

Proof by Deduction

1 (2 marks)

Let the numbers be $2m$ and $2n$, where m and n are integers

B1

$2m + 2n = 2(m + n)$, which is even since $m + n$ is an integer

E1

Answer: proved

2 marks

2 (2 marks)

$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$

M1

$2n + 1$ is of the form $2 \times \text{integer} + 1$, so it is odd

E1

Answer: proved

2 marks

3 (3 marks)

Let the numbers be $2m + 1$ and $2n + 1$

B1

$(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1$

M1

$= 2(2mn + m + n) + 1$, which is odd

E1

Answer: proved

3 marks

4 (2 marks)

$(x + 3)^2 = x^2 + 6x + 9$

M1

$(x + 3)^2 - 6x - 9 = x^2$

A1

Answer: shown

2 marks

5 (3 marks)

Let the integers be n , $n + 1$, $n + 2$

B1

Sum $= 3n + 3$

M1

$= 3(n + 1)$, a multiple of 3

E1

Answer: proved

3 marks

6 (2 marks)

$2^{n+1} = 2 \times 2^n$

M1

$2 \times 2^n + 2^n = 3 \times 2^n$

A1

Answer: shown

2 marks

7 (3 marks)

Let the numbers be $2n + 1$ and $2n + 3$

B1

Sum $= 4n + 4$

M1

$= 4(n + 1)$, a multiple of 4

E1

Answer: proved

3 marks

8 (3 marks)

$n^2 + n = n(n + 1)$

M1

One of n and $n + 1$ is even

E1

so the product is even

E1

Answer: proved

3 marks

9 (3 marks)

$x^2 - 6x + 10 = (x - 3)^2 + 1$

M1 A1

$(x - 3)^2 \geq 0$ for all real x , so the expression is at least $1 > 0$

E1

Answer: proved

3 marks

10 (3 marks)

$$T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \quad \text{M1}$$

$$= \frac{(n+1)(n+(n+2))}{2} = \frac{(n+1)(2n+2)}{2} \quad \text{A1}$$

$$= (n+1)^2, \text{ which is a square number} \quad \text{A1}$$

Answer: proved 3 marks

11 (3 marks)

Arithmetic series with $a = 1$, $d = 2$ B1

$$S_n = \frac{n}{2}(2 \times 1 + (n-1) \times 2) \quad \text{M1}$$

$$= \frac{n}{2} \times 2n = n^2 \quad \text{A1}$$

Answer: proved 3 marks

12 (3 marks)

$$(a+b)^2 - 4ab = a^2 - 2ab + b^2 = (a-b)^2 \quad \text{M1 A1}$$

$$(a-b)^2 \geq 0 \text{ so } (a+b)^2 \geq 4ab, \text{ with equality when } a = b \quad \text{E1}$$

Answer: proved; equality iff $a = b$ 3 marks

13 (4 marks)

$$n^3 - n = n(n-1)(n+1) \quad \text{M1}$$

This is a product of three consecutive integers B1
 so one of them is even and one is a multiple of 3 E1
 2 and 3 are coprime, so the product is divisible by 6 E1

Answer: proved 4 marks

14 (4 marks)

Odd number $2n+1$: $(2n+1)^2 = 4n^2 + 4n + 1 = 4n(n+1) + 1$ M1 A1
 $n(n+1)$ is a product of consecutive integers, so even: $n(n+1) = 2k$ E1
 $(2n+1)^2 = 8k + 1$ A1

Answer: proved 4 marks

15 (4 marks)

Numbers $2n$ and $2n+2$ B1
 Product $= 4n^2 + 4n = 4n(n+1)$ M1 A1
 $n(n+1)$ is even, so the product is $4 \times 2k = 8k$ E1

Answer: proved 4 marks

16 (4 marks)

(a) $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$ M1
 which is ≥ 0 , so $a + b \geq 2\sqrt{ab}$ A1

Answer: shown 2 marks

(b) Divide by 2: $\frac{a+b}{2} \geq \sqrt{ab}$ M1
 Equality when $\sqrt{a} - \sqrt{b} = 0$, i.e. $a = b$ A1

Answer: equality iff $a = b$ 2 marks

17 (4 marks)

(a) $\text{LHS} = m^4 - 2m^2n^2 + n^4 + 4m^2n^2$ M1 A1
 $= m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2$ A1

Answer: proved 3 marks

(b) $m^2 - n^2 = 21$, $2mn = 20$, $m^2 + n^2 = 29$: 20, 21, 29 B1

Answer: 20, 21, 29 1 mark

18 (4 marks)**(a)** The step 'so $a + b = b$ ' divides both sides by $(a - b)$

B1

Since $a = b$, $(a - b) = 0$ and division by zero is not allowed

B1

Answer: division by $(a - b) = 0$

2 marks

(b) $n = 1$ gives $1 = 1$, so n^2 is not greater than n

B1

 $n = 0$ gives $0 = 0$, so n^2 is not greater than n

B1

Answer: $n = 0$ and $n = 1$

2 marks

19 (4 marks)Divide by a : $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

M1

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

M1 A1

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}, \text{ so } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A1

Answer: proved

4 marks

20 (5 marks)

$$\text{(a) } M = \left(\frac{0+2b}{2}, \frac{0+2c}{2}\right)$$

M1

 $M(b, c), N(a + b, c)$

A1

Answer: $M(b, c), N(a + b, c)$

2 marks

(b) Both M and N have y -coordinate c , so MN is horizontal, as is OA

B1

$$MN = (a + b) - b = a$$

M1

$$OA = 2a, \text{ so } MN = \frac{1}{2}OA$$

A1

Answer: proved

3 marks

21 (5 marks)Let the integers be $n - 1, n, n + 1$

B1

$$(n - 1)^2 + n^2 + (n + 1)^2 = 3n^2 + 2$$

M1 A1

 $3n^2$ is a multiple of 3

B1

so the sum leaves remainder 2 when divided by 3, and is not a multiple of 3

E1

Answer: proved

5 marks

22 (5 marks)

$$\tan x = \frac{\sin x}{\cos x}, \text{ so } \tan x + \frac{1}{\tan x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

M1

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

M1 A1

$$\sin^2 x + \cos^2 x = 1$$

B1

$$= \frac{1}{\sin x \cos x}$$

A1

Answer: proved

5 marks

23 (5 marks)

$$\frac{dy}{dx} = 3x^2 + 3$$

M1 A1

$$x^2 \geq 0 \text{ for all real } x$$

B1

$$\text{so } \frac{dy}{dx} \geq 3 > 0$$

A1

The gradient is always positive, so y is increasing

E1

Answer: proved

5 marks

24 (4 marks)

$$x^2 + 1 = kx + 2$$

M1

$$x^2 - kx - 1 = 0$$

A1

$$\text{Discriminant} = k^2 - 4(1)(-1) = k^2 + 4$$

M1

$k^2 \geq 0$ for all real k , so $k^2 + 4 > 0$; the equation always has two distinct real roots, so there are two distinct intersections

A1

Answer: proved

4 marks

25 (6 marks)

Real roots when the discriminant $b^2 - 4ac \geq 0$

B1

$$k^2 - 4(k + 3) \geq 0, \text{ i.e. } k^2 - 4k - 12 \geq 0$$

M1 A1

$$(k - 6)(k + 2) \geq 0$$

M1

Sketch of an upward parabola with roots -2 and 6 ; the product is non-negative outside the roots

A1

So $k \leq -2$ or $k \geq 6$; every step is reversible, giving 'if and only if'

E1

Answer: proved

6 marks

26 (6 marks)

$$(a) \frac{1}{n} - \frac{1}{n+1} = \frac{(n+1) - n}{n(n+1)}$$

M1

$$= \frac{1}{n(n+1)}$$

A1

Answer: proved

2 marks

$$(b) \text{Sum} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{N} - \frac{1}{N+1})$$

M1

Each middle term cancels with the next

A1

$$= 1 - \frac{1}{N+1} = \frac{N}{N+1}$$

A1

Answer: proved

3 marks

$$(c) \frac{N}{N+1} = 1 - \frac{1}{N+1} < 1 \text{ since } \frac{1}{N+1} > 0$$

E1

Answer: proved

1 mark

27 (7 marks)

$$(a) (n+1)^3 = n^3 + 3n^2 + 3n + 1$$

M1

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1 = 3n(n+1) + 1$$

A1

Answer: shown

2 marks

(b) $n(n+1)$ is even, so $n(n+1) = 2k$

E1

$$3n(n+1) + 1 = 6k + 1$$

M1 A1

Answer: proved

3 marks

(c) $6k + 1$ leaves remainder 1 on division by 2, and by 3

M1

so it is neither even nor a multiple of 3

E1

Answer: proved

2 marks

28 (8 marks)

(a) $a^L = b$ by the definition of a logarithm

B1

Take logs to base b : $L \log_b a = \log_b b = 1$

M1

$$\log_b a = \frac{1}{L}$$

A1

Answer: proved

3 marks

(b) Let $\log_b c = M$, so $b^M = c$

B1

$$b = a^L, \text{ so } c = (a^L)^M = a^{LM}$$

M1

$$\log_a c = LM = \log_a b \times \log_b c$$

A1

Answer: proved

3 marks

(c) By part (b), $\log_2 5 \times \log_5 8 = \log_2 8$

M1

$$= 3$$

A1

Answer: 3

2 marks

29 (8 marks)

(a) $\text{RHS} = a^2c^2 - 2abcd + b^2d^2$

M1

$$+ a^2d^2 + 2abcd + b^2c^2$$

M1

$$= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$

A1

$$= (a^2 + b^2)(c^2 + d^2)$$

A1

Answer: proved

4 marks

(b) $ac - bd$ and $ad + bc$ are integers whenever a, b, c, d are

E1

Answer: proved

1 mark

(c) $a = 2, b = 3, c = 1, d = 2: (2 - 6)^2 + (4 + 3)^2 = 4^2 + 7^2 = 65$

M1 A1

Taking $c = 2, d = 1: (4 - 3)^2 + (2 + 6)^2 = 1^2 + 8^2 = 65$

A1

Answer: $65 = 4^2 + 7^2 = 1^2 + 8^2$

3 marks

30 (8 marks)

(a) $(n - 1)^3 + (n + 1)^3 = 2n^3 + 6n$

M1 A1

Adding $n^3: 3n^3 + 6n = 3n(n^2 + 2)$

A1

Answer: shown

3 marks

(b) $n = 3m: n$ is a multiple of 3, so $n(n^2 + 2)$ is a multiple of 3

B1

$n = 3m + 1: n^2 + 2 = 9m^2 + 6m + 3 = 3(3m^2 + 2m + 1)$

M1 A1

$n = 3m + 2: n^2 + 2 = 9m^2 + 12m + 6 = 3(3m^2 + 4m + 2)$

A1

Answer: proved

4 marks

(c) The sum is $3 \times n(n^2 + 2) = 3 \times 3t = 9t$

E1

Answer: proved

1 mark

Proof by Exhaustion**1 (2 marks)**

$n = 1, 2, 3, 4, 5$ give 11, 13, 17, 23, 31

M1

Each of these has no factors other than 1 and itself, so each is prime

A1

Answer: 11, 13, 17, 23, 31 are all prime

2 marks

2 (3 marks)

The last digit of n^2 depends only on the last digit of n

B1

Last digits 0, 1, 2, ..., 9 give squares ending 0, 1, 4, 9, 6, 5, 6, 9, 4, 1

M1

The possible final digits are 0, 1, 4, 5, 6, 9, so 2, 3, 7, 8 never occur

E1

Answer: proved

3 marks

3 (3 marks)

$n = 1, 2, 3: 1 < 2, 2 < 4, 6 < 8$, so false

M1

$n = 4: 24 > 16, n = 5: 120 > 32, n = 6: 720 > 64$

M1

True for $n = 4, 5, 6$ only

A1

Answer: true for $n = 4, 5, 6$

3 marks

4 (3 marks)

The primes are 5, 7, 11, 13, 17, 19, 23, 29

B1

$5 = 6 - 1, 7 = 6 + 1, 11 = 12 - 1, 13 = 12 + 1$

M1

$17 = 18 - 1, 19 = 18 + 1, 23 = 24 - 1, 29 = 30 - 1$

A1

Answer: shown

3 marks

5 (3 marks)

Every integer is $3k, 3k + 1$ or $3k + 2$

B1

$(3k)^2 = 9k^2 = 3(3k^2); (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$

M1

$(3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, so all cases covered

A1

Answer: proved

3 marks

6 (3 marks)

$x \geq 0$: $x = 2x - 3$ gives $x = 3$, which satisfies $x \geq 0$

M1 A1

$x < 0$: $-x = 2x - 3$ gives $x = 1$, which is not less than 0, so rejected

A1

Answer: $x = 3$ only

3 marks

7 (3 marks)

Even integer $2k$: $(2k)^2 = 4k^2$, a multiple of 4

B1 M1

Odd integer $2k + 1$: $(2k + 1)^2 = 4k(k + 1) + 1$, one more than a multiple of 4

A1

Answer: proved

3 marks

8 (3 marks)

$n = 2k$: $4k^2 + 2k + 1 = 2(2k^2 + k) + 1$, odd

M1 A1

$n = 2k + 1$: $(2k + 1)^2 + (2k + 1) + 1 = 4k^2 + 6k + 3 = 2(2k^2 + 3k + 1) + 1$, odd

A1

Answer: proved

3 marks

9 (4 marks)

$n^3 - n = n(n - 1)(n + 1)$

M1

$n = 3k$: n is a multiple of 3

B1

$n = 3k + 1$: $n - 1 = 3k$ is a multiple of 3

B1

$n = 3k + 2$: $n + 1 = 3(k + 1)$ is a multiple of 3

A1

Answer: proved

4 marks

10 (4 marks)

Case $2x - 1 \geq 0$: $2x - 1 = x + 4$ gives $x = 5$, which satisfies $x \geq \frac{1}{2}$

M1 A1

Case $2x - 1 < 0$: $-(2x - 1) = x + 4$ gives $1 - 2x = x + 4$, $x = -1$, which satisfies $x < \frac{1}{2}$

M1

Check: $|2(-1) - 1| = 3 = -1 + 4$ and $|9| = 9 = 5 + 4$, so both valid

A1

Answer: $x = 5$ or $x = -1$

4 marks

11 (4 marks)

An integer is $2a$ or $2a + 1$: $(2a)^2 = 4a^2$ and $(2a + 1)^2 = 4a(a + 1) + 1$, so every square is of the form $4k$ or $4k + 1$

B1

Possible sums: $4k + 4k' = 4(k + k')$, $(4k) + (4k' + 1) = 4(k + k') + 1$, $(4k + 1) + (4k' + 1) = 4(k + k') + 2$

M1 A1

None leaves remainder 3 on division by 4

E1

Answer: proved

4 marks

12 (4 marks)

n even, $n = 2k$: $n^2 + 3 = 4k^2 + 3$, remainder 3, so not divisible

M1 A1

n odd, $n = 2k + 1$: $n^2 + 3 = 4k^2 + 4k + 4 = 4(k^2 + k + 1)$

M1

Divisible by 4 exactly when n is odd

A1

Answer: all odd integers n

4 marks

13 (4 marks)

Case $3x - 2 \geq 0$: $3x - 2 > 4$

M1

$x > 2$, which satisfies $x \geq \frac{2}{3}$

A1

Case $3x - 2 < 0$: $-(3x - 2) > 4$, so $3x < -2$

M1

$x < -\frac{2}{3}$; the solution is $x < -\frac{2}{3}$ or $x > 2$

A1

Answer: $x < -\frac{2}{3}$ or $x > 2$

4 marks

14 (5 marks)

Write $n = 5k + r$ with $r = 0, 1, 2, 3, 4$, so $n^5 \equiv r^5$ on division by 5

M1 A1

$r^5 = 0, 1, 32, 243, 1024$

M1

These leave remainders 0, 1, 2, 3, 4, which equal r in each case

A1

so $n^5 - n \equiv r - r = 0$ and is divisible by 5

E1

Answer: proved

5 marks

15 (4 marks)

If m is even, $m = 2k$: $m^2 = 4k^2$ is a multiple of 4

M1 A1

If m is odd, $m = 2k + 1$: $m^2 = 4(k^2 + k) + 1$, remainder 1

A1

Neither case gives remainder 2 on division by 4, so no solutions

E1

Answer: proved

4 marks

16 (5 marks)

$p = 3$: 5 and 7 are prime, so $p = 3$ works

B1

$p = 2$: 4 is not prime

B1

Otherwise $p = 3k + 1$ or $3k + 2$ (p is not a multiple of 3)

M1

$p = 3k + 1$: $p + 2 = 3(k + 1)$, a multiple of 3 greater than 3

A1

$p = 3k + 2$: $p + 4 = 3(k + 2)$, a multiple of 3 greater than 3. So only $p = 3$

A1

Answer: $p = 3$

5 marks

17 (5 marks)

$x < 1$: $(1 - x) + (4 - x) = 5 - 2x > 3$

M1 A1

$1 \leq x \leq 4$: $(x - 1) + (4 - x) = 3$

M1 A1

$x > 4$: $(x - 1) + (x - 4) = 2x - 5 > 3$, so ≥ 3 always, equality for $1 \leq x \leq 4$

A1

Answer: proved; equality for $1 \leq x \leq 4$

5 marks

18 (5 marks)

$(x - y)(x + y) = 7$

M1

7 is prime, so $x - y$ and $x + y$ are 1, 7, 7, 1, -1, -7 or -7, -1

A1

$x - y = 1$, $x + y = 7$: $x = 4$, $y = 3$; $x - y = 7$, $x + y = 1$: $x = 4$, $y = -3$

M1

$x - y = -1$, $x + y = -7$: $x = -4$, $y = -3$; $x - y = -7$, $x + y = -1$: $x = -4$, $y = 3$

A1

(4, 3), (4, -3), (-4, 3), (-4, -3)

A1

Answer: ($\pm 4, \pm 3$)

5 marks

19 (4 marks)

Every integer x has the form $5k$, $5k \pm 1$ or $5k \pm 2$

B1

$(5k)^2 = 5(5k^2)$, $(5k \pm 1)^2 = 5(5k^2 \pm 2k) + 1$

M1

$(5k \pm 2)^2 = 5(5k^2 \pm 4k) + 4$

A1

So x^2 leaves remainder 0, 1 or 4 on division by 5, never 2; hence $x^2 = 5y + 2$ is impossible

A1

Answer: no integer solutions

4 marks

20 (4 marks)

(a) All 36 outcomes are equally likely

B1

Odd products arise only when both scores are odd: $3 \times 3 = 9$ outcomes

M1

$P(\text{odd}) = \frac{9}{36} = \frac{1}{4}$

A1

Answer: $\frac{1}{4}$

3 marks

(b) even $\times$ anything is even, so odd $\times$ odd is the only case giving an odd product

E1

Answer: proved

1 mark

21 (5 marks)

$$x^2 = 3y^2 + 1 \text{ for } y = 1, 2, 3, 4$$

M1

$$y = 1: x^2 = 4, x = 2$$

A1

$$y = 2: x^2 = 13, \text{ not a perfect square}$$

A1

$$y = 3: x^2 = 28, \text{ not a perfect square}$$

A1

$$y = 4: x^2 = 49, x = 7. \text{ Solutions } (2, 1), (7, 4)$$

A1

Answer: (2, 1) and (7, 4)

5 marks

22 (5 marks)Squares of integers are of the form $4k$ or $4k + 1$

B1

Possible values of $x^2 + y^2$ on division by 4: $0 + 0, 0 + 1, 1 + 1$, giving remainders 0, 1, 2

M1 A1

$$2023 = 4 \times 505 + 3, \text{ remainder } 3$$

M1

No combination gives remainder 3, so no integer solutions

E1

Answer: proved

5 marks

23 (6 marks)

$$\text{Even } n = 2k: n^4 = 16k^4, \text{ remainder } 0$$

B1 M1

$$\text{Odd } n = 2k + 1: n^2 = 4k(k + 1) + 1 = 8t + 1 \text{ (since } k(k + 1) \text{ is even)}$$

M1 A1

$$n^4 = (8t + 1)^2 = 64t^2 + 16t + 1$$

M1

$$= 16(4t^2 + t) + 1, \text{ remainder } 1$$

A1

Answer: proved

6 marks

24 (6 marks)

$$\text{Let } 3p + 1 = m^2 \text{ with } m \text{ a positive integer}$$

B1

$$3p = m^2 - 1 = (m - 1)(m + 1)$$

M1 A1

Since p is prime, the factors of $3p$ are 1, 3, p , $3p$. $(m - 1)(m + 1)$ with $m + 1 > m - 1$

M1

$$m - 1 = 1: m + 1 = 3 = 3p \text{ gives } p = 1, \text{ not prime; } m - 1 = 3: m + 1 = 5 = p, \text{ prime}$$

A1

$$m - 1 = p: m + 1 = p + 2 = 3 \text{ gives } p = 1, \text{ rejected. Only } p = 5 \text{ (} 16 = 4^2 \text{)}$$

A1

Answer: $p = 5$

6 marks

25 (6 marks)

$$\text{Critical values } x = 1 \text{ and } x = -2$$

B1

$$x < -2: (1 - x) + (-2x - 4) = -3x - 3 = 9, x = -4, \text{ valid}$$

M1 A1

$$-2 \leq x < 1: (1 - x) + (2x + 4) = x + 5 = 9, x = 4, \text{ not in this range}$$

M1 A1

$$x \geq 1: (x - 1) + (2x + 4) = 3x + 3 = 9, x = 2, \text{ valid}$$

A1

Answer: $x = -4$ or $x = 2$

6 marks

26 (5 marks)

$$n^3 - 1 = (n - 1)(n^2 + n + 1)$$

M1 A1

$$n = 1: n^3 - 1 = 0, \text{ not prime}$$

B1

$$n = 2: 7, \text{ which is prime}$$

B1

$$n \geq 3: n - 1 \geq 2 \text{ and } n^2 + n + 1 \geq 13 \text{ are both greater than } 1, \text{ so the number is composite}$$

E1

Answer: proved

5 marks

27 (8 marks)**(a)** p is not even and not a multiple of 3, so $p \bmod 6 \in \{1, 5\}$

M1

$$\text{That is } p = 6k + 1 \text{ or } p = 6k - 1 = 6(k - 1) + 5$$

E1

Answer: explained

2 marks

$$\textbf{(b)} \quad p = 6k \pm 1: p^2 - 1 = 36k^2 \pm 12k$$

M1 A1

$$= 12k(3k \pm 1)$$

A1

$$\text{If } k \text{ is even, } k(3k \pm 1) \text{ is even}$$

B1

$$\text{If } k \text{ is odd, } 3k \pm 1 \text{ is even, so } k(3k \pm 1) \text{ is even}$$

M1

$$\text{Hence } 12k(3k \pm 1) = 12 \times 2t = 24t$$

A1

Answer: proved

6 marks

28 (8 marks)

(a) Even $n = 2m$: $n^2 = 4m^2$: remainder 0 if m even, 4 if m odd

M1 A1

Odd n : $n^2 = 8t + 1$ (since $n^2 - 1 = 4k(k + 1)$ is a multiple of 8)

M1 A1

Answer: proved

4 marks

(b) Each square is 0, 1 or 4 modulo 8

B1

Adding three such remainders, the totals modulo 8 are 0, 1, 2, 3, 4, 5, 6 (for example $4 + 4 + 4 = 12 \equiv 4$, $4 + 4 + 1 = 9 \equiv 1$, $4 + 1 + 1 = 6$)

M1 A1

Listing all ten combinations of 0, 1, 4 shows 7 never occurs, so $8k + 7$ is impossible

E1

Answer: proved

4 marks

29 (7 marks)

$n = 3k$, $3k + 1$ or $3k + 2$

B1

$n = 3k$: $n^3 = 27k^3 = 9(3k^3)$, remainder 0

M1 A1

$n = 3k + 1$: $n^3 = 27k^3 + 27k^2 + 9k + 1 = 9(3k^3 + 3k^2 + k) + 1$, remainder 1

M1 A1

$n = 3k + 2$: $n^3 = 27k^3 + 54k^2 + 36k + 8 = 9(3k^3 + 6k^2 + 4k) + 8$, remainder 8

M1 A1

Answer: proved

7 marks

30 (7 marks)

$n^2 + 1 = (n + 1)(n - 1) + 2$

M1 A1

$$\frac{n^2 + 1}{n + 1} = n - 1 + \frac{2}{n + 1}$$

A1

This is an integer exactly when $n + 1$ divides 2

M1

$n + 1 \in \{1, 2, -1, -2\}$

A1

$n = 0, 1, -2, -3$

A1

Check $n = -3$: $\frac{10}{-2} = -5$, an integer

A1

Answer: $n = -3, -2, 0, 1$

7 marks

Proof by Contradiction**1 (3 marks)**

Assume n^2 is odd but n is even, so $n = 2k$

B1

$n^2 = 4k^2 = 2(2k^2)$, which is even

M1

This contradicts n^2 being odd, so n must be odd

E1

Answer: proved

3 marks

2 (2 marks)

Assume there is a greatest even integer N . Then $N + 2$ is even and $N + 2 > N$

M1

This contradicts N being the greatest, so no greatest even integer exists

E1

Answer: proved

2 marks

3 (3 marks)

Assume $x \leq 5$ and $y \leq 5$

B1

Then $x + y \leq 10$

M1

This contradicts $x + y > 10$, so at least one of x and y is greater than 5

E1

Answer: proved

3 marks

4 (3 marks)

Assume integers x, y exist with $2x + 4y = 7$

B1

Then $2(x + 2y) = 7$, so 7 is even

M1

7 is odd: contradiction, so no such integers exist

E1

Answer: proved

3 marks

5 (3 marks)

Assume a real solution exists. Complete the square: $(x - 1)^2 + 2 = 0$

M1 A1

Then $(x - 1)^2 = -2$, but a real square cannot be negative: contradiction

E1

Answer: proved

3 marks

6 (3 marks)

Assume $3n + 1 = 3k$ for some integer k

B1

Then $3(k - n) = 1$, so $k - n = \frac{1}{3}$

M1

$k - n$ is an integer, but $\frac{1}{3}$ is not: contradiction

E1

Answer: proved

3 marks

7 (4 marks)

(a) Assume $r + s = q$ with q rational

B1

Then $s = q - r$

M1

The difference of two rationals is rational, so s is rational: contradiction

E1

Answer: proved

3 marks

(b) 3 is rational and $\sqrt{2}$ is irrational, so by (a) $3 + \sqrt{2}$ is irrational

E1

Answer: proved

1 mark

8 (3 marks)

Let $r \neq 0$ be rational and s irrational. Assume $rs = q$ is rational

B1

Then $s = \frac{q}{r}$

M1

The quotient of two rationals ($r \neq 0$) is rational, so s is rational: contradiction

E1

Answer: proved

3 marks

9 (6 marks)

(a) Suppose m is odd: $m = 2k + 1$, $m^2 = 4k^2 + 4k + 1$ is odd

M1

So m^2 even forces m even

E1

Answer: proved

2 marks

(b) $p^2 = 2q^2$, so p^2 is even and p is even: $p = 2r$

M1 A1

$4r^2 = 2q^2$, so $q^2 = 2r^2$ is even and q is even

M1

p and q both even contradicts 'no common factor'. Hence $\sqrt{2}$ is irrational

E1

Answer: proved

4 marks

10 (4 marks)

Assume $a^2 - b^2 = 2$, so $(a - b)(a + b) = 2$ with $a + b > a - b > 0$

M1

The only positive integer factor pair of 2 is 1×2 , so $a - b = 1$ and $a + b = 2$

A1

Adding: $2a = 3$, so $a = \frac{3}{2}$, not an integer

M1

Contradiction, so no such integers exist

E1

Answer: proved

4 marks

11 (3 marks)

Assume $a \leq 10$ and $b \leq 10$

B1

Both are positive, so $ab \leq 10 \times 10 = 100$

M1

This contradicts $ab > 100$, so at least one of a and b is greater than 10

A1

Answer: proved

3 marks

12 (3 marks)

Assume the lines meet at a point (x, y)

B1

Substituting $y = 3x - 2$: $6x - 2(3x - 2) = 7$, so $4 = 7$

M1

This is a contradiction, so the lines do not intersect

A1

Answer: proved

3 marks

13 (3 marks)

Assume s is the smallest positive real number

B1

Then $\frac{s}{2}$ is positive and $\frac{s}{2} < s$

M1

This contradicts s being the smallest. Hence no smallest positive real number exists

E1

Answer: proved

3 marks

14 (7 marks)

(a) Assume $\sqrt{3} = \frac{p}{q}$, p and q integers with highest common factor 1

B1

$p^2 = 3q^2$, so p^2 is a multiple of 3

M1

Hence $p = 3r$, so $9r^2 = 3q^2$ and $q^2 = 3r^2$

A1

So q^2 , and hence q , is a multiple of 3

M1

p and q share the factor 3: contradiction. So $\sqrt{3}$ is irrational

E1

Answer: proved

5 marks

(b) If $2\sqrt{3}$ were rational, $\sqrt{3} = \frac{1}{2} \times 2\sqrt{3}$ would be rational

M1

This contradicts (a), so $2\sqrt{3}$ is irrational

E1

Answer: proved

2 marks

15 (4 marks)

Assume $\log_{10} 2 = \frac{p}{q}$ with p, q positive integers

B1

Then $10^{p/q} = 2$, so $10^p = 2^q$

M1

$10^p = 2^p \times 5^p$ is a multiple of 5, but 2^q is not

A1

Contradiction, so $\log_{10} 2$ is irrational

E1

Answer: proved

4 marks

16 (3 marks)

Assume no month contains three or more of the people, so each of the 12 months has at most 2

B1

Then there are at most $12 \times 2 = 24$ people

M1

This contradicts there being 25 people, so some month contains at least three

A1

Answer: proved

3 marks

17 (3 marks)

Assume $\frac{1}{x} = \frac{p}{q}$ is rational, with $p \neq 0$ since $\frac{1}{x} \neq 0$

B1

Then $x = \frac{q}{p}$

M1

That is rational, contradicting x being irrational

E1

Answer: proved

3 marks

18 (5 marks)

Assume $\sqrt{6} = \frac{p}{q}$ in lowest terms

B1

$p^2 = 6q^2$ is even, so p is even: $p = 2r$

M1 A1

$4r^2 = 6q^2$, so $3q^2 = 2r^2$ is even, hence q^2 is even and q is even

M1

p and q share the factor 2: contradiction. So $\sqrt{6}$ is irrational

E1

Answer: proved

5 marks

19 (4 marks)

Assume $a + \frac{1}{a} < 2$ for some $a > 0$

B1

Multiply by $a > 0$: $a^2 + 1 < 2a$

M1

$a^2 - 2a + 1 < 0$, i.e. $(a - 1)^2 < 0$

A1

A real square cannot be negative: contradiction

E1

Answer: proved

4 marks

20 (5 marks)

Assume $\sqrt[3]{2} = \frac{p}{q}$, with p and q integers having no common factor

B1

$p^3 = 2q^3$, so p^3 is even and p is even: $p = 2r$

M1 A1

$8r^3 = 2q^3$, so $q^3 = 4r^3$ is even and q is even

M1

p and q share the factor 2: contradiction, so $\sqrt[3]{2}$ is irrational

E1

Answer: proved

5 marks

21 (6 marks)

Assume there are finitely many primes $p_1, p_2, \dots, p_n$

B1

Let $N = p_1 p_2 \dots p_n + 1$

M1

$N > 1$, so N has at least one prime factor p, and p must be one of $p_1, \dots, p_n$

B1

p divides N and p divides $p_1 p_2 \dots p_n$

M1

so p divides the difference $N - p_1 p_2 \dots p_n = 1$

A1

No prime divides 1: contradiction, so there are infinitely many primes

E1

Answer: proved

6 marks

22 (6 marks)

(a) $p^2 = 6q^2$, so p is even, $p = 2r$

M1

$4r^2 = 6q^2$, $3q^2 = 2r^2$, so q is even

A1

Common factor 2 contradicts lowest terms

E1

Answer: proved

3 marks

(b) Assume $\sqrt{2} + \sqrt{3} = r$, rational. Square: $5 + 2\sqrt{6} = r^2$

M1

$\sqrt{6} = \frac{r^2 - 5}{2}$, which is rational

A1

This contradicts part (a), so $\sqrt{2} + \sqrt{3}$ is irrational

E1

Answer: proved

3 marks

23 (5 marks)

Assume $4k + 3 = a^2 + b^2$ for integers a and b

B1

The square of an integer is $4m$ (even) or $4m + 1$ (odd)

M1

So $a^2 + b^2$ has remainder 0, 1 or 2 when divided by 4

A1

But $4k + 3$ has remainder 3: contradiction

M1

So $4k + 3$ is never a sum of two squares

E1

Answer: proved

5 marks

24 (6 marks)

Assume a and b are both odd, so a^2 and b^2 are each of the form $4m + 1$

B1

$a^2 + b^2 = 4(m + n) + 2$, remainder 2 on division by 4

M1 A1

c^2 is the sum, so c^2 has remainder 2

M1

A square has remainder 0 or 1 on division by 4 (even: $4t$, odd: $4t + 1$)

A1

Contradiction, so a and b are not both odd

E1

Answer: proved

6 marks

25 (5 marks)

(a) Assume $\log_2 3 = \frac{p}{q}$ with positive integers p, q

B1

$2^{p/q} = 3$, so $2^p = 3^q$

M1

The left side is even and the right side is odd

A1

Contradiction, so $\log_2 3$ is irrational

E1

Answer: proved

4 marks

(b) If $2^p = 3^q$ with p and q positive integers, then $\log_2 3 = p/q$ would be rational, contradicting (a)

E1

Answer: proved

1 mark

26 (5 marks)**(a)** Assume a and b are both rational

B1

Then $a + b$ is rational (sum of two rationals)

M1

This contradicts $a + b$ being irrational

E1

Answer: proved

3 marks

(b) Take $a = \sqrt{2}$, $b = -\sqrt{2}$: a is irrational

M1

but $a + b = 0$ is rational

A1

Answer: $a = \sqrt{2}$, $b = -\sqrt{2}$

2 marks

27 (7 marks)Assume a rational root $x = \frac{p}{q}$ in lowest terms, so p and q are not both even

B1

$$a p^2 + b p q + c q^2 = 0$$

M1

 p, q both odd: $a p^2 + b p q + c q^2$ is odd + odd + odd = odd

A1

 p odd, q even: the first term is odd and the other two are even, so the sum is odd

A1

 p even, q odd: only $c q^2$ is odd, so the sum is odd

A1

In every case the left side is odd, but it equals 0 (even): contradiction

M1

So there are no rational roots

E1

Answer: proved

7 marks

28 (8 marks)Assume a root $x = \frac{p}{q}$, integers p, q with no common factor, $q > 0$

B1

Multiply by q^3 : $p^3 + p q^2 + q^3 = 0$

M1 A1

 p, q both odd: odd + odd + odd is odd

M1

 p odd, q even: odd + even + even is odd

A1

 p even, q odd: even + even + odd is odd

A1

 p, q both even is excluded (no common factor)

B1

Left side is always odd but equals 0: contradiction

E1

Answer: proved

8 marks

29 (9 marks)

(a) $(4m + 1)(4n + 1) = 16mn + 4m + 4n + 1$

M1

$$= 4(4mn + m + n) + 1$$

A1

Answer: proved

2 marks

(b) $N = 4(p_1 \dots p_n - 1) + 3$

M1

 N is odd, so every prime factor is odd

A1

Answer: shown

2 marks

(c) If every prime factor of N were of the form $4m + 1$, then N would be of the form $4t + 1$ by (a)

M1 A1

 N has remainder 3, so some prime factor q is of the form $4m + 3$

A1

So q is one of $p_1, \dots, p_n$; then q divides $4p_1 \dots p_n$ and q divides N

M1

so q divides $4p_1 \dots p_n - N = 1$: contradiction

E1

Answer: proved

5 marks

30 (6 marks)

(a) $z^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2} \times \sqrt{2}}$

M1

$$= (\sqrt{2})^2 = 2$$

A1

Answer: shown

2 marks

(b) Take $a = b = \sqrt{2}$: both irrational and $a^b = z$ is rational

E1

Answer: $a = b = \sqrt{2}$

1 mark

(c) Take $a = z$ (irrational) and $b = \sqrt{2}$ (irrational)

M1

$$a^b = z^{\sqrt{2}} = 2 \text{ by (a), which is rational}$$

A1

 z is either rational or irrational, so in both cases such a and b exist

E1

Answer: proved

3 marks

Proof by Counterexample

1 (2 marks)

Take 2

M1

2 is prime but even, so the statement is false

A1

Answer: 2

2 marks

2 (2 marks)

Take $x = \frac{1}{2}$ (or $x = 1$, or $x = 0$)

M1

$x^2 = \frac{1}{4} < \frac{1}{2} = x$, so the statement is false

A1

Answer: $x = \frac{1}{2}$

2 marks

3 (2 marks)

Take $a = 1$ and $b = -2$: $a > b$

M1

$a^2 = 1 < 4 = b^2$, so the statement is false

A1

Answer: $a = 1$, $b = -2$

2 marks

4 (2 marks)

Take $a = b = 1$: $\sqrt{a+b} = \sqrt{2} \approx 1.414$

M1

$\sqrt{a} + \sqrt{b} = 2$, which is different: the statement is false

A1

Answer: $a = b = 1$

2 marks

5 (3 marks)

Try $n = 40$

M1

$40^2 + 40 + 41 = 1681$

A1

$1681 = 41^2$, which is not prime

A1

Answer: $n = 40$ gives 41^2

3 marks

6 (3 marks)

$p = 2, 3, 5, 7$ give 3, 7, 31, 127, all prime, so test $p = 11$

M1

$2^{11} - 1 = 2047$

A1

$2047 = 23 \times 89$, which is not prime

A1

Answer: $p = 11$

3 marks

7 (3 marks)

(a) Take $\sqrt{2}$ and $-\sqrt{2}$, both irrational

M1

Their sum is 0, which is rational

A1

Answer: $\sqrt{2} + (-\sqrt{2}) = 0$

2 marks

(b) $\sqrt{2} \times \sqrt{2} = 2$, which is rational

B1

Answer: $\sqrt{2} \times \sqrt{2} = 2$

1 mark

8 (3 marks)

Take $A = B = 30^\circ$

M1

$\sin 60^\circ = \frac{\sqrt{3}}{2} \approx 0.866$

A1

$\sin 30^\circ + \sin 30^\circ = 1 \neq 0.866$

A1

Answer: $A = B = 30^\circ$

3 marks

9 (3 marks)

Take $a = 1$ and $b = -1$, so $a > b$

M1

$\frac{1}{a} = 1$ and $\frac{1}{b} = -1$

A1

$1 < -1$ is false, so the statement is false

A1

Answer: $a = 1$, $b = -1$

3 marks

10 (2 marks)Counterexample: a rhombus with sides 5 cm and angles 60° and 120°

B1

All four sides are equal but its angles are not 90° , so it is not a square

B1

Answer: a rhombus

2 marks

11 (2 marks)Take $a = 1$ and $b = 1$: $\ln(a + b) = \ln 2 = 0.693\dots$

B1

 $\ln a + \ln b = 0 + 0 = 0 \neq \ln 2$, so the statement is false

B1

Answer: $a = b = 1$

2 marks

12 (3 marks)Take $a = 1$, $b = -1$

M1

 $|a + b| = 0$

A1

 $|a| + |b| = 2 \neq 0$

A1

Answer: $a = 1$, $b = -1$

3 marks

13 (3 marks)

Take 2 and 3, both prime

M1

 $2 + 3 = 5$

A1

5 is prime, so the statement is false

A1

Answer: $2 + 3 = 5$

3 marks

14 (4 marks)(a) $f'(x) = 3x^2$, so $f'(0) = 0$

M1 A1

For $x < 0$, $f(x) < 0 = f(0)$ and for $x > 0$, $f(x) > 0 = f(0)$: neither a maximum nor a minimum

A1

Answer: $x = 0$, $f = x^3$

3 marks

(b) A point of inflection

B1

Answer: point of inflection

1 mark

15 (3 marks)

Take 1, 2, 3

M1

 $1 \times 2 \times 3 = 6$

A1

6 is not divisible by 12

A1

Answer: $1 \times 2 \times 3 = 6$

3 marks

16 (3 marks)Choose the cubic $y = x^3 + x$

B1

 $x^3 + x = x(x^2 + 1) = 0$, and $x^2 + 1 > 0$ for all real x

M1

So the only root is $x = 0$: the graph crosses the x -axis once, not three times

A1

Answer: $y = x^3 + x$

3 marks

17 (4 marks)(a) x^3 is increasing: $x_1 < x_2$ implies $x_1^3 < x_2^3$

M1

 $f'(x) = 3x^2$, $f'(0) = 0$, not greater than 0

M1 A1

Answer: $f = x^3$ at $x = 0$

3 marks

(b) If f is increasing then $f'(x) \geq 0$ for all x

B1

Answer: $f'(x) \geq 0$

1 mark

18 (3 marks)Take $a = 0$: $x^2 = 0$ has only one solution $x = 0$

M1 A1

(or $a = -1$: $x^2 = -1$ has no real solutions)

A1

Answer: $a = 0$ or $a = -1$

3 marks

19 (3 marks)

$n = 1, 2, 3: 1 \leq 2, 2 \leq 4, 6 \leq 8$

B1

$n = 4: 4! = 24$

M1

$2^4 = 16$ and $24 > 16$, so the claim is false

A1

Answer: $n = 4$

3 marks

20 (3 marks)

$k = 1, 2, 3, 4, 5$ give 5, 11, 17, 23, 29, all prime

M1

$k = 6$ gives 35

A1

$35 = 5 \times 7$, not prime

A1

Answer: $k = 6$

3 marks

21 (4 marks)

(a) Take $m = 2$ and $n = 8$

M1

$mn = 16 = 4^2$ but neither 2 nor 8 is a perfect square

A1

Answer: $m = 2, n = 8$

2 marks

(b) No prime appears in both m and n , so the primes of m each appear an even number of times in mn , hence in m

M1

Similarly for n , so both are perfect squares

E1

Answer: shown

2 marks

22 (6 marks)

(a) $x^2 = 2x - 5$, so $x^2 - 2x + 5 = 0$

M1

Discriminant $= (-2)^2 - 4 \times 1 \times 5 = -16 < 0$

A1

No real solutions, so the line never meets the curve

A1

Answer: discriminant -16

3 marks

(b) $x^2 - mx - c = 0$

M1

Discriminant $m^2 + 4c < 0$

A1

$$c < -\frac{m^2}{4}$$

A1

$$\text{Answer: } c < -\frac{m^2}{4}$$

3 marks

23 (6 marks)

(a) $n = 1: 1^4 + 4 = 5$

M1

5 is prime

A1

Answer: $n = 1$

2 marks

(b) $(n^2 + 2)^2 - (2n)^2 = n^4 + 4n^2 + 4 - 4n^2$

M1

$= n^4 + 4$, and a difference of squares factorises as stated

A1

Answer: shown

2 marks

(c) $n^2 + 2n + 2 > 1$ and $n^2 - 2n + 2 = (n - 1)^2 + 1 \geq 2$ for $n \geq 2$

M1

Both factors exceed 1, so the number is composite

E1

Answer: proved

2 marks

24 (5 marks)

(a) Take $a = b = 4$

M1

$$\frac{a+b}{2} = 4 \text{ and } \sqrt{ab} = 4, \text{ equal, not greater}$$

A1

Answer: $a = b = 4$

2 marks

(b) $(\sqrt{a} - \sqrt{b})^2 \geq 0$

M1

$$a - 2\sqrt{ab} + b \geq 0$$

A1

$$\text{so } \frac{a+b}{2} \geq \sqrt{ab}$$

A1

Answer: proved

3 marks

25 (4 marks)

(a) $e^{-1} = 0.368$

M1

$x^2 = 1$ and $0.368 < 1$, so $e^x < x^2$ at $x = -1$

A1

Answer: $x = -1$

2 marks

(b) The curves cross once, at a negative value of x

B1

For x less than that value the parabola is above the exponential: the claim fails for all $x < -0.70$ (approximately)

B1

Answer: $x < \text{about } -0.7$

2 marks

26 (6 marks)

(a) False: $n = 2$, $n^2 = 4$ is divisible by 4 but 2 is not divisible by 4

M1 A1

Answer: false, $n = 2$

2 marks

(b) True. If n were odd, $n = 2k + 1$, $n^2 = 4k^2 + 4k + 1$ is odd, not divisible by 4

M1

So n^2 divisible by 4 forces n even

E1

Answer: true

2 marks

(c) True. If n were odd, n^3 would be odd (odd $\times$ odd $\times$ odd), so not divisible by 8

M1

So n is even

E1

Answer: true; note $n = 2$ gives 8, $n = 4$ gives 64

2 marks

27 (7 marks)

(a) Let $n = ab$ with $a, b > 1$ integers

B1

$2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + \dots + 2^a + 1)$

M1 A1

Both factors exceed 1 ($2^a - 1 \geq 3$), so composite

E1

Answer: proved

4 marks

(b) $n = 11$

B1

$2^{11} - 1 = 2047$

M1

$2047 = 23 \times 89$, composite

A1

Answer: $n = 11$

3 marks

28 (7 marks)

(a) $a = 3$, $b = 4$, $c = 5$

M1

$3^2 + 4^2 = 5^2$ but neither 3 nor 4 is divisible by 5

A1

Answer: 3, 4, 5

2 marks

(b) Squares of integers leave remainder 0, 1 or 4 on division by 5 (check $n = 5k$, $5k \pm 1$, $5k \pm 2$)

M1 A1

Suppose neither a nor b is divisible by 5: $a^2, b^2 \in 1, 4 \pmod{5}$

M1

$a^2 + b^2 \in 2, 0, 3 \pmod{5}$

A1

c^2 must be 0, 1 or 4, so $c^2 \equiv 0$ and 5 divides c

E1

Answer: proved

5 marks

29 (7 marks)

(a) $5 \times 128 + 1 = 641$

B1

$16 + 625 = 641$

B1

Answer: shown

2 marks

(b) From $5 \times 2^7 + 1 = 641$: $5 \times 2^7 \equiv -1 \pmod{641}$

M1

Raise to the 4th power: $5^4 \times 2^{28} \equiv 1$

M1

From $641 = 2^4 + 5^4$: $5^4 \equiv -2^4$

M1

So $-2^4 \times 2^{28} \equiv 1$, i.e. $2^{32} \equiv -1 \pmod{641}$

A1

Hence 641 divides $2^{32} + 1$, which is larger than 641, so F_5 is composite

E1

Answer: proved

5 marks

30 (5 marks)

(a) $80^2 - 79 \times 80 + 1601 = 6400 - 6320 + 1601$

M1

$= 1681 = 41^2$, not prime

A1

Answer: $n = 80$

2 marks

(b) $(n + 1)(n + 2) - 41(2n - 39) = n^2 + 3n + 2 - 82n + 1599 = n^2 - 79n + 1601$

M1 A1

$n = 41k - 1$ or $41k - 2$ makes $(n + 1)(n + 2)$ a multiple of 41, so the expression is a multiple of 41 and is larger than 41 when $k \geq 2$: composite (e.g. $n = 80$ gives 1681, $n = 81$ gives $1763 = 41 \times 43$)

E1

Answer: shown

3 marks