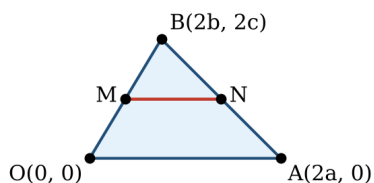


Proof by Deduction

- 1 Prove that the sum of any two even numbers is even. [2]
- 2 Show that $(n + 1)^2 - n^2 \equiv 2n + 1$ and hence that the difference between the squares of two consecutive integers is always odd. [2]
- 3 Prove that the product of any two odd numbers is odd. [3]
- 4 Show that $(x + 3)^2 - 6x - 9 \equiv x^2$. [2]
- 5 Prove that the sum of any three consecutive integers is a multiple of 3. [3]
- 6 Show that $2^{n+1} + 2^n = 3 \times 2^n$. [2]
- 7 Prove that the sum of any two consecutive odd numbers is a multiple of 4. [3]
- 8 Prove that $n^2 + n$ is even for every integer n . [3]
- 9 Prove that $x^2 - 6x + 10 > 0$ for all real values of x . [3]
- 10 The n th triangular number is $T_n = \frac{n(n+1)}{2}$, so the first few triangular numbers are 1, 3, 6, 10, ...
Prove that the sum of two consecutive triangular numbers is always a square number. [3]
- 11 Prove that the sum of the first n odd numbers $1 + 3 + 5 + \dots + (2n - 1)$ is n^2 . [3]
- 12 Prove that $(a + b)^2 \geq 4ab$ for all real numbers a and b .
State when equality holds. [3]
- 13 Prove that $n^3 - n$ is divisible by 6 for every positive integer n . [4]
- 14 Prove that the square of any odd number is of the form $8k + 1$, where k is an integer. [4]
- 15 Prove that the product of any two consecutive even numbers is a multiple of 8. [4]
- 16 (This question is about the arithmetic mean and the geometric mean of two positive numbers.)
(a) Show that $(\sqrt{a} - \sqrt{b})^2 \geq 0$ gives $a + b \geq 2\sqrt{ab}$ for positive a and b . [2]
(b) Hence prove that $\frac{a+b}{2} \geq \sqrt{ab}$, and find the condition for equality. [2]
- 17 Positive integers m and n satisfy $m > n$.
(a) Prove that $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$. [3]
(b) Use $m = 5$, $n = 2$ to write down a Pythagorean triple. [1]
- 18 A student 'proves' that $1 = 2$ as follows.
Let $a = b$. Then $a^2 = ab$, so $a^2 - b^2 = ab - b^2$.
Factorising gives $(a + b)(a - b) = b(a - b)$, so $a + b = b$.
Since $a = b$, this says $2b = b$, and so $2 = 1$.
(a) Identify the step in which the argument fails and explain why it is not valid. [2]
(b) Another student writes: ' $n^2 > n$ for every integer n , because $n^2 = n \times n$ and multiplying makes a number bigger.' Give two values of n for which the statement is false. [2]

- 19** Prove that the solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, by completing the square. [4]

- 20** The diagram shows triangle OAB with O(0, 0), A(2a, 0) and B(2b, 2c). The points M and N are the midpoints of OB and AB respectively.



- (a) Find the coordinates of M and N. [2]
 (b) Hence prove that MN is parallel to OA and that $MN = \frac{1}{2}OA$. [3]

- 21** Prove that the sum of the squares of three consecutive integers is never a multiple of 3. [5]

- 22** Prove that $\tan x + \frac{1}{\tan x} \equiv \frac{1}{\sin x \cos x}$. [5]

- 23** Prove that the curve $y = x^3 + 3x + 5$ is increasing for all real x . [5]

- 24** Prove that, for every real value of k , the line $y = kx + 2$ meets the curve $y = x^2 + 1$ at two distinct points. [4]

- 25** Prove that the equation $x^2 + kx + k + 3 = 0$ has real roots if and only if $k \leq -2$ or $k \geq 6$. [6]

- 26** (This question is about the sum of the series $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{N(N+1)}$.)

- (a) Prove that $\frac{1}{n(n+1)} \equiv \frac{1}{n} - \frac{1}{n+1}$. [2]
 (b) Hence prove that the sum of the first N terms is $\frac{N}{N+1}$. [3]
 (c) Deduce that the sum of the series is always less than 1. [1]

- 27** (This question is about the difference between the cubes of consecutive integers.)

- (a) Show that $(n+1)^3 - n^3 = 3n(n+1) + 1$. [2]
 (b) Hence prove that the difference between the cubes of two consecutive integers is of the form $6k + 1$. [3]
 (c) Hence prove that the difference between the cubes of two consecutive positive integers is never divisible by 2 or by 3. [2]

- 28** For positive numbers a , b and c , none of them equal to 1, let $L = \log_a b$.

- (a) Prove that $\log_b a = \frac{1}{L}$. [3]
 (b) Prove that $\log_a b \times \log_b c = \log_a c$. [3]
 (c) Hence evaluate $\log_2 5 \times \log_5 8$. [2]

- 29** (This question is about numbers which are the sum of two squares.)

- (a) Prove that $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$. [4]
 (b) Deduce that the product of two numbers, each of which is the sum of two squares of integers, is also the sum of two squares of integers. [1]
 (c) Given that $13 = 2^2 + 3^2$ and $5 = 1^2 + 2^2$, use the identity to write 65 as a sum of two squares in two different ways. [3]

- 30** Prove that the sum of the cubes of any three consecutive integers is divisible by 9.
- (a) Show that $(n - 1)^3 + n^3 + (n + 1)^3 = 3n(n^2 + 2)$. [3]
- (b) Prove that $n(n^2 + 2)$ is a multiple of 3 for every integer n , by considering the possible remainders of n on division by 3. [4]
- (c) Hence complete the proof. [1]

Proof by Exhaustion

- 1** Show that $n^2 - n + 11$ is a prime number for each of $n = 1, 2, 3, 4, 5$. [2]
- 2** Prove that no square number ends in the digit 2, 3, 7 or 8. [3]
- 3** For each of $n = 1, 2, 3, 4, 5, 6$, determine whether $n! > 2^n$. [3]
- 4** Show that every prime number p with $5 \leq p < 30$ can be written as $6k - 1$ or $6k + 1$ for some integer k . [3]
- 5** Prove that the square of any integer is either a multiple of 3 or one more than a multiple of 3. [3]
- 6** Show that the equation $|x| = 2x - 3$ has exactly one solution, by considering the cases $x \geq 0$ and $x < 0$. [3]
- 7** Prove that the square of any integer is either a multiple of 4 or one more than a multiple of 4. [3]
- 8** Prove that $n^2 + n + 1$ is odd for every integer n , by considering the cases n even and n odd. [3]
- 9** Prove that $n^3 - n$ is a multiple of 3 for every integer n , by considering $n = 3k, 3k + 1$ and $3k + 2$. [4]
- 10** Solve $|2x - 1| = x + 4$. [4]
- 11** Prove that the sum of the squares of two integers is never of the form $4k + 3$. [4]
- 12** Find all integers n for which $n^2 + 3$ is divisible by 4, and prove your answer. [4]
- 13** By considering the cases $3x - 2 \geq 0$ and $3x - 2 < 0$ separately, solve the inequality $|3x - 2| > 4$. [4]
- 14** Prove that $n^5 - n$ is divisible by 5 for every integer n . [5]
- 15** Prove that there are no integers m and n such that $m^2 = 4n + 2$. [4]
- 16** Find all prime numbers p such that $p + 2$ and $p + 4$ are also prime, and prove that there are no others. [5]
- 17** Prove that $|x - 1| + |x - 4| \geq 3$ for all real x , and state when equality holds. [5]
- 18** Find all integer solutions of $x^2 - y^2 = 7$. [5]
- 19** Prove that the equation $x^2 = 5y + 2$ has no solutions in which x and y are both integers. [4]

- 20** Two fair six-sided dice are rolled and the scores are multiplied. The table shows all 36 products.

		second die					
		1	2	3	4	5	6
first die	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

- (a) Use the table to show that the probability that the product is odd is $\frac{1}{4}$. [3]
- (b) Prove, by considering the parity of each score, that the product is odd only when both scores are odd. [1]
-
- 21** Find all positive integer solutions of $x^2 - 3y^2 = 1$ with $y \leq 4$. [5]
-
- 22** Prove that the equation $x^2 + y^2 = 2023$ has no solutions in integers. [5]
-
- 23** Prove that the fourth power of any integer leaves remainder 0 or 1 when divided by 16. [6]
-
- 24** Find all prime numbers p for which $3p + 1$ is a perfect square, justifying that there are no others. [6]
-
- 25** Solve $|x - 1| + |2x + 4| = 9$, considering the ranges $x < -2$, $-2 \leq x < 1$ and $x \geq 1$. [6]
-
- 26** Prove that the only prime number of the form $n^3 - 1$, where n is a positive integer, is 7. [5]
-
- 27** Let p be a prime number with $p > 3$.
 (a) Explain why p can be written as $6k + 1$ or $6k - 1$ for some integer k . [2]
 (b) Hence prove that $p^2 - 1$ is divisible by 24. [6]
-
- 28** (This question is about numbers of the form $8k + 7$.)
 (a) Prove that the square of any integer leaves remainder 0, 1 or 4 when divided by 8. [4]
 (b) Hence prove that no number of the form $8k + 7$ can be written as the sum of the squares of three integers. [4]
-
- 29** Prove that the cube of any integer leaves remainder 0, 1 or 8 when divided by 9. [7]
-
- 30** Find all integers n , other than $n = -1$, for which $\frac{n^2 + 1}{n + 1}$ is an integer. [7]

Proof by Contradiction

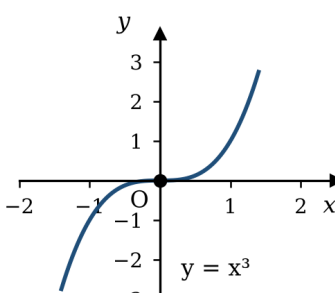
- 1** Prove by contradiction that if n^2 is odd, where n is an integer, then n is odd. [3]
-
- 2** Prove by contradiction that there is no greatest even integer. [2]
-
- 3** Two real numbers x and y satisfy $x + y > 10$. Prove by contradiction that at least one of them is greater than 5. [3]
-
- 4** Prove by contradiction that there are no integers x and y such that $2x + 4y = 7$. [3]
-
- 5** Prove by contradiction that there is no real number x for which $x^2 - 2x + 3 = 0$. [3]

6	Prove by contradiction that $3n + 1$ is never a multiple of 3, for any integer n .	[3]
7	(This question is about the sum of a rational number and an irrational number.) (a) Prove by contradiction that if r is rational and s is irrational then $r + s$ is irrational. (b) Deduce that $3 + \sqrt{2}$ is irrational, given that $\sqrt{2}$ is irrational.	[3] [1]
8	Prove by contradiction that the product of a non-zero rational number and an irrational number is irrational.	[3]
9	(This question proves that $\sqrt{2}$ is irrational.) (a) Prove that if m is an integer and m^2 is even, then m is even. (b) Assume $\sqrt{2} = \frac{p}{q}$ where p and q are integers with no common factor. Use (a) to obtain a contradiction.	[2] [4]
10	Prove by contradiction that there are no positive integers a and b such that $a^2 - b^2 = 2$.	[4]
11	a and b are positive integers with $ab > 100$. Prove by contradiction that at least one of a and b is greater than 10.	[3]
12	Prove by contradiction that the lines $y = 3x - 2$ and $6x - 2y = 7$ do not intersect.	[3]
13	Prove by contradiction that there is no smallest positive real number.	[3]
14	(This question is about irrational numbers.) (a) Prove by contradiction that $\sqrt{3}$ is irrational. You may assume that if m^2 is a multiple of 3 then m is a multiple of 3. (b) Deduce that $2\sqrt{3}$ is irrational.	[5] [2]
15	Prove by contradiction that $\log_{10} 2$ is irrational.	[4]
16	Prove by contradiction that, among any 25 people, at least three were born in the same calendar month.	[3]
17	Given that x is a non-zero irrational number, prove by contradiction that $\frac{1}{x}$ is irrational.	[3]
18	Prove by contradiction that $\sqrt{6}$ is irrational. You may assume that if a prime p divides ab then p divides a or p divides b .	[5]
19	Prove by contradiction that $a + \frac{1}{a} \geq 2$ for every positive real number a .	[4]
20	Prove by contradiction that the cube root of 2 is irrational. You may assume that if m^3 is even then m is even.	[5]
21	Prove by contradiction that there are infinitely many prime numbers.	[6]
22	(This question is about the number $\sqrt{2} + \sqrt{3}$.) (a) Prove by contradiction that $\sqrt{6}$ is irrational: assume $\sqrt{6} = \frac{p}{q}$ in lowest terms and show that p and q must both be even. (b) Hence prove by contradiction that $\sqrt{2} + \sqrt{3}$ is irrational.	[3] [3]
23	Prove by contradiction that no integer of the form $4k + 3$ can be written as the sum of two squares of integers.	[5]

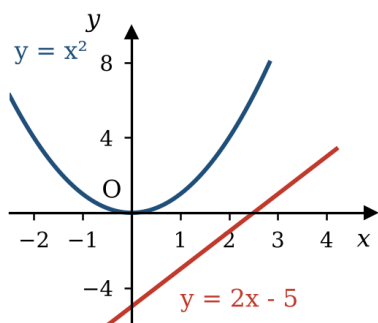
- 24** a, b and c are positive integers with $a^2 + b^2 = c^2$. Prove by contradiction that a and b cannot both be odd. [6]
- 25** (This question is about the number $\log_2 3$.)
 (a) Prove by contradiction that $\log_2 3$ is irrational. [4]
 (b) Deduce that there are no positive integers p and q with $2^p = 3^q$. [1]
- 26** (This question is about two real numbers a and b.)
 (a) Prove by contradiction that if $a + b$ is irrational then at least one of a and b is irrational. [3]
 (b) Show that the converse is false by giving a counterexample. [2]
- 27** The quadratic equation $ax^2 + bx + c = 0$ has odd integer coefficients a, b and c. Prove by contradiction that the equation has no rational roots. [7]
- 28** Prove that the equation $x^3 + x + 1 = 0$ has no rational roots. [8]
- 29** (This question proves that there are infinitely many primes of the form $4k + 3$.)
 (a) Prove that the product of two numbers of the form $4m + 1$ is of the form $4t + 1$. [2]
 (b) Assume there are finitely many such primes, $3 = p_1, 7 = p_2, \dots, p_n$. Let $N = 4p_1p_2\dots p_n - 1$. Show that N has remainder 3 on division by 4 and that all its prime factors are odd. [2]
 (c) Hence complete the proof by contradiction. [5]
- 30** (This question is about irrational powers.) It is given that $\sqrt{2}$ is irrational. Let $z = (\sqrt{2})^{\sqrt{2}}$.
 (a) Show that $z^{\sqrt{2}} = 2$. [2]
 (b) Suppose z is rational. Explain what this shows about irrational numbers a and b with a^b rational. [1]
 (c) Suppose z is irrational. Show that there are again irrational numbers a and b with a^b rational, and hence prove that there exist irrational numbers a and b such that a^b is rational. [3]

Proof by Counterexample

- 1** Show, by giving a counterexample, that the statement 'all prime numbers are odd' is false. [2]
- 2** Show that the statement ' $x^2 > x$ for all real numbers x' is false. [2]
- 3** Disprove the statement 'if $a > b$ then $a^2 > b^2$ '. [2]
- 4** Disprove the statement ' $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$ '. [2]
- 5** A student claims that $n^2 + n + 41$ is prime for every positive integer n. Show that the claim is false. [3]
- 6** It is claimed that $2^p - 1$ is prime for every prime number p. Show that the claim is false. [3]
- 7** (This question is about the sum of two irrational numbers.)
 (a) Show that the statement 'the sum of any two irrational numbers is irrational' is false. [2]
 (b) Give a counterexample to 'the product of any two irrational numbers is irrational'. [1]
- 8** Show that the statement ' $\sin(A + B) = \sin A + \sin B$ for all angles A and B' is false. [3]

- 9** Disprove the statement 'if $a > b$ then $\frac{1}{a} < \frac{1}{b}$ '. [3]
- 10** Disprove the statement 'a quadrilateral whose four sides are all equal in length is a square'. [2]
- 11** Show that the statement ' $\ln(a + b) = \ln a + \ln b$ for all positive real numbers a and b ' is false. [2]
- 12** Disprove the statement ' $|a + b| = |a| + |b|$ for all real numbers a and b '. [3]
- 13** A student says: 'The sum of two prime numbers is always composite.' Disprove this. [3]
- 14** A student claims: 'If $f'(a) = 0$ then the graph of $y = f(x)$ has a maximum or a minimum at $x = a$.' The diagram shows the graph of $y = x^3$.
- 
- (a) Show that $f(x) = x^3$ is a counterexample to the claim. [3]
- (b) State the name of the type of stationary point at $x = 0$. [1]
- 15** Disprove the statement 'the product of any three consecutive positive integers is divisible by 12'. [3]
- 16** Disprove the statement 'the graph of every cubic function $y = ax^3 + bx^2 + cx + d$ crosses the x -axis at exactly three points'. [3]
- 17** A student writes: 'If f is an increasing function then $f'(x) > 0$ for all x .'
- (a) Use $f(x) = x^3$ to show that the statement is false. [3]
- (b) State a correct version of the statement. [1]
- 18** Disprove the statement 'the equation $x^2 = a$ has two real solutions for every real number a '. [3]
- 19** It is claimed that $n! \leq 2^n$ for every positive integer n .
Show that the claim is true for $n = 1, 2, 3$ but false in general. [3]
- 20** A student claims that every number of the form $6k - 1$, where k is a positive integer, is prime.
Show that the claim is false. [3]
- 21** (This question is about the statement 'if mn is a perfect square then m and n are both perfect squares', where m and n are positive integers.)
- (a) Show that the statement is false. [2]
- (b) Show that the statement is true when m and n have no common factor other than 1, given that every prime in the factorisation of a perfect square occurs an even number of times. [2]

- 22** A student claims that every straight line of the form $y = mx + c$ meets the curve $y = x^2$. The diagram shows $y = x^2$ and the line $y = 2x - 5$.

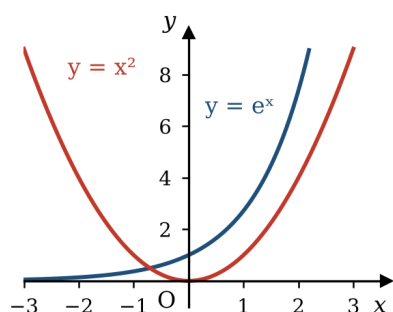


- (a) Show that the line $y = 2x - 5$ is a counterexample. [3]
 (b) Find, in terms of m , the condition on c for the line $y = mx + c$ to miss the curve $y = x^2$. [3]

- 23** A student claims that $n^4 + 4$ is composite for every positive integer n .
 (a) Show that the claim is false. [2]
 (b) Show that $n^4 + 4 = (n^2 + 2n + 2)(n^2 - 2n + 2)$. [2]
 (c) Hence prove that $n^4 + 4$ is composite for every integer $n \geq 2$. [2]

- 24** A student claims that $\frac{a+b}{2} > \sqrt{ab}$ for all positive real numbers a and b .
 (a) Show that the claim is false. [2]
 (b) Prove the corrected statement $\frac{a+b}{2} \geq \sqrt{ab}$ for all positive a and b . [3]

- 25** A student claims that $e^x > x^2$ for all real x . The diagram shows $y = e^x$ and $y = x^2$.



- (a) Show that the claim is false, by evaluating both sides at $x = -1$. [2]
 (b) Use the diagram to describe the set of values of x for which the claim fails. [2]

- 26** n is an integer. For each statement, either give a counterexample or prove that it is true.
 (a) If n^2 is divisible by 4 then n is divisible by 4. [2]
 (b) If n^2 is divisible by 4 then n is even. [2]
 (c) If n^3 is divisible by 8 then n is divisible by 2. [2]

- 27** (This question is about numbers of the form $2^n - 1$.)
 (a) Prove that if n is composite, then $2^n - 1$ is composite. [4]
 (b) A student suggests the converse: if n is prime then $2^n - 1$ is prime. Disprove this with a counterexample. [3]

- 28** a , b and c are positive integers with $a^2 + b^2 = c^2$.
 (a) A student claims that one of a and b is always divisible by 5. Give a counterexample. [2]
 (b) Prove the corrected claim: at least one of a , b and c is divisible by 5. [5]

- 29** Fermat conjectured that $F_n = 2^{2^n} + 1$ is prime for every $n \geq 0$. F_0 to F_4 (3, 5, 17, 257, 65 537) are all prime.
- (a) Show that $641 = 5 \times 2^7 + 1 = 2^4 + 5^4$. [2]
- (b) Hence prove that 641 divides $F_5 = 2^{32} + 1$, so that Fermat's conjecture is false. [5]
-
- 30** A student discovers that the expression $n^2 - 79n + 1601$ gives a prime number for every integer n from 0 to 79, and concludes that it is prime for every positive integer n .
- (a) Show that the conclusion is false by evaluating the expression at $n = 80$. [2]
- (b) Show that $n^2 - 79n + 1601 = (n + 1)(n + 2) - 41(2n - 39)$, and hence that the expression is composite for $n = 41k - 1$ and $n = 41k - 2$ for every integer $k \geq 2$. [3]
-