4]
- (b) Sara wants exactly £20 000 after 60 months. Find x , giving your answer to the nearest penny above. [3]
- (c) Sara's friend says that the interest rate is really 4.8% a year. Explain why the true annual growth factor is greater than 1.048, and find the annual percentage rate, correct to 3 significant figures. [2]

Sum to Infinity

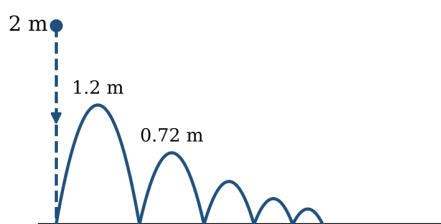
- 1** A geometric series has first term 12 and common ratio $\frac{1}{3}$.
Find the sum to infinity of the series. [2]
-
- 2** Determine which of these geometric series are convergent. For each convergent series, find its sum to infinity.
(i) $6 + 3 + 1.5 + \dots$
(ii) $2 - 6 + 18 - \dots$
(iii) $10 - 5 + 2.5 - \dots$ [3]
-
- 3** Find the sum to infinity of the geometric series $20 + 10 + 5 + \dots$ [2]
-
- 4** State, giving a reason, whether each of these geometric series converges: (i) $r = 1.2$, (ii) $r = -0.5$, (iii) $r = 1$. [2]
-
- 5** A geometric series has first term 8 and sum to infinity 10. Find the common ratio. [2]
-
- 6** The sum to infinity of a geometric series with common ratio 0.3 is 14. Find the first term. [2]

- 7 The diagram shows a square of side 1 unit. It is divided into vertical strips of widths $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, ... and so on, with each strip a third as wide as the one before. The strips are shaded alternately, but together they continue indefinitely towards the right-hand edge of the square.



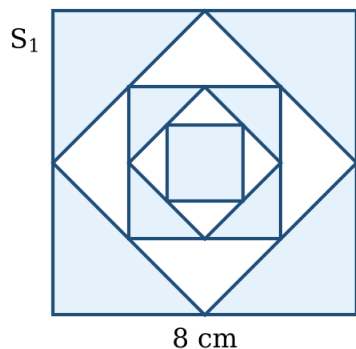
- (a) Find the width of the n th strip. [1]
- (b) Find the sum to infinity of the strip widths, and hence the width of the part of the square that is not covered by any strip. [3]
-
- 8 (This question is about recurring decimals.)
- (a) Write $0.363636\dots$ as the sum of a geometric series, stating the first term and common ratio. [2]
- (b) Hence show that $0.363636\dots = \frac{4}{11}$. [2]
-
- 9 A geometric series has common ratio $\frac{1}{4}$. The sum of its first three terms is 21. Find the first term of the series and the sum to infinity. [4]
-
- 10 Express the recurring decimal $0.1666\dots$ as a fraction in its simplest form. [4]
-
- 11 The geometric series $1 + 2x + 4x^2 + 8x^3 + \dots$ is convergent. Find the range of values of x and the sum to infinity when $x = \frac{1}{4}$. [4]
-
- 12 The common ratio of a geometric series is the square of its first term a . The sum to infinity of the series is $\frac{2}{3}$. Find the value of a , giving a reason for rejecting any other solution. [4]
-
- 13 Find the sum of all the terms of the geometric series $64 + 32 + 16 + \dots$ from the fourth term onwards. [3]
-
- 14 A geometric series has sum to infinity 27 and the sum of its first two terms is 24. Find the two possible pairs of values of the first term a and the common ratio r . [6]
-

- 15** A ball is dropped from a height of 2 m onto a horizontal floor. After each bounce it rises to 0.6 of the height from which it fell, as shown in the diagram. The ball is modelled as a particle and it continues to bounce indefinitely.



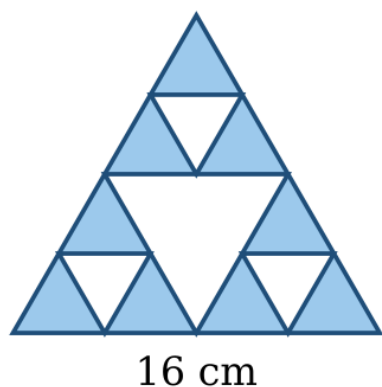
- (a) Find the greatest height the ball reaches after the third bounce. [2]
 (b) Show that the total distance the ball travels is 8 m. [4]
-
- 16** A geometric series has first term 800 and sum to infinity 1000. The r th term of the series is u_r .
 (a) Find the common ratio. [2]
 (b) Find the sum of the first five terms. [2]
 (c) Hence evaluate $\sum_{r=6}^{\infty} u_r$. [2]
-
- 17** A pendulum swings through an arc of length 40 cm on its first swing. Each later swing has an arc 5% shorter than the one before it.
 (a) Find the total distance travelled by the pendulum bob before it stops, according to this model. [3]
 (b) Find the number of the first swing whose arc is less than 1 cm. [3]
-
- 18** The second term of a geometric series is 48 and the fifth term is -6 .
 (a) Find the common ratio and the first term. [3]
 (b) Show that the sum to infinity less the sum of the first n terms is $-64(-\frac{1}{2})^n$. [3]
-
- 19** A geometric series has all terms positive. The sum of its first three terms is $\frac{19}{27}$ of its sum to infinity.
 Find the common ratio. [4]
-
- 20** The sum to infinity of a convergent geometric series is four times its second term.
 (a) Show that the common ratio of the series is $\frac{1}{2}$. [3]
 (b) The sum of the first five terms of the series is 1550. Find the first term. [2]
 (c) Find the least value of n for which the sum of the first n terms exceeds 1599. [2]
-

- 21** The diagram shows a square S_1 of side 8 cm. A second square S_2 is formed by joining the midpoints of the sides of S_1 . A third square S_3 is formed by joining the midpoints of the sides of S_2 , and so on indefinitely.



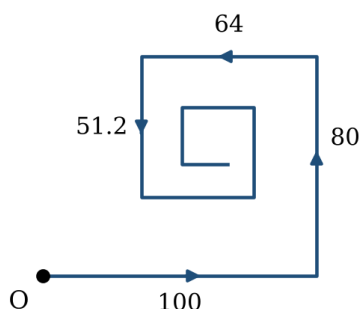
- (a) Show that the areas of the squares form a geometric sequence and state its common ratio. [3]
- (b) Find the sum of the areas of all the squares. [2]
- (c) Find the sum of the perimeters of all the squares, giving your answer in the form $p + q\sqrt{2}$. [3]
-
- 22** The geometric series $1 + \frac{2x}{x+3} + \left(\frac{2x}{x+3}\right)^2 + \dots$ is convergent.
- (a) Show that $-1 < x < 3$. [4]
- (b) Show that the sum to infinity of the series is $\frac{x+3}{3-x}$. [2]
- (c) Given that the sum to infinity is 7, find x , and verify that your value is valid. [2]
-
- 23** The geometric series $1 + \cos \theta + \cos^2 \theta + \cos^3 \theta + \dots$ is considered for $0 < \theta < \pi$.
- (a) Explain why the series converges for every θ in this interval. [2]
- (b) Find the sum to infinity in terms of θ . [1]
- (c) Find the value of θ for which the sum to infinity is 2. [3]
-
- 24** (This question is about the infinite series $\sum_{r=1}^{\infty} \frac{2^r + 3^r}{5^r}$.)
- (a) Show that the series can be written as the sum of two geometric series. [2]
- (b) Hence find the exact sum of the series. [3]
-
- 25** Consider the series $5^x + 25^x + 125^x + \dots$
- (a) Show that the series is geometric and find the set of values of x for which it converges. [2]
- (b) Given that the sum to infinity is $\frac{3}{2}$, find the exact value of x and give it to 3 significant figures. [4]

- 26** The diagram shows an equilateral triangle of side 16 cm. At stage 1 the triangle formed by joining the midpoints of the sides is removed. At stage 2 the same is done to each of the three remaining triangles at the corners, and the process continues for ever. Let the area of the original triangle be A .



- (a) Show that the area removed at stage 1 is $\frac{1}{4}A$ and that the area removed at stage 2 is $\frac{3}{16}A$. [3]
- (b) Show that the areas removed at the successive stages form a geometric sequence and find the total area removed. [3]
- (c) Hence find the total length of all the sides of the removed triangles after stage n , in terms of n , and explain why the total length tends to infinity. [3]
-
- 27** A geometric series G has first term a and common ratio r , where $|r| < 1$. The sum to infinity of G is 12. The squares of the terms of G form a second series H , and the sum to infinity of H is 72. [2]
- (a) Show that H is a geometric series, and state its first term and common ratio. [2]
- (b) Find the values of a and r . [5]
- (c) Find the sum to infinity of the series formed by the cubes of the terms of G . [3]
-
- 28** A patient takes a 200 mg tablet of a drug at 8 am every day. During each 24 hour period, 30% of the drug present in the body just before a tablet is taken (and which came from earlier tablets) is removed, and 70% remains. Let A_n mg be the amount of the drug in the body immediately after the n th tablet is taken. [2]
- (a) Find A_3 . [2]
- (b) Show that $A_n = \frac{2000}{3}(1 - 0.7^n)$. [3]
- (c) Hence show that the amount of the drug in the body immediately after a tablet never reaches 700 mg. [2]
- (d) The patient becomes ill if the amount of the drug immediately after a tablet exceeds 600 mg. After how many tablets does this first happen? [4]
- (e) State one assumption of this model that might not hold in practice. [1]

- 29** A snail starts at O and crawls 100 cm east, then turns left through 90° and crawls 80 cm, turns left through 90° again and crawls 64 cm, and so on. Each section is 0.8 times as long as the one before and the snail always turns left through 90° . The diagram shows the start of its path.

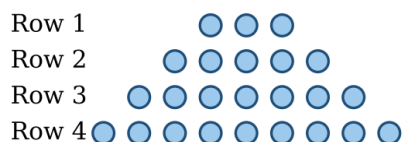


- (a) Write down the lengths of the 4th and 5th sections. [1]
- (b) Show that the total distance travelled east or west of O (taking east as positive) tends to a limit of $\frac{2500}{41}$ cm. [4]
- (c) Find the limiting position north or south of O, and hence the limiting distance of the snail from O, to 3 significant figures. [4]
-
- 30** (This question is about recurring decimals.)
- (a) Prove that the recurring decimal $0.d_1d_2\dots d_k d_1d_2\dots d_k \dots$ (a block of k digits repeating) equals $\frac{N}{10^k - 1}$, where N is the integer with digits $d_1d_2\dots d_k$. [4]
- (b) Hence write $0.142857142857\dots$ as a fraction in its simplest form. [2]
- (c) Use the result of part (a) with $N = 9$ and $k = 1$ to show that $0.999\dots = 1$, and comment on the claim that $0.999\dots$ is 'just less than 1'. [3]

Sigma Notation and Standard Series

- 1** Evaluate $\sum_{r=1}^5 (3r + 1)$. [2]
-
- 2** Write the series $3 + 6 + 9 + \dots + 60$ using sigma notation. [2]
-
- 3** Evaluate $\sum_{r=1}^4 (r^2 + 1)$. [2]
-
- 4** Find the value of $\sum_{r=1}^6 2^r$. [2]
-
- 5** Write the series $2 + 5 + 8 + \dots + 29$ using sigma notation. [2]
-
- 6** Use the formula for the sum of an arithmetic series to find $\sum_{r=1}^{100} r$. [2]
-
- 7** Evaluate each of the following.
- (a) $\sum_{r=1}^{12} (5r - 2)$ [2]
- (b) $\sum_{r=1}^{10} 3 \times 2^r$ [2]

- 8** The diagram shows the first four rows of a pattern of dots. Row k contains $2k + 1$ dots and the pattern continues in the same way.



- (a) Show that the total number of dots in the first n rows is $n^2 + 2n$. [3]
 (b) Find the number of rows needed to give a total of 224 dots. [2]

- 9** A geometric series is $5 + 15 + 45 + \dots + 3645$.

- (a) Write the series in the form $\sum_{r=1}^N ab^{r-1}$, stating the values of a , b and N . [2]
 (b) Hence find the value of the sum. [2]

- 10** Evaluate $\sum_{r=5}^{15} (3r - 4)$. [3]

- 11** Show that $\sum_{r=1}^n (4r - 3) = n(2n - 1)$. [3]

- 12** Find $\sum_{r=1}^{10} 3(-2)^r$. [3]

- 13** Given that $\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$, evaluate $\sum_{r=1}^{12} (r^2 + 2r)$. [4]

- 14** k is a constant.

- (a) Given that $\sum_{r=1}^{10} (3r + k) = 215$, find the value of k . [3]
 (b) Using this value of k , evaluate $\sum_{r=11}^{30} (3r + k)$. [3]

- 15** A series is given by $\sum_{r=1}^n 3\left(\frac{1}{2}\right)^r$.

- (a) Find the exact value of the sum when $n = 8$. [2]
 (b) Find the sum to infinity of the series. [1]
 (c) Find the smallest value of n for which the sum of the first n terms is greater than 2.99. [3]

- 16** Given that $\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{r=1}^n r = \frac{n(n+1)}{2}$, show that $\sum_{r=1}^n r(r+1) = \frac{n(n+1)(n+2)}{3}$. [4]

- 17** Given that $\sum_{r=1}^n r^3 = \left(\frac{n(n+1)}{2}\right)^2$, evaluate $\sum_{r=11}^{20} r^3$. [4]

- 18** (This question is about a change of lower limit.)

- (a) By writing $s = r - 1$, show that $\sum_{r=2}^{n+1} (4r - 3) = \sum_{s=1}^n (4s + 1)$. [2]
 (b) Given that $\sum_{r=2}^{n+1} (4r - 3) = 1325$, find the value of n . [4]

19

This question is about the series $\sum_{r=1}^n (5r - 1)$.

(a) Show that $\sum_{r=1}^n (5r - 1) = \frac{n(5n + 3)}{2}$. [3]

(b) Hence find, in terms of n , $\sum_{r=n+1}^{2n} (5r - 1)$, simplifying your answer. [3]

20

Show that $\sum_{r=1}^n (r + 1)(r + 3) = \frac{n(2n^2 + 15n + 31)}{6}$.

You may use $\sum_{r=1}^n r^2 = \frac{n(n + 1)(2n + 1)}{6}$. [5]

21

The sum of the first n terms of a sequence is $S_n = 2^{n+2} + n^2 - 4$.

(a) Find the first term of the sequence. [1]

(b) Use $u_n = S_n - S_{n-1}$ to show that $u_n = 2^{n+1} + 2n - 1$ for $n \geq 2$, and show that this formula also gives the first term. [4]

(c) Hence evaluate $\sum_{r=4}^8 u_r$. [2]

22

Let $S_n = \sum_{r=1}^n (2^r + 3r)$.

(a) Show that $S_n = 2^{n+1} - 2 + \frac{3n(n + 1)}{2}$. [4]

(b) Hence evaluate $\sum_{r=6}^{10} (2^r + 3r)$. [2]

(c) A student claims that $\sum_{r=1}^n r \times 2^r = (\sum_{r=1}^n r)(\sum_{r=1}^n 2^r)$. Show, by considering $n = 2$, that this is not true in general. [2]

23

You may use $\sum_{r=1}^n r^3 = \frac{n^2(n + 1)^2}{4}$, $\sum_{r=1}^n r^2 = \frac{n(n + 1)(2n + 1)}{6}$ and $\sum_{r=1}^n r = \frac{n(n + 1)}{2}$.

(a) Show that $\sum_{r=1}^n r(r + 1)(r + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}$. [6]

(b) Hence evaluate $\sum_{r=5}^{10} r(r + 1)(r + 2)$. [3]

24

(This question is about the series $\sum_{r=1}^n \frac{1}{r(r + 2)}$.)

(a) Show that $\frac{1}{r(r + 2)} = \frac{1}{2}(\frac{1}{r} - \frac{1}{r + 2})$. [2]

(b) Hence show that $\sum_{r=1}^n \frac{1}{r(r + 2)} = \frac{1}{2}(\frac{3}{2} - \frac{1}{n + 1} - \frac{1}{n + 2})$. [4]

(c) Find the sum to infinity of the series. [1]

25

Consider the series $2 + 22 + 222 + 2222 + \dots$, in which the r th term has r digits, all equal to 2.

(a) Show that the r th term of the series is $\frac{2}{9}(10^r - 1)$. [2]

(b) Hence show that the sum of the first n terms is $\frac{2}{81}(10^{n+1} - 10 - 9n)$. [4]

(c) Find the sum of the first six terms. [1]

26

It is given that $\sum_{r=1}^n (ar + b) = n(3n + 2)$ for all positive integers n , where a and b are constants.

- (a) Find the values of a and b . [4]
- (b) Hence evaluate $\sum_{r=11}^{20} (ar + b)$. [3]

27

$f(r)$ is a function defined for $r = 1, 2, 3, \dots, 12$ and it is given that

$$\sum_{r=1}^{12} (f(r) - 5) = 60 \text{ and } \sum_{r=1}^{12} (f(r) - 5)^2 = 700.$$

- (a) Find the value of $\sum_{r=1}^{12} f(r)$. [2]
- (b) Find the value of $\sum_{r=1}^{12} (f(r))^2$. [4]
- (c) Hence find the variance of the 12 values $f(1), f(2), \dots, f(12)$. [3]

28

$$\text{Let } T_n = \sum_{r=1}^{2n} (-1)^r (5r + 1).$$

- (a) Write out the terms of T_3 and hence evaluate it. [2]
- (b) By pairing the terms, prove that $T_n = 5n$. [4]
- (c) Hence evaluate $\sum_{r=1}^{40} (-1)^r (5r + 1)$. [1]
- (d) Find, in terms of n , the value of $\sum_{r=1}^{2n+1} (-1)^r (5r + 1)$. [2]

29

(This question is about the series $\sum_{r=1}^n r \times r!$)

- (a) Show that $r \times r! = (r + 1)! - r!$. [2]
- (b) Hence show that $\sum_{r=1}^n r \times r! = (n + 1)! - 1$. [3]
- (c) Evaluate $\sum_{r=1}^6 r \times r!$. [1]
- (d) Find the least n for which $\sum_{r=1}^n r \times r!$ exceeds 10^9 . [3]

30

(This question is about the series $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)}$.)

- (a) Find constants A and B such that $\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} - \frac{B}{(r+1)(r+2)}$. [3]
- (b) Hence show that $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$. [4]
- (c) State the sum to infinity of the series. [1]
- (d) Find the smallest n for which the sum to n terms is within 0.001 of the sum to infinity. [3]

Recurrence Relations

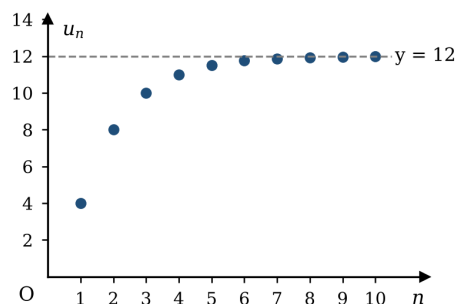
1

A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 3$.
Find u_2, u_3 and u_4 .

[2]

- 2** The sequence $a_1, a_2, a_3, \dots$ is given by $a_1 = 2$ and $a_{n+1} = (a_n)^2 - 1$.
Find the value of a_5 . [2]
-
- 3** A sequence is defined by $u_1 = 2$ and $u_{n+1} = 3u_n - 2n + 1$.
Find u_2, u_3 and u_4 . [2]
-
- 4** A sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + n$. Write down the first five terms. [2]
-
- 5** A sequence is defined by $u_1 = 1$ and $u_{n+1} = \frac{1}{1 + u_n}$. Find u_4 as a fraction. [2]
-
- 6** For each sequence, state whether it is increasing, decreasing, periodic or none of these:
(i) $u_{n+1} = u_n + 3, u_1 = 2$ (ii) $u_{n+1} = -u_n, u_1 = 5$ (iii) $u_{n+1} = \frac{1}{2}u_n, u_1 = 8$ [3]
-
- 7** A sequence is defined by $u_1 = 3$ and $u_{n+1} = ku_n + 2$, where k is a constant. Given that $u_2 = 11$:
(a) Find the value of k . [2]
(b) Find u_3 and the value of $u_1 + u_2 + u_3$. [2]
-
- 8** A sequence is defined by $u_1 = 2$ and $u_{n+1} = 3u_n$ for $n \geq 1$.
(a) Describe the sequence and write down a formula for u_n . [2]
(b) Find $\sum_{r=1}^8 u_r$. [2]
-
- 9** A sequence is defined by $u_1 = 2$ and $u_{n+1} = ku_n + 1$, where k is a constant.
(a) Show that $u_3 = 2k^2 + k + 1$. [2]
(b) Given that $u_1 + u_2 + u_3 = 18$, find the two possible values of k . [3]
-
- 10** A sequence is defined by $u_1 = 2$ and $u_{n+1} = \frac{u_n - 1}{u_n + 1}$.
(a) Find u_2, u_3, u_4 and u_5 and describe the behaviour of the sequence. [3]
(b) Find u_{2025} . [2]
(c) Find the sum of the first 100 terms. [2]
-
- 11** A sequence is defined by $u_{n+1} = 0.6u_n + 8, u_1 = 5$.
Show that the sequence converges, and find its limit. [3]
-
- 12** (This question is about forming recurrence relations.)
(a) Write down a recurrence relation for the sequence 5, 8, 11, 14, ... and another for 2, 6, 18, 54, ... [2]
(b) A sequence has n th term $u_n = 2n^2 - 3$. Show that $u_{n+1} = u_n + 4n + 2$. [2]
-
- 13** A sequence is defined by $u_{n+1} = 2u_n - 3$ and $u_3 = 5$.
Work backwards to find u_2 and u_1 , and find u_4 . [4]
-

- 14** A sequence is defined by $u_1 = 4$ and $u_{n+1} = \frac{1}{2}u_n + 6$. The diagram shows the first ten terms of the sequence, together with the line $y = 12$.



- (a) Find u_2 and u_3 . [2]
 (b) Assuming that the sequence converges to a limit L , find the value of L . [2]
 (c) Show that $12 - u_{n+1} = \frac{1}{2}(12 - u_n)$, and explain how this confirms what the diagram suggests. [2]

- 15** A sequence is defined by $u_1 = 3$ and $u_{n+1} = \frac{1}{1 - u_n}$.
 (a) Find u_2 , u_3 and u_4 , and hence state the period of the sequence. [2]
 (b) Find u_{100} . [2]
 (c) Find $\sum_{r=1}^{100} u_r$. [2]

- 16** The number of fish in a lake at the start of year n is P_n . Each year the population grows by 10% and then 300 fish are caught, so $P_{n+1} = 1.1P_n - 300$.
 (a) Show that if $P_1 = 3000$ then the population stays constant. [2]
 (b) Given instead that $P_1 = 2800$, find P_2 , P_3 and P_4 and comment. [3]
 (c) Explain why a limit of 3000 is not approached when $P_1 = 2800$. [1]

- 17** A sequence is defined by $u_1 = 1$ and $u_{n+1} = \frac{u_n}{1 + 2u_n}$.
 (a) Find u_2 , u_3 and u_4 . [2]
 (b) Write down a formula for u_n . [1]
 (c) Show that your formula satisfies the recurrence relation $u_{n+1} = \frac{u_n}{1 + 2u_n}$. [3]

- 18** A sequence is defined by $u_1 = 2$, $u_2 = 5$ and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$.
 (a) Find the first eight terms. [2]
 (b) Find the smallest n for which the sum of the first n terms exceeds 200. [3]

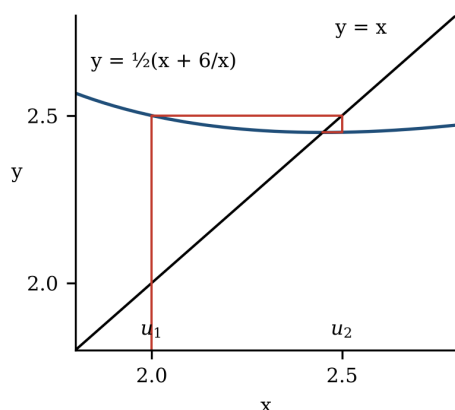
- 19** A sequence is defined by $u_1 = 3$ and $u_{n+1} = u_n + 2n + 1$.
 Find u_2 , u_3 and u_4 , and prove that $u_n = n^2 + 2$. [5]

- 20** A sequence is defined by $u_1 = 10$ and $u_{n+1} = 0.4u_n + c$, where c is a constant. The sequence converges to 6.
 Find the value of c and find u_3 . [4]

- 21** A sequence is defined by $u_1 = 2$ and $u_{n+1} = ku_n + 4$, where k is a constant. Given that $u_3 = 34$:
 (a) Show that $2k^2 + 4k - 30 = 0$ and find the possible values of k . [4]
 (b) Given that the sequence is increasing, find u_4 and $u_1 + u_2 + u_3 + u_4$. [2]
 (c) Explain why the sequence has no limit when $k = 3$. [1]

- 22** A fish farm contains 10 000 fish at the start of year 1. At the end of each year, 15% of the fish present are sold and then 2000 young fish are added. Let P_n be the number of fish at the start of year n .
- Write down a recurrence relation for P_{n+1} in terms of P_n . [1]
 - Find P_2 and P_3 . [2]
 - Find the limit to which the number of fish tends in the long run, giving your answer to the nearest whole number. [2]
 - It can be shown that $P_n = 13\,333.3 - 3333.3 \times 0.85^{n-1}$. Find the first year in which the number of fish at the start of the year exceeds 13 000. [3]
-
- 23** The Fibonacci sequence is defined by $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$.
- Find u_{10} and u_{11} , and calculate $\frac{u_{11}}{u_{10}}$ to 5 decimal places. [3]
 - Let $r_n = \frac{u_{n+1}}{u_n}$. Show that $r_{n+1} = 1 + \frac{1}{r_n}$. [2]
 - Given that r_n tends to a limit $\varphi > 0$, show that $\varphi^2 - \varphi - 1 = 0$ and find φ in exact form. [3]
-
- 24** A sequence is defined by $u_1 = 5$ and $u_{n+1} = (-1)^n u_n + k$, where k is a constant.
- Show that $u_5 = 5$. [2]
 - Write down, in terms of k , the value of u_{26} . [1]
 - Given that $\sum_{r=1}^4 u_r = 14$, find the value of k . [2]
 - Evaluate $\sum_{r=1}^{26} u_r$. [2]
-
- 25** A sequence satisfies $u_{n+1} = pu_n + q$ for constants p and q . It is given that $u_2 = 8$, $u_3 = 20$ and $u_4 = 44$.
- Find p and q , and hence u_1 . [4]
 - Show that $u_n + 4$ is a geometric sequence and hence that $u_n = 3 \times 2^n - 4$. [3]
-
- 26** A sequence is defined by $u_1 = k$ and $u_{n+1} = u_n^2 - 2u_n + 2$.
- Find the two values of k for which the sequence is constant. [3]
 - Find the first four terms when $k = 3$ and describe the behaviour. [2]
 - Find the first four terms when $k = 1.5$ and state what the sequence appears to approach. [2]
-
- 27** A sequence is defined by $u_1 = 5$ and $u_{n+1} = 2u_n - 3$.
- Find u_2 , u_3 and u_4 . [2]
 - Let $v_n = u_n - 3$. Show that $v_{n+1} = 2v_n$, and hence prove that $u_n = 3 + 2^n$. [3]
 - Show that $\sum_{r=1}^n u_r = 3n + 2^{n+1} - 2$. [3]
 - Find the least value of n for which $\sum_{r=1}^n u_r > 1\,000\,000$. [2]
-

A sequence is defined by $u_1 = 2$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{6}{u_n})$. The diagram shows the curve $y = \frac{1}{2}(x + \frac{6}{x})$ and the line $y = x$, together with a cobweb staircase starting from u_1 .



- Find u_2 , u_3 and u_4 , giving u_4 to 5 decimal places. [3]
- Assuming the sequence converges to a limit L , show that $L = \sqrt{6}$. [3]
- Show that, for $u_n > 0$, $u_{n+1} - \sqrt{6} = \frac{(u_n - \sqrt{6})^2}{2u_n}$, and hence that $u_{n+1} \geq \sqrt{6}$ for all n . [4]
- Explain why this shows that every term from u_2 onwards is greater than or equal to the limit. [1]

A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 2^n$.

- Find u_2 , u_3 and u_4 . [2]
- Let $v_n = \frac{u_n}{2^n}$. Show that $v_{n+1} = v_n + \frac{1}{2}$. [3]
- Hence find a formula for u_n . [3]
- Hence find the least value of n for which u_n exceeds 1 000 000. [4]

Kim retires with a pension pot of £250 000. At the end of each year the pot is increased by 3% and then Kim withdraws £W. Let F_n be the value of the pot (in pounds) after n withdrawals, so $F_0 = 250\,000$.

- Write down a recurrence relation for F_{n+1} in terms of F_n . [1]
- Given that $W = 9000$, show that $F_n = 300\,000 - 50\,000 \times 1.03^n$. [4]
- Find the number of the withdrawal after which the pot first becomes negative (i.e. the pension fails to pay in full). [3]
- Find the largest annual withdrawal W for which the pot never decreases, and explain your answer. [3]

Binomial Expansion (Positive Integer Powers)

Evaluate (a) ${}^7C_3$, (b) ${}^{10}C_2$.

The diagram shows the first six rows of Pascal's triangle.

Use it to expand $(2 + x)^4$, giving each coefficient as a number.

n = 0						1					
n = 1					1		1				
n = 2				1		2		1			
n = 3			1		3		3		1		
n = 4		1		4		6		4		1	
n = 5	1		5		10		10		5		1

[3]

- | | | |
|----|--|-------------------|
| 3 | Use the binomial theorem to expand $(1 + x)^5$. | [2] |
| 4 | Expand $(x + 2)^3$. | [2] |
| 5 | Find the coefficient of x^2 in the expansion of $(1 + 3x)^5$. | [2] |
| 6 | Find the coefficient of x^3 in the expansion of $(2 + x)^5$. | [2] |
| 7 | Find the first four terms, in ascending powers of x , of the binomial expansion of $(1 + 2x)^7$. | [4] |
| 8 | Find the coefficient of:
(a) x^4 in the expansion of $(2 - x)^7$,
(b) x^3 in the expansion of $(1 + 3x)^6$. | [2]
[2] |
| 9 | Expand $(3 - 2x)^4$, giving each term in its simplest form. | [4] |
| 10 | Find the coefficient of x^5 in the expansion of $(1 - \frac{1}{2}x)^8$. | [3] |
| 11 | The coefficient of x^2 in the expansion of $(1 + x)^n$, where n is a positive integer, is 45. Find the value of n . | [3] |
| 12 | Find the term in x^3 in the expansion of $(2x + 3)^5$. | [3] |
| 13 | Expand $(2x - \frac{1}{x})^4$, simplifying each term. | [4] |
| 14 | (This question is about the product $(2 + x)(1 - 3x)^5$.)
(a) Find the first three terms, in ascending powers of x , of the expansion of $(1 - 3x)^5$.
(b) Hence find the first three terms of the expansion of $(2 + x)(1 - 3x)^5$. | [3]
[3] |
| 15 | In the expansion of $(2 + ax)^6$, where a is a non-zero constant, the coefficients of x^2 and x^3 are equal.
(a) Write down, in terms of a , the coefficient of x^2 and of x^3 .
(b) Find the value of a .
(c) Hence find the coefficient of x^4 . | [2]
[2]
[2] |
| 16 | (This question is about the expansion of $(2x + 3)^4$.)
(a) Expand $(2x + 3)^4$ in descending powers of x .
(b) Hence find the coefficient of x^2 in the expansion of $(x - 2)(2x + 3)^4$. | [3]
[3] |
| 17 | In the expansion of $(k + 2x)^5$, where k is a positive constant, the coefficient of x^3 is 720. Find the value of k and the coefficient of x^2 . | [5] |
| 18 | Find the term independent of x in the expansion of $(2x + \frac{1}{x^2})^6$. | [5] |
| 19 | (This question is about the expansion of $(2 + 5x)^4$.)
(a) Expand $(2 + 5x)^4$ in ascending powers of x .
(b) By choosing a suitable value of x , and without using a calculator, find the exact value of 102^4 . | [3]
[2] |

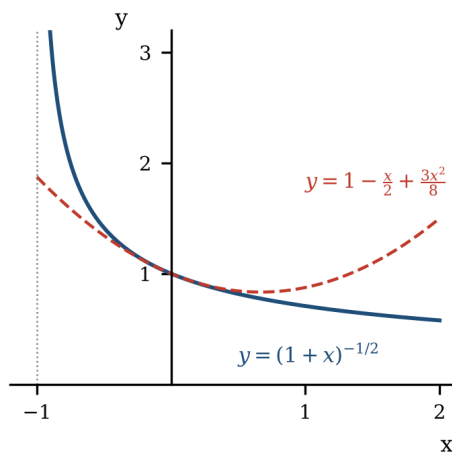
- 20** (This question is about the expansion of $(3 + x)^4$.)
- (a) Expand $(3 + x)^4$ in ascending powers of x . [2]
- (b) Hence write down the expansion of $(3 - 4x^2)^4$ in ascending powers of x . [2]
- (c) Use the first three terms of your answer to (b), with $x = 0.05$, to estimate 2.99^4 to 4 decimal places. [2]
-
- 21** (This question is about $(1 + x)^8$.)
- (a) Find the first four terms in the expansion of $(1 + x)^8$ in ascending powers of x . [2]
- (b) By substituting $x = 0.02$, find an estimate for 1.02^8 , and give the percentage error of the estimate to 2 significant figures, given that $1.02^8 = 1.1716594$ to 7 decimal places. [3]
- (c) Use a suitable value of x to estimate 0.98^8 using the same expansion. [2]
- (d) A student says: "Adding more terms of the expansion will always make the estimate of 1.02^8 bigger." Explain why this is correct. [1]
-
- 22** Consider the expansion of $(x + \frac{2}{x})^8$.
- (a) Find the term independent of x . [4]
- (b) Find the coefficient of x^4 . [2]
- (c) Find the sum of all the coefficients in the expansion. [2]
-
- 23** (This question is about the identity ${}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$.)
- (a) By choosing a suitable value of x in the expansion of $(1 + x)^n$, show that the sum of all the binomial coefficients is 2^n . [2]
- (b) Find the value of ${}^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^nnC_n$ for $n \geq 1$. [2]
- (c) Hence show that the sum of the coefficients with even lower index equals 2^{n-1} , and evaluate ${}^8C_0 + {}^8C_2 + {}^8C_4 + {}^8C_6 + {}^8C_8$. [3]
-
- 24** (This question is about the expansion of $(2x - \frac{3}{x})^7$.)
- (a) Show that the general term is ${}^7C_r 2^{7-r}(-3)^r x^{7-2r}$. [2]
- (b) Find the coefficient of x and the coefficient of $\frac{1}{x}$. [4]
- (c) Explain why there is no constant term. [1]
-
- 25** (This question is about $(1 + \sqrt{3})^5$.)
- (a) Use the binomial theorem to show that $(1 + \sqrt{3})^5 = 76 + 44\sqrt{3}$. [4]
- (b) Write down $(1 - \sqrt{3})^5$ in the form $p + q\sqrt{3}$. [1]
- (c) Hence show that $76^2 - 3 \times 44^2 = -32$. [2]
-
- 26** The expansion of $(1 + ax)^5(2 - x)^4$ has no term in x .
- (a) Find the value of a . [4]
- (b) Find the coefficient of x^2 in the expansion. [3]
-
- 27** (This question is about the number $(2 + \sqrt{3})^6$.)
- (a) Use the binomial expansion to show that $(2 + \sqrt{3})^6 = 1351 + 780\sqrt{3}$. [4]
- (b) Write down the value of $(2 - \sqrt{3})^6$ in the form $a + b\sqrt{3}$. [1]
- (c) Hence show that $(2 + \sqrt{3})^6 + (2 - \sqrt{3})^6 = 2702$. [2]
- (d) Using the fact that $0 < 2 - \sqrt{3} < 0.3$, show that $(2 + \sqrt{3})^6$ is less than 2702 but by less than 0.001. [3]

- 28** In the binomial expansion of $(1 + ax)^n$, where n is a positive integer, the coefficient of x is 12 and the coefficient of x^2 is 60.
- (a) Find the values of a and n . [5]
- (b) Find the coefficient of x^3 . [1]
- (c) Prove that ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$ for $1 \leq r \leq n - 1$. [4]
-
- 29** (This question proves that ${}^nC_0^2 + {}^nC_1^2 + \dots + {}^nC_n^2 = {}^{2n}C_n$, i.e. the sum of the squares of the binomial coefficients.)
- (a) Explain why ${}^nC_r = {}^nC_{n-r}$. [1]
- (b) By considering the coefficient of x^n in $(1 + x)^n(1 + x)^n = (1 + x)^{2n}$, show that the sum of squares equals ${}^{2n}C_n$. [5]
- (c) Verify the result for $n = 4$. [2]
-
- 30** The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 2 : 3. Find the value of n and the three coefficients. [8]

Binomial Expansion (Fractional and Negative Powers)

- 1** Find the first three terms, in ascending powers of x , of the binomial expansion of $(1 + x)^{-2}$. [2]
-
- 2** (a) Find the first three terms of the expansion of $(1 + x)^{1/2}$ in ascending powers of x .
(b) State the range of values of x for which the expansion is valid. [3]
-
- 3** State the range of values of x for which the binomial expansion of $(1 + 4x)^{-3}$ is valid. [2]
-
- 4** Find the first three terms in the binomial expansion of $(1 - x)^{-1}$ in ascending powers of x . [2]
-
- 5** Find the coefficient of x^2 in the expansion of $(1 + x)^{-3}$. [2]
-
- 6** Find the first three terms in the expansion of $(1 + 2x)^{1/2}$ in ascending powers of x . [3]
-
- 7** Find the first three terms in the binomial expansion of $(1 - 3x)^{-2}$ in ascending powers of x , and state the values of x for which the expansion is valid. [4]
-
- 8** Find the first three terms in the expansion of $\sqrt{4 + x}$ in ascending powers of x , and state the range of x for which the expansion is valid. [5]
-
- 9** Find the first three terms in the expansion of $(1 - \frac{1}{2}x)^{-3}$ in ascending powers of x and state the range of x for which it is valid. [4]
-
- 10** Expand $\frac{1}{\sqrt{1 + 2x}}$ in ascending powers of x up to and including the term in x^2 . [4]
-
- 11** Find the first three terms in the expansion of $(1 + 3x)^{1/3}$ and the range of x for which the expansion is valid. [4]
-
- 12** Find the first three terms in the expansion of $\sqrt{9 - 4x}$ in ascending powers of x . [4]
-
- 13** Find the coefficient of x^4 in the expansion of $\frac{1}{(1 - 2x)^2}$. [3]

- 14** The diagram shows the graph of $y = (1 + x)^{-1/2}$ (solid curve) and the graph of the quadratic that approximates it near $x = 0$ (dashed curve).



- (a) Find the first three terms in the binomial expansion of $(1 + x)^{-1/2}$ in ascending powers of x . [3]
- (b) Use your expansion with $x = 0.02$ to estimate $\frac{1}{\sqrt{1.02}}$ to 5 decimal places. [2]
- (c) Explain, with reference to the diagram, why the expansion should not be used to estimate $(1 + x)^{-1/2}$ when $x = 2$. [1]
-
- 15** (This question is about $\frac{1}{(2 + x)^2}$.)
- (a) Find the first three terms in the expansion of $\frac{1}{(2 + x)^2}$ in ascending powers of x , and state the range of validity. [2]
- (b) Find the coefficient of x^3 in the expansion. [2]
-
- 16** $f(x) = \frac{(2 + 3x)^2}{1 - 2x}$.
- (a) Find the first three terms in the binomial expansion of $(1 - 2x)^{-1}$ in ascending powers of x . [2]
- (b) Hence find the first three terms in the expansion of $f(x)$ in ascending powers of x . [3]
- (c) State the range of values of x for which the expansion is valid. [1]
-
- 17** (This question is about an estimate of the cube root of 65.)
- (a) Find the first three terms in the binomial expansion of $(1 + x)^{1/3}$ in ascending powers of x . [2]
- (b) Write $\sqrt[3]{65}$ in the form $4(1 + x)^{1/3}$ and hence estimate $\sqrt[3]{65}$ to 4 decimal places. [3]
-
- 18** The coefficient of x^2 in the expansion of $(1 + ax)^{-2}$ is 12.
- (a) Find the two possible values of a . [3]
- (b) For the positive value of a , find the coefficient of x^3 and the range of x for which the expansion is valid. [3]
-
- 19** $f(x) = \frac{1}{(1 + x)(1 - 2x)}$.
- (a) Express $f(x)$ in partial fractions. [3]
- (b) Hence find the expansion of $f(x)$ up to and including the term in x^2 . [3]
- (c) State the range of x for which the expansion is valid. [1]
-
- 20** Show that, up to and including the term in x^2 , $\sqrt{\frac{1 + x}{1 - x}} = 1 + x + \frac{1}{2}x^2$. [5]

- 21** (This question is about the function $f(x) = \frac{3-x}{(1-2x)^2}$.)
- Find the first four terms in the binomial expansion of $(1-2x)^{-1}$ in ascending powers of x . [2]
 - By differentiating both sides of your expansion in (a), show that $(1-2x)^{-2} = 1 + 4x + 12x^2 + \dots$ [3]
 - Hence find the first three terms in the expansion of $f(x)$ in ascending powers of x , and state the range of x for which it is valid. [3]
-
- 22** $f(x) = \frac{5+x}{(1-x)(2+x)}$.
- Express $f(x)$ in partial fractions. [3]
 - Hence find the first three terms in the expansion of $f(x)$ in ascending powers of x . [4]
 - State the range of x for which your expansion is valid. [1]
-
- 23** (This question is about the expansion of $(1+4x)^{-1/2}$.)
- Show that the expansion of $(1+4x)^{-1/2}$ in ascending powers of x begins $1 - 2x + 6x^2 - 20x^3$. [3]
 - The expression $(1+ax)(1+bx^2) - (1+4x)^{-1/2}$ has no terms in x or x^2 , so its term of lowest degree is the term in x^3 . Find the values of a and b , and the coefficient of x^3 . [4]
-
- 24** The expansion of $(1+px)^q$, in ascending powers of x , begins $1 - 4x + 24x^2 + \dots$
Given that q is not a positive integer, find the values of p and q and state the range of x for which the expansion is valid. [7]
-
- 25** (This question is about an approximation to $\sqrt{2}$.)
- Show that $\sqrt{2} = \frac{7}{5}(1 + \frac{1}{49})^{1/2}$. [2]
 - Use the first three terms of the binomial expansion of $(1+x)^{1/2}$ to find an approximation to $\sqrt{2}$, correct to 6 decimal places. [3]
 - Explain why this approximation is much more accurate than using $(1+x)^{1/2}$ with $x = 1$ to estimate $\sqrt{2}$. [2]
-
- 26** A student expands $(1-4x)^{-1/2}$ as $1 + 2x + 6x^2 + \dots$ and writes: 'This is valid for $|x| < 1$, so I can put $x = 0.3$ to estimate $\frac{1}{\sqrt{-0.2}}$.'
- Show that the expansion is correct as far as the x^2 term. [2]
 - Explain the two errors in the student's statement. [2]
 - Use a suitable value of x to estimate $\frac{1}{\sqrt{0.96}}$ to 4 decimal places, and compare with the calculator value. [3]
-
- 27** $f(x) = \frac{12}{(1-2x)(1+x)^2}$, $|x| < \frac{1}{2}$.
- Express $f(x)$ in partial fractions. [5]
 - Hence show that, up to and including the term in x^2 , $f(x) = 12 + 36x^2$. [4]
 - Explain why the expansion is valid for $|x| < \frac{1}{2}$ and not for a larger range. [1]

- 28** The first three terms in the expansion of $(1 + ax)^n$, in ascending powers of x , are $1 - 6x + 27x^2$. Here n is not a positive integer.
- Find the values of n and a . [5]
 - State the range of values of x for which the expansion is valid. [1]
 - Hence find the first three terms in the expansion of $\frac{1}{(1 + 3x)^2(1 - x)}$. [3]
 - Use your answer to (c) with $x = 0.01$ to estimate $\frac{1}{(1.03)^2 \times 0.99}$, giving your answer to 4 decimal places. [2]
-
- 29** (This question is about the coefficients in the expansion of $(1 - 4x)^{-1/2}$.)
- Show that the expansion begins $1 + 2x + 6x^2 + 20x^3$. [3]
 - Show that the coefficient of x^n is $\frac{2^n \times 1 \times 3 \times 5 \times \dots \times (2n - 1)}{n!}$. [3]
 - Hence prove that the coefficient of x^n is ${}^{2n}C_n$, using $1 \times 3 \times 5 \times \dots \times (2n - 1) = \frac{(2n)!}{2^n n!}$. [2]
-
- 30** In relativity, a particle of mass m moving at speed v has kinetic energy $E = (\gamma - 1)mc^2$, where $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$ and c is the speed of light.
- Let $x = \frac{v^2}{c^2}$. Show that, for v much less than c , $\gamma \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2$. [3]
 - Hence show that $E \approx \frac{1}{2}mv^2(1 + \frac{3v^2}{4c^2})$. [3]
 - For $v = 0.1c$, find the percentage by which the classical value $\frac{1}{2}mv^2$ underestimates the exact value of E , to 2 significant figures. [4]
 - Find the percentage error when the corrected formula in part (b) is used for $v = 0.1c$. [1]