

Discrete Random Variables

1 (2 marks)

(a) The probabilities sum to 1, so $p = 1 - 0.1 - 0.3 - 0.2 = 0.4$

M1

Answer: $p = 0.4$

1 mark

(b) $P(X > 2) = P(X = 3) + P(X = 4) = 0.4 + 0.2 = 0.6$

A1

Answer: 0.6

1 mark

2 (3 marks)

Sum of probabilities = 1: $k(1 + 2 + 3 + 4) = 10k = 1$

M1

So $k = \frac{1}{10}$

A1

$P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{10} + \frac{1}{5} = \frac{3}{10}$

A1

Answer: $k = \frac{1}{10}$, $P(X < 2) = \frac{3}{10}$

3 marks

3 (3 marks)

(a) The probabilities sum to $0.3 + 0.4 + 0.2 + 0.3 = 1.2$, which is not 1

M1 A1

Answer: The probabilities sum to 1.2, not 1

2 marks

(b) $P(X = 4) = 1 - 0.3 - 0.4 - 0.2 = 0.1$

A1

Answer: 0.1

1 mark

4 (4 marks)

(a) $P(1 \leq X < 3) = P(X = 1) + P(X = 2) = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$

M1 A1

Answer: $\frac{5}{8}$

2 marks

(b) $F(1.5) = P(X \leq 1.5) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

M1 A1

Answer: $\frac{1}{2}$

2 marks

5 (2 marks)

There are 8 equally likely outcomes; the numbers with 0, 1, 2, 3 heads are 1, 3, 3, 1

M1

$P(X = 0, 1, 2, 3) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$

A1

Answer: $P(X = 0, 1, 2, 3) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$

2 marks

6 (5 marks)

Each value is equally likely, so $P(X = 7) = \frac{1}{10}$

B1

Multiples of 3 are 3, 6, 9 so $P = \frac{3}{10}$

M1 A1

$X^2 > 40$ requires $X \geq 7$, so $X = 7, 8, 9, 10$ and $P = \frac{4}{10} = \frac{2}{5}$

M1 A1

Answer: (a) $\frac{1}{10}$, (b) $\frac{3}{10}$, (c) $\frac{2}{5}$

5 marks

7 (4 marks)

(a) There are 16 equally likely outcomes (a, b). X can be 0, 1, 2 or 3

B1

$X = 0$: (1,1), (2,2), (3,3), (4,4) so 4 outcomes. $X = 1$: 6 outcomes. $X = 2$: 4 outcomes. $X = 3$: 2 outcomes

M1

$x = 0, 1, 2, 3$ with $P(X = x) = \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}$ respectively

A1

Answer: $P(X = 0, 1, 2, 3) = \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}$

3 marks

(b) $P(X \geq 2) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

A1

Answer: $\frac{3}{8}$ **1 mark****8 (4 marks)**

(a) $P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.55 - 0.3 = 0.25$

A1**Answer: 0.25****1 mark**

(b) $P(X > 2) = 1 - P(X \leq 2) = 1 - 0.3 = 0.7$

A1**Answer: 0.7****1 mark**

(c) Subtract consecutive cumulative values: $P(X = 1) = 0.1$, $P(X = 2) = 0.2$, $P(X = 3) = 0.25$

M1

$P(X = 4) = 0.8 - 0.55 = 0.25$, $P(X = 5) = 1 - 0.8 = 0.2$

A1**Answer: 0.1, 0.2, 0.25, 0.25, 0.2 for x = 1 to 5****2 marks****9 (5 marks)**

(a) The values are 4, 5, 6, 7, 8, five values, each with probability $\frac{1}{12}$

M1

$P = \frac{5}{12}$

A1**Answer: $\frac{5}{12}$** **2 marks**

(b) X is even: 2, 4, 6, 8, 10, 12, six values

B1

$X^2 > 50$ needs $X \geq 8$, so 8, 10, 12 are the values that qualify

M1

$P = \frac{3}{6} = \frac{1}{2}$

A1**Answer: $\frac{1}{2}$** **3 marks****10 (4 marks)**

(a) $c = P(X \leq 3) = P(X \leq 2) + P(X = 3) = 0.4 + 0.35 = 0.75$

M1 A1**Answer: c = 0.75****2 marks**

(b) $P(X = 1) = 0.15$, $P(X = 2) = 0.4 - 0.15 = 0.25$

M1

$P(X = 3) = 0.35$, $P(X = 4) = 1 - 0.75 = 0.25$

A1**Answer: 0.15, 0.25, 0.35, 0.25****2 marks****11 (3 marks)**There are $4 \times 3 = 12$ equally likely ordered pairs**M1** $X = 2$: (1,2), (2,1) so 2 outcomes; $X = 3$: 4 outcomes; $X = 4$: 6 outcomes**M1**

$P(X = 2) = \frac{1}{6}$, $P(X = 3) = \frac{1}{3}$, $P(X = 4) = \frac{1}{2}$

A1**Answer: $P(X = 2, 3, 4) = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}$** **3 marks****12 (4 marks)**

$5k + 4k + 3k + 2k + k = 15k = 1$ so $k = \frac{1}{15}$

M1 A1The largest probability is at $x = 1$, so the mode is 1**B1**

$P(X \text{ even}) = P(X = 2) + P(X = 4) = \frac{4}{15} + \frac{2}{15} = \frac{2}{5}$

A1**Answer: $k = \frac{1}{15}$, mode 1, $P(\text{even}) = \frac{2}{5}$** **4 marks****13 (3 marks)**

Sum 4 from (1, 3), (3, 1) and (2, 2)

M1

$P = 2(0.5)(0.2) + 0.3^2 = 0.2 + 0.09$

M1

$= \frac{29}{100} = 0.29$

A1**Answer: 0.29****3 marks**

14 (7 marks)**(a)** $x(6 - x)$ takes the values 5, 8, 9, 8 for $x = 1, 2, 3, 4$

M1

$$30k = 1 \text{ so } k = \frac{1}{30}$$

A1

Answer: $k = \frac{1}{30}$

2 marks

(b) $P(X \geq 2 | X \leq 3) = \frac{P(2 \leq X \leq 3)}{P(X \leq 3)} = \frac{17}{30} \div \frac{22}{30}$

M1

$$= \frac{17}{22}$$

A1

Answer: $\frac{17}{22}$

2 marks

(c) Sum is 5 for the pairs (1, 4), (2, 3), (3, 2), (4, 1)

M1

$$P = 2 \times \frac{1}{6} \times \frac{4}{15} + 2 \times \frac{4}{15} \times \frac{3}{10}$$

M1

$$= \frac{56}{225}$$

A1

Answer: $\frac{56}{225}$

3 marks

15 (6 marks)

(a) $P(X = 0) = P(BB) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$

M1 A1

$$P(X = 1) = P(RB) + P(BR) = \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{4}{7}$$

M1 A1

Answer: $P(X = 0, 1, 2) = \frac{1}{7}, \frac{4}{7}, \frac{2}{7}$

4 marks

(b) $P(X = 2 | X \geq 1) = \frac{P(X = 2)}{P(X \geq 1)} = \frac{2}{7} \div \frac{6}{7}$

M1

$$= \frac{1}{3}$$

A1

Answer: $\frac{1}{3}$

2 marks

16 (6 marks)

(a) $P(X = 2) = P(\text{not six, then six}) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

M1 A1

$$X = 3 \text{ whenever the first two rolls are not sixes: } P(X = 3) = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

M1 A1

Answer: $P(X = 3) = \frac{25}{36}$

4 marks

(b) $P(X \geq 2) = 1 - P(X = 1) = 1 - \frac{1}{6} = \frac{5}{6}$

M1 A1

Answer: $\frac{5}{6}$

2 marks

17 (6 marks)**(a)** Every probability must be at least 0: $0.35 - a \geq 0$, $2a \geq 0$ and $0.65 - a \geq 0$

M1

So $0 \leq a \leq 0.35$

A1

Answer: $0 \leq a \leq 0.35$

2 marks

(b) A total of 4 comes from (1, 3), (3, 1) and (2, 2)

M1

$$P = 2(0.35 - a)(0.65 - a) + (2a)^2$$

M1 A1

$$= 2(0.2275 - a + a^2) + 4a^2 = 6a^2 - 2a + 0.455$$

A1

Answer: shown

4 marks

18 (5 marks)Probabilities sum to 1: $a + b = 0.5$

M1

$$E(X) = 0.3 + 2a + 3b + 0.8 = 2.4 \text{ so } 2a + 3b = 1.3$$

M1 A1

Solving: $b = 1.3 - 2(0.5) = 0.3$ and $a = 0.2$

M1 A1

Answer: $a = 0.2, b = 0.3$

5 marks

19 (5 marks)

$$(a) F(4) = \frac{16+4}{20} = 1$$

B1

$$P(X = 3) = F(3) - F(2) = \frac{12}{20} - \frac{6}{20} = \frac{3}{10}$$

M1 A1

$$\text{Answer: } \frac{3}{10}$$

3 marks

$$(b) P(2 < X \leq 4) = F(4) - F(2) = 1 - \frac{3}{10}$$

M1

$$= \frac{7}{10}$$

A1

$$\text{Answer: } \frac{7}{10}$$

2 marks**20 (5 marks)**

$$(a) \text{ The probabilities sum to } \frac{1+2+3+4}{6} = \frac{10}{6} \neq 1$$

M1 E1

$$\text{Answer: Sum is } \frac{5}{3}, \text{ not } 1$$

2 marks

$$(b) \text{ Sum} = \frac{10}{10} = 1 \text{ and each value lies between 0 and 1, so valid}$$

B1

$$P(X \text{ prime}) = P(X = 2) + P(X = 3) = \frac{2}{10} + \frac{3}{10} = \frac{1}{2}$$

M1 A1

$$\text{Answer: } \frac{1}{2}$$

3 marks**21 (9 marks)**

$$(a) \text{ The probabilities sum to 1, a geometric series with first term } \frac{k}{3} \text{ and common ratio } \frac{1}{3}$$

M1

$$S_{\infty} = \frac{k}{3} \div (1 - \frac{1}{3}) = \frac{k}{2}$$

M1

$$\text{So } \frac{k}{2} = 1 \text{ and } k = 2$$

A1

$$\text{Answer: } k = 2$$

3 marks

$$(b) P(X \text{ even}) = P(X = 2) + P(X = 4) + \dots = 2(\frac{1}{3})^2 + 2(\frac{1}{3})^4 + \dots$$

M1

$$\text{Geometric series with first term } \frac{2}{9} \text{ and common ratio } \frac{1}{9}$$

M1

$$\text{Sum} = \frac{2}{9} \div (1 - \frac{1}{9}) = \frac{1}{4}$$

A1

$$\text{Answer: } \frac{1}{4}$$

3 marks

$$(c) P(X > n) = 2(\frac{1}{3})^{n+1} + 2(\frac{1}{3})^{n+2} + \dots = \frac{2(\frac{1}{3})^{n+1}}{(1 - \frac{1}{3})} = (\frac{1}{3})^n$$

M1 A1

$$(\frac{1}{3})^n < 0.001 \text{ so } n > \frac{\ln 1000}{\ln 3} = 6.29\dots; \text{ the least integer is } n = 7$$

A1

$$\text{Answer: } n = 7$$

3 marks**22 (9 marks)**

$$(a) 0.2 + k + k^2 + 0.24 = 1$$

M1

$$k^2 + k - 0.56 = 0$$

A1

$$(k - 0.4)(k + 1.4) = 0 \text{ so } k = 0.4 \text{ or } k = -1.4$$

M1

$$k = -1.4 \text{ is rejected because a probability cannot be negative, so } k = 0.4$$

E1

$$\text{Answer: } k = 0.4$$

4 marks

$$(b) P(X = 2) = 0.4, P(X = 3) = 0.16, P(X = 4) = 0.24$$

B1

$$P(X > 2 | X \geq 2) = \frac{P(X > 2)}{P(X \geq 2)} = \frac{0.4}{0.8}$$

M1

$$= 0.5$$

A1

$$\text{Answer: } 0.5$$

3 marks

$$(c) P(\text{equal}) = \sum P(X = x)^2 = 0.04 + 0.16 + 0.0256 + 0.0576 = 0.2832$$

M1

A1

Answer: 0.2832

2 marks

23 (8 marks)

$$(a) \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

M1 A1

Answer: $\frac{1}{x} - \frac{1}{x+1}$

2 marks

$$(b) \text{Sum of probabilities} = k\left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

M1

$$\text{Terms cancel in pairs leaving } k\left(1 - \frac{1}{n+1}\right) = k\frac{n}{n+1}$$

M1 A1

$$\text{This equals 1, so } k = \frac{n+1}{n}$$

A1

Answer: k = $\frac{n+1}{n}$

4 marks

$$(c) k = \frac{5}{4} \text{ so } P(X = 3) = \frac{5}{48}, P(X = 4) = \frac{5}{80} = \frac{1}{16}$$

M1

$$P(X \geq 3) = \frac{5}{48} + \frac{1}{16} = \frac{8}{48} = \frac{1}{6}$$

A1

Answer: $\frac{1}{6}$

2 marks

24 (9 marks)

(a) Y can take the values 0, 2, 3, 4, 5, 6

B1

$$P(Y = 0) = 0.4^2 = \frac{4}{25}, P(Y = 2) = 2 \times 0.4 \times 0.4 = \frac{8}{25}, P(Y = 3) = 2 \times 0.4 \times 0.2 = \frac{4}{25}$$

M1 A1

$$P(Y = 4) = 0.4^2 = \frac{4}{25}, P(Y = 5) = 2 \times 0.4 \times 0.2 = \frac{4}{25}, P(Y = 6) = 0.2^2 = \frac{1}{25}$$

A1

Answer: P(Y = 0, 2, 3, 4, 5, 6) = $\frac{4}{25}, \frac{8}{25}, \frac{4}{25}, \frac{4}{25}, \frac{4}{25}, \frac{1}{25}$

4 marks

(b) $X_1 > X_2$ for (2, 0), (3, 0) and (3, 2)

M1

$$P = 0.4 \times 0.4 + 0.2 \times 0.4 + 0.2 \times 0.4 = \frac{8}{25}$$

A1

Answer: $\frac{8}{25}$

2 marks

$$(c) P(Y \geq 3) = \frac{4}{25} + \frac{4}{25} + \frac{4}{25} + \frac{1}{25} = \frac{13}{25}$$

M1

$$X_1 > X_2 \text{ and } Y \geq 3 \text{ for } (3, 0) \text{ and } (3, 2), \text{ probability } 0.2 \times 0.4 + 0.2 \times 0.4 = \frac{4}{25}$$

M1

$$P = \frac{4}{25} \div \frac{13}{25} = \frac{4}{13}$$

A1

Answer: $\frac{4}{13}$

3 marks

25 (8 marks)

$$(a) P(\text{equal}) = a^2 + 0.5^2 + (0.5 - a)^2$$

M1 A1

$$= 2a^2 - a + 0.5 = 0.38, \text{ so } 2a^2 - a + 0.12 = 0$$

M1 A1

Answer: shown

4 marks

(b) $(10a - 2)(10a - 3) = 0$ gives $a = 0.2$ or $a = 0.3$

M1 A1

For $a = 0.3$: $P(X = 2) = 0.2$ and $E(X) = 0.5 + 0.4 = 0.9$

M1 A1

Answer: a = 0.2 or 0.3; E(X) = 0.9

4 marks

26 (9 marks)

$$(a) \text{Both scores equal: 6 of the 36 outcomes, so } P(X = 0) = \frac{6}{36} = \frac{1}{6}$$

M1 A1

$$X = 4 \text{ needs a 4 first and a different score second: } \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$$

M1 A1

Answer: shown

4 marks

$$(b) P(X \text{ even}) = P(X = 0) + P(X = 2) + P(X = 4) + P(X = 6)$$

M1

$$= \frac{6}{36} + 3 \times \frac{5}{36} = \frac{21}{36} = \frac{7}{12}$$

M1 A1

Answer: $\frac{7}{12}$

3 marks

(c) $P(X \neq 0) = \frac{30}{36}$, $P(X > 3) = 3 \times \frac{5}{36} = \frac{15}{36}$

M1

Conditional probability = $\frac{15}{30} = \frac{1}{2}$

A1

Answer: $\frac{1}{2}$

2 marks

27 (10 marks)

(a) $\sum_{r=1}^n P(X = r) = k \sum_{r=1}^n r^2 = 1$

M1

$$\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

B1

So $k = \frac{6}{n(n+1)(2n+1)}$

A1

Answer: $k = \frac{6}{n(n+1)(2n+1)}$

3 marks

(b) $P(X = n) = kn^2 = \frac{6n}{(n+1)(2n+1)}$

M1 A1

$$\frac{6n}{(n+1)(2n+1)} = \frac{8}{15} \text{ so } 90n = 8(2n^2 + 3n + 1)$$

M1

$$8n^2 - 33n + 4 = 0 \text{ so } (8n - 1)(n - 4) = 0$$

M1

$n = \frac{1}{8}$ or $n = 4$; n is a positive integer, so $n = 4$

A1

Answer: $n = 4$

5 marks

(c) $k = \frac{6}{4 \times 5 \times 9} = \frac{1}{30}$

M1

$P(X \text{ odd}) = k(1^2 + 3^2) = \frac{10}{30} = \frac{1}{3}$

A1

Answer: $\frac{1}{3}$

2 marks

28 (12 marks)

(a) $X \leq x$ means both dice show at most x

M1

$$P(X \leq x) = \frac{x}{n} \times \frac{x}{n} = \frac{x^2}{n^2}$$

A1

$P(X = x) = P(X \leq x) - P(X \leq x - 1)$

M1

$$= \frac{x^2 - (x-1)^2}{n^2} = \frac{2x-1}{n^2}$$

A1

Answer: $P(X = x) = \frac{2x-1}{n^2}$

4 marks

(b) For $n = 4$: $P(X = 3) = \frac{5}{16}$, $P(X = 4) = \frac{7}{16}$

B1

A sum of 7 needs the values 3 and 4 in either order

M1

$$P = 2 \times \frac{5}{16} \times \frac{7}{16} = \frac{35}{128}$$

A1

Answer: $\frac{35}{128}$

3 marks

(c) $P(X = n) = \frac{2n-1}{n^2} < 0.1$

M1

$$2n - 1 < 0.1n^2 \text{ so } n^2 - 20n + 10 > 0$$

M1 A1

Roots $n = 10 \pm \sqrt{90}$, so $n > 10 + \sqrt{90} = 19.49...$

M1

$n = 20$ (check: $\frac{39}{400} = 0.0975 < 0.1$, but $\frac{37}{361} = 0.1025$ for $n = 19$)

A1

Answer: $n = 20$

5 marks

29 (14 marks)**(a)** The three values must be 1, 2 and 4 in some order, and there are $3! = 6$ orders**M1**

$$P = 6 \times 0.5 \times 0.3 \times 0.2 = \frac{9}{50}$$

M1 A1**Answer:** $\frac{9}{50}$ **3 marks****(b)** A total of 6 comes from 2, 2, 2 or from 1, 1, 4 in any order**M1**

$$P = 0.3^3 + 3 \times 0.5^2 \times 0.2$$

M1 A1

$$= \frac{177}{1000}$$

A1**Answer:** $\frac{177}{1000}$ **4 marks****(c)** With at most one 4 the largest total is $4 + 2 + 2 = 8$, so a total of at least 9 needs at least two 4s: 4, 4, 1 (total 9), 4, 4, 2 (total 10) or 4, 4, 4 (total 12)**M1**

$$P(4, 4, 1) = 3 \times 0.2^2 \times 0.5 = 0.06, P(4, 4, 2) = 3 \times 0.2^2 \times 0.3 = 0.036, P(4, 4, 4) = 0.2^3 = 0.008$$

M1 A1

$$P(\text{total} \geq 9) = 0.104 = \frac{13}{125}$$

A1

$$\text{Exactly two 4s means } 4, 4, 1 \text{ or } 4, 4, 2, \text{ probability } 0.096 = \frac{12}{125}$$

B1

$$P = \frac{12}{125} \div \frac{13}{125} = \frac{12}{13}$$

M1 A1**Answer:** $P(\text{total} \geq 9) = \frac{13}{125}$; conditional probability $\frac{12}{13}$ **7 marks****30 (13 marks)**

$$\text{(a)} P(X > x) = \left(\frac{1}{2}\right)^{x+1} + \left(\frac{1}{2}\right)^{x+2} + \dots$$

M1

$$\text{Geometric series with first term } \left(\frac{1}{2}\right)^{x+1} \text{ and ratio } \frac{1}{2}: \text{sum} = \frac{\left(\frac{1}{2}\right)^{x+1}}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^x$$

M1 A1**Answer: shown****3 marks****(b)** $M \geq m$ means both $X_1 \geq m$ and $X_2 \geq m$, each with probability $P(X > m - 1) = \left(\frac{1}{2}\right)^{m-1}$ **M1 A1**

$$\text{By independence, } P(M \geq m) = \left(\frac{1}{2}\right)^{2(m-1)} = \left(\frac{1}{4}\right)^{m-1}$$

A1**Answer: shown****3 marks**

$$\text{(c)} P(M = m) = P(M \geq m) - P(M \geq m + 1) = \left(\frac{1}{4}\right)^{m-1} - \left(\frac{1}{4}\right)^m$$

M1 A1

$$= \left(\frac{1}{4}\right)^{m-1} \left(1 - \frac{1}{4}\right) = \frac{3}{4} \left(\frac{1}{4}\right)^{m-1}, \text{ a geometric distribution with } p = \frac{3}{4}$$

M1 E1

$$\text{Answer: } P(M = m) = \frac{3}{4} \left(\frac{1}{4}\right)^{m-1}$$

4 marks

$$\text{(d)} P(M = 2) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$$

A1

$$P(M \text{ even}) = \frac{3}{16} + \frac{3}{4} \left(\frac{1}{4}\right)^3 + \dots, \text{ first term } \frac{3}{16}, \text{ ratio } \frac{1}{16}$$

M1

$$\text{Sum} = \frac{3}{16} \div \frac{15}{16} = \frac{1}{5}$$

A1

$$\text{Answer: } \frac{3}{16}, \frac{1}{5}$$

3 marks**Expectation and Variance****1 (2 marks)**

$$E(X) = \sum x P(X = x) = 1(0.1) + 2(0.2) + 3(0.3) + 4(0.4)$$

M1

$$= 0.1 + 0.4 + 0.9 + 1.6 = 3$$

A1**Answer: $E(X) = 3$** **2 marks**

2 (3 marks)

$$E(X) = 0(0.5) + 1(0.3) + 2(0.2) = 0.7$$

B1

$$E(X^2) = 0 + 1(0.3) + 4(0.2) = 1.1$$

M1

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 1.1 - 0.49 = 0.61$$

A1

Answer: $E(X) = 0.7$, $\text{Var}(X) = 0.61$

3 marks

3 (4 marks)

$$E(X) = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \frac{21}{6} = \frac{7}{2}$$

M1 A1

$$E(X^2) = \frac{1 + 4 + 9 + 16 + 25 + 36}{6} = \frac{91}{6}$$

M1 A1

Answer: $E(X) = \frac{7}{2}$, $E(X^2) = \frac{91}{6}$

4 marks

4 (2 marks)

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

M1

$$E(X^2) = 7 + 9 = 16$$

A1

Answer: 16

2 marks

5 (3 marks)

$$E(X) = -0.2 + 0 + 0.3 = 0.1$$

M1 A1

$$E(X^2) = 0.2 + 0 + 0.3 = 0.5, \text{ so } \text{Var}(X) = 0.5 - 0.1^2 = 0.49$$

A1

Answer: $E(X) = 0.1$, $\text{Var}(X) = 0.49$

3 marks

6 (2 marks)

$$E(W) = 4(0.5) + (-1)(0.5)$$

M1

$$= £1.50$$

A1

Answer: £1.50

2 marks

7 (5 marks)

$$(a) E(3X - 2) = 3(4) - 2 = 10$$

A1

Answer: 10

1 mark

$$(b) \text{Var}(aX + b) = a^2 \text{Var}(X)$$

M1

$$\text{Var}(3X - 2) = 9 \times 9 = 81$$

A1

Answer: 81

2 marks

$$(c) E(X^2) = \text{Var}(X) + [E(X)]^2$$

M1

$$= 9 + 16 = 25$$

A1

Answer: 25

2 marks

8 (5 marks)

$$(a) P(W = 5) = \frac{1}{6}, P(W = 1) = \frac{2}{6} = \frac{1}{3}, P(W = -2) = \frac{3}{6} = \frac{1}{2}$$

B1

$$E(W) = 5\left(\frac{1}{6}\right) + 1\left(\frac{1}{3}\right) - 2\left(\frac{1}{2}\right)$$

M1

$$= \frac{5}{6} + \frac{2}{6} - \frac{6}{6} = \frac{1}{6}$$

A1

Answer: $E(W) = £\frac{1}{6}$

3 marks

$$(b) \text{Expected total} = 120 \times \frac{1}{6} = £20$$

M1

 $E(W) \neq 0$, so the game is not fair (it favours the player)

E1

Answer: £20; not fair

2 marks

9 (4 marks)

$$(a) E = 300 \times \frac{1}{1000} + 50 \times \frac{5}{1000} + 5 \times \frac{20}{1000}$$

M1

$$= 0.30 + 0.25 + 0.10 = £0.65$$

A1

Answer: £0.65

2 marks

$$(b) \text{Expected profit} = 2 - 0.65 = £1.35$$

M1 A1

Answer: £1.35**2 marks****10 (5 marks)**

$$E(X) = 0.3 + 1.0 + 0.6 = 1.9$$

M1 A1

$$E(X^2) = 0.3 + 2.0 + 1.8 = 4.1, \text{Var}(X) = 4.1 - 1.9^2 = 0.49$$

M1 A1

$$\text{Standard deviation} = \sqrt{0.49} = 0.7$$

A1**Answer: 1.9, 0.49, 0.7****5 marks****11 (4 marks)**

$$E(X^2 - 3) = \sum (x^2 - 3)P(X = x) = (-3)(0.2) + (-2)(0.3) + 1(0.4) + 6(0.1)$$

M1

$$= -\frac{1}{5}$$

A1

$$E\left(\frac{1}{X+1}\right) = 1(0.2) + \frac{1}{2}(0.3) + \frac{1}{3}(0.4) + \frac{1}{4}(0.1)$$

M1

$$= \frac{61}{120}$$

A1**Answer: $-\frac{1}{5}$, $\frac{61}{120}$** **4 marks****12 (8 marks)****(a)** Subtract consecutive values: $P(X = 1, 2, 3, 4, 5) = 0.2, 0.3, 0.2, 0.2, 0.1$ **M1 A1**

$$E(X) = 0.2 + 0.6 + 0.6 + 0.8 + 0.5 = 2.7$$

M1

$$E(X^2) = 0.2 + 1.2 + 1.8 + 3.2 + 2.5 = 8.9 \text{ so } \text{Var}(X) = 8.9 - 2.7^2 = 1.61$$

M1 A1**Answer: $E(X) = 2.7$, $\text{Var}(X) = 1.61$** **5 marks****(b)** $E(Y) = 3 \times 2.7 - 2 = 6.1$ **A1**

$$\text{Var}(Y) = 3^2 \times 1.61 = 14.49$$

M1 A1**Answer: $E(Y) = 6.1$, $\text{Var}(Y) = 14.49$** **3 marks****13 (7 marks)**

$$\text{(a)} P(X = 1) = \frac{4C1 \times 2C2}{6C3} = \frac{4}{20} = \frac{1}{5}$$

M1 A1

$$P(X = 2) = \frac{4C2 \times 2C1}{6C3} = \frac{12}{20} = \frac{3}{5} \text{ and } P(X = 3) = \frac{4C3}{6C3} = \frac{4}{20} = \frac{1}{5}$$

M1 A1**Answer: $P(X = 1, 2, 3) = \frac{1}{5}, \frac{3}{5}, \frac{1}{5}$** **4 marks**

$$\text{(b)} E(X) = \frac{1}{5} + \frac{6}{5} + \frac{3}{5} = 2$$

M1

$$E(X^2) = \frac{1}{5} + \frac{12}{5} + \frac{9}{5} = \frac{22}{5}, \text{ so } \text{Var}(X) = \frac{22}{5} - 4 = \frac{2}{5}$$

M1 A1**Answer: $E(X) = 2$, $\text{Var}(X) = \frac{2}{5}$** **3 marks****14 (8 marks)****(a)** By symmetry about $x = 3$, $\mu = 3$ (or $E(X) = 0.1 + 0.5 + 0.9 + 1 + 0.5 = 3$)**M1 A1**

$$E(X^2) = 0.1 + 1 + 2.7 + 4 + 2.5 = 10.3, \text{ so } \text{Var}(X) = 10.3 - 9 = 1.3 \text{ and } \sigma = 1.14$$

M1 A1**Answer: $\mu = 3$, $\sigma = 1.14$** **4 marks****(b)** $\mu - \sigma = 1.86$ and $\mu + \sigma = 4.14$, so $X = 2, 3, 4$ qualify**M1**

$$P = 0.25 + 0.3 + 0.25 = 0.8$$

A1**Answer: 0.8****2 marks**

$$\text{(c)} E(2X^2 + 1) = 2 \times 10.3 + 1 = 21.6$$

M1 A1**Answer: 21.6****2 marks****15 (7 marks)**

$$\text{(a)} k(1 + 2 + 3 + 4) = 10k = 1 \text{ so } k = 0.1$$

M1**Answer: $k = 0.1$** **1 mark**

$$\text{(b)} E(X) = \sum x(0.1x) = 0.1(1 + 4 + 9 + 16)$$

M1

$$= 3$$

A1**Answer: 3****2 marks**

$$\text{(c)} E(X^2) = 0.1(1 + 8 + 27 + 64) = 10$$

M1

$$\text{Var}(X) = 10 - 3^2 = 1$$

Answer: 1

A1
2 marks

$$(d) E\left(\frac{1}{X}\right) = \sum \frac{1}{x}(0.1x) = 0.1 \times 4$$

$$= 0.4$$

Answer: 0.4

M1
A1
2 marks

16 (8 marks)

$$(a) E(C) = 500(0.08) + 2000(0.02) = 40 + 40 = 80$$

M1 A1

$$E(C^2) = 250000(0.08) + 4000000(0.02) = 20000 + 80000 = 100000$$

M1

$$\text{Var}(C) = 100000 - 6400 = 93600, \text{sd} = \text{£}306$$

M1 A1

Answer: E(C) = £80, sd = £306

5 marks

$$(b) \text{Expected profit} = p - 80 = 20 \text{ so } p = 100$$

M1 A1

$$\text{Var}(\text{profit}) = \text{Var}(C) = 93600, \text{ since subtracting a constant does not change variance}$$

B1

Answer: p = £100, variance 93600

3 marks

17 (7 marks)

$$(a) E(X) = \frac{1 \times 1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + 5 \times 9 + 6 \times 11}{36}$$

M1

$$= \frac{1 + 6 + 15 + 28 + 45 + 66}{36} = \frac{161}{36}$$

M1 A1

Answer: $\frac{161}{36}$

3 marks

$$(b) E(X^2) = \frac{1 + 12 + 45 + 112 + 225 + 396}{36} = \frac{791}{36}$$

M1 A1

$$\text{Var}(X) = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = 1.97$$

M1 A1

Answer: 1.97

4 marks

18 (6 marks)

$$k\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12}k = 1 \text{ so } k = \frac{12}{25}$$

M1 A1

$$E(X) = \sum x \times \frac{k}{x} = 4k = \frac{48}{25}$$

M1 A1

$$E(X^2) = k(1 + 2 + 3 + 4) = 10k = \frac{120}{25}, \text{Var}(X) = \frac{120}{25} - \left(\frac{48}{25}\right)^2 = \frac{696}{625}$$

M1 A1

Answer: $k = \frac{12}{25}$, $E(X) = \frac{48}{25}$, $\text{Var}(X) = \frac{696}{625}$

6 marks

19 (5 marks)

$$\text{Var}(Y) = a^2 \times 16 = 1 \text{ so } a = \frac{1}{4}$$

M1 A1

$$E(Y) = 10a + b = 0 \text{ so } b = -\frac{5}{2}$$

M1 A1

$$E(Y^2) = \text{Var}(Y) + [E(Y)]^2 = 1 + 0 = 1$$

B1

Answer: $a = \frac{1}{4}$, $b = -\frac{5}{2}$

5 marks

20 (5 marks)

$$E(X) = \frac{3+5}{2} = 4 \text{ and } E(Y) = 0 + 2 + 2 = 4$$

M1 A1

$$\text{Var}(X) = \frac{9+25}{2} - 16 = 1; E(Y^2) = 0 + 8 + 16 = 24 \text{ so } \text{Var}(Y) = 24 - 16 = 8$$

M1 A1

Game A: it has the smaller variance, so its results are less spread out

E1

Answer: Both 4; Game A (variance 1 against 8)

5 marks

21 (8 marks)

$$(a) \text{The formula is the wrong way round: } \text{Var}(X) = E(X^2) - [E(X)]^2$$

E1

A variance is a mean of squared deviations, so it cannot be negative

E1

Answer: Formula reversed; variance cannot be negative

2 marks

$$(b) \text{Var}(X) = 19.6 - 4.2^2$$

$$= 19.6 - 17.64 = 1.96$$

Answer: 1.96

M1

A1

2 marks

$$(c) E(Y) = 5 - 2(4.2) = -3.4$$

M1 A1

$$\text{Var}(Y) = (-2)^2 \text{Var}(X) = 4 \times 1.96 = 7.84$$

M1

$$\text{Standard deviation} = \sqrt{7.84} = 2.8$$

A1

Answer: E(Y) = -3.4, sd = 2.8

4 marks

22 (9 marks)

$$(a) P(X = 0) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$$

M1 A1

$$P(X = 2) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10} \text{ and } P(X = 1) = 1 - 0.3 - 0.1 = 0.6$$

A1

Answer: 0.3, 0.6, 0.1

3 marks

$$(b) E(X) = 0.6 + 2(0.1) = 0.8$$

M1

$$\text{Expected payout} = 3E(X) = \text{£}2.40$$

M1

For a fair game, $c = 2.40$

A1

Answer: c = £2.40

3 marks

$$(c) E(X^2) = 0.6 + 4(0.1) = 1.0, \text{ so } \text{Var}(X) = 1 - 0.8^2 = 0.36$$

M1

$$\text{Profit} = 3X - 2.40, \text{ so } \text{Var} = 3^2 \text{Var}(X)$$

M1

$$= 9 \times 0.36 = 3.24$$

A1

Answer: 3.24

3 marks

23 (9 marks)

$$(a) E(Y) = aE(X) + b = a\mu + b$$

B1

Answer: $a\mu + b$

1 mark

$$(b) Y - E(Y) = aX + b - a\mu - b = a(X - \mu)$$

M1 A1

$$\text{Var}(Y) = \sum a^2(x - \mu)^2 P(X = x)$$

M1

$$= a^2 \sum (x - \mu)^2 P(X = x) = a^2 \text{Var}(X)$$

M1 A1

Answer: proved

5 marks

$$(c) \text{sd}(Y) = \sqrt{a^2 \text{Var}(X)} = |a| \text{sd}(X), \text{ since a square root is never negative}$$

M1 E1

$$\text{sd} = |-3| \times 2 = 6$$

A1

Answer: 6

3 marks

24 (11 marks)

$$(a) X = 3 \text{ means TTH: } \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

M1 A1

$$X = 4 \text{ means TTT followed by anything (the fourth toss is the last): } \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

M1 A1

Answer: shown

4 marks

$$(b) E(X) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{8} = \frac{15}{8}$$

M1 A1

$$E(X^2) = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{8} = \frac{37}{8}, \text{ Var}(X) = \frac{37}{8} - \left(\frac{15}{8}\right)^2 = \frac{71}{64}$$

M1 A1

$$\text{Answer: } E(X) = \frac{15}{8}, \text{ Var}(X) = \frac{71}{64}$$

4 marks

$$(c) \text{Expected return} = 2E(X) - 1 = \frac{15}{4} - 1 = \frac{11}{4}$$

M1 A1

$$\text{Expected profit} = \frac{11}{4} - 1 = \text{£}1.75$$

A1

Answer: £1.75

3 marks

25 (12 marks)

$$(a) E(X) = 0.25 + 1 + 1 = 2.25$$

M1 A1

$$E(X^2) = 0.25 + 2 + 4 = 6.25, \text{ Var}(X) = 6.25 - 2.25^2 = \frac{19}{16}$$

M1 A1

$$\text{Answer: } E(X) = \frac{9}{4}, \text{ Var}(X) = \frac{19}{16}$$

4 marks

$$(b) E(X_1 X_2) = E(X_1)E(X_2) = \frac{81}{16}$$

M1 A1

$$E(|X_1 - X_2|) = 2[(1)(0.25)(0.5) + (3)(0.25)(0.25) + (2)(0.5)(0.25)]$$

$$= \frac{9}{8}$$

M1

A1

Answer: $\frac{81}{16}, \frac{9}{8}$

4 marks

$$(c) \text{Var}(X_1 + X_2) = 2\text{Var}(X) = \frac{19}{8}$$

M1 A1

$$\text{Var}(X_1 - X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{19}{8}, \text{ the same, since variances add for independent variables}$$

M1 E1

Answer: $\frac{19}{8}$ for both

4 marks

26 (7 marks)

$$a + b + c = 1$$

B1

$$E(X) = a + 2b + 3c = 2.1$$

B1

$$E(X^2) = 0.49 + 2.1^2 = 4.9, \text{ so } a + 4b + 9c = 4.9$$

M1 A1

$$\text{Subtracting equations: } b + 2c = 1.1 \text{ and } 3b + 8c = 3.9, \text{ so } 2c = 0.6$$

M1

$$c = 0.3, b = 0.5, a = 0.2$$

A1 A1

Answer: $a = 0.2, b = 0.5, c = 0.3$

7 marks

27 (10 marks)

$$(a) E(X) = \frac{1}{n} \sum_{r=1}^n r = \frac{1}{n} \times \frac{n(n+1)}{2}$$

M1

$$= \frac{n+1}{2}$$

A1

Answer: $\frac{n+1}{2}$

2 marks

$$(b) E(X^2) = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

M1 A1

$$\text{Var}(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)[2(2n+1) - 3(n+1)]}{12}$$

M1

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

A1

Answer: $\text{Var}(X) = \frac{n^2-1}{12}$

4 marks

$$(c) \frac{n^2-1}{12} = 8.25 \text{ so } n^2 = 100$$

M1

$$n = 10 \text{ (} n > 0 \text{)}$$

A1

Answer: $n = 10$

2 marks

$$(d) E(Y) = 2(5.5) - 7 = 4$$

B1

$$\text{Var}(Y) = 4 \times 8.25 = 33$$

B1

Answer: $E(Y) = 4, \text{Var}(Y) = 33$

2 marks

28 (15 marks)

$$(a) P(X=1) + \dots + P(X=4) = \frac{1+2+3+4}{24} = \frac{5}{12}$$

M1 A1

$$\frac{5}{12} + \frac{7}{12} = 1 \text{ so the distribution is valid, and } P(X \leq 4) = \frac{5}{12}$$

A1

Answer: valid; $P(X \leq 4) = \frac{5}{12}$

3 marks

$$(b) E\left(\frac{1}{X}\right) = \sum \frac{1}{x} P(X=x) = \frac{1}{24} \times 4 + \frac{7}{12} \times \frac{1}{6}$$

M1 A1

$$= \frac{1}{6} + \frac{7}{72} = \frac{19}{72}$$

A1

Answer: shown

3 marks

$$(c) E\left(\frac{1}{X^2}\right) = \sum \frac{1}{x^2} P(X=x) = \frac{1}{24}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{7}{12} \times \frac{1}{36}$$

M1 A1

$$= \frac{89}{864}$$

A1

$$\text{Var}\left(\frac{1}{X}\right) = \frac{89}{864} - \left(\frac{19}{72}\right)^2 = \frac{173}{5184}$$

M1 A1

$$\text{Answer: } \frac{173}{5184}$$

5 marks

$$(d) Y = 1 + \frac{2}{X}$$

M1

$$E(Y) = 1 + 2E\left(\frac{1}{X}\right) = \frac{55}{36}$$

A1

$$\text{Var}(Y) = 2^2 \text{Var}\left(\frac{1}{X}\right) = \frac{173}{1296}$$

M1 A1

$$\text{Answer: } E(Y) = \frac{55}{36}, \text{Var}(Y) = \frac{173}{1296}$$

4 marks

29 (15 marks)

$$(a) E(X) = \sum x \frac{1}{2^n} C(n, x) = \frac{n}{2^n} \sum C(n-1, x-1)$$

M1 A1

The sum over $x = 1$ to n is the sum of all binomial coefficients for $n-1$, which is 2^{n-1}

M1

$$E(X) = \frac{n \times 2^{n-1}}{2^n} = \frac{n}{2}$$

A1

$$\text{Answer: } \frac{n}{2}$$

4 marks

$$(b) E[X(X-1)] = \frac{n(n-1)}{2^n} \sum C(n-2, x-2) = \frac{n(n-1) \times 2^{n-2}}{2^n}$$

M1 A1

$$= \frac{n(n-1)}{4}$$

A1

$$\text{Answer: } \frac{n(n-1)}{4}$$

3 marks

$$(c) E(X^2) = E[X(X-1)] + E(X) = \frac{n(n-1)}{4} + \frac{n}{2} = \frac{n^2 + n}{4}$$

M1 A1

$$\text{Var}(X) = \frac{n^2 + n}{4} - \left(\frac{n}{2}\right)^2 = \frac{n}{4}$$

M1 A1

$$\text{Answer: } \frac{n}{4}$$

4 marks

$$(d) E(X) = 6, \text{Var}(X) = 3$$

B1

$$E(3X - 5) = 13$$

M1

$$\text{sd} = 3\sqrt{3} = 5.20$$

M1 A1

$$\text{Answer: } 13 \text{ and } 5.20$$

4 marks

30 (16 marks)

$$(a) E(\text{second roll}) = \frac{1+2+3+4+5+6}{6} = 3.5$$

M1 A1

Rejecting gives an expected 3.5, which beats 1, 2 or 3 but not 4, 5 or 6

E1 E1

$$\text{Answer: } 3.5$$

4 marks

$$(b) \text{Accept } 4, 5, 6 \text{ (each with probability } \frac{1}{6} \text{): contributes } \frac{4+5+6}{6} = 2.5$$

M1 A1

Reject with probability $\frac{1}{2}$, expected prize 3.5: contributes 1.75

M1

$$\text{Total} = 2.5 + 1.75 = 4.25$$

A1

$$\text{Answer: } \pounds 4.25$$

4 marks

$$(c) E(\text{prize}^2) = \frac{16+25+36}{6} + \frac{1}{2} \times \frac{91}{6} = \frac{77}{6} + \frac{91}{12} = \frac{245}{12}$$

M1 A1 A1

$$\text{Var} = \frac{245}{12} - \left(\frac{17}{4}\right)^2 = \frac{113}{48}$$

M1 A1

$$\text{Answer: } \frac{113}{48}$$

5 marks

$$(d) \text{Expected profit} = 4.25 - 4 = \pounds 0.25$$

M1 A1

This is positive, so the game favours the player and is not fair

E1

$$\text{Answer: } \pounds 0.25 \text{ for the player; not fair}$$

3 marks

Binomial Distribution

1 (2 marks)

$$P(X = 3) = C(10, 3) (0.3)^3 (0.7)^7$$

$$= 120 \times 0.027 \times 0.0823543 = 0.267$$

M1

A1

2 marks

Answer: 0.267

2 (2 marks)

$$X \sim B(8, 0.5); P(X \leq 2) = \frac{1}{256}(1 + 8 + 28)$$

$$= \frac{37}{256} = 0.145$$

M1

A1

2 marks

Answer: $\frac{37}{256} = 0.145$

3 (2 marks)

Using the binomial distribution (calculator): $P(X \leq 2) = P(0) + P(1) + P(2)$

$$= 0.544$$

M1

A1

2 marks

Answer: 0.544

4 (2 marks)

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.9^{20}$$

$$= 0.878$$

M1

A1

2 marks

Answer: 0.878

5 (2 marks)

$$E(X) = np = 30 \times 0.2 = 6$$

$$\text{Var}(X) = npq = 30 \times 0.2 \times 0.8 = 4.8$$

B1

B1

2 marks

Answer: $E(X) = 6$, $\text{Var}(X) = 4.8$

6 (4 marks)

(a) $X \sim B(12, \frac{1}{6})$

B1 B1

Answer: $X \sim B(12, \frac{1}{6})$

2 marks

(b) $P(X = 3) = C(12, 3)(\frac{1}{6})^3(\frac{5}{6})^9 = 0.197$

M1 A1

2 marks

Answer: 0.197

7 (5 marks)

(a) The guesses are independent of one another

The probability of a correct guess is constant (0.25) for each question

B1

B1

2 marks

Answer: Independence; constant probability 0.25

(b) $X \sim B(12, 0.25)$

B1

$$P(X \geq 5) = 1 - P(X \leq 4)$$

M1

$$= 1 - 0.8424 = 0.158$$

A1

3 marks

Answer: 0.158

8 (5 marks)

(a) $np = 6$ and $np(1 - p) = 4.2$

B1 B1

Dividing: $1 - p = \frac{4.2}{6} = 0.7$, so $p = 0.3$

M1

$$n = \frac{6}{0.3} = 20$$

A1

4 marks

Answer: $n = 20$, $p = 0.3$

(b) $P(X = 6) = C(20, 6)(0.3)^6(0.7)^{14} = 0.192$

A1

1 mark

Answer: 0.192

9 (3 marks)

$$\begin{aligned}
 P(4 \leq X \leq 8) &= P(X \leq 8) - P(X \leq 3) \\
 &= 0.9578 - 0.1727 \\
 &= 0.785
 \end{aligned}$$

M1

M1

A1

Answer: 0.785

3 marks

10 (3 marks)

$$\begin{aligned}
 \text{(a)} \quad X &\sim B(12, 0.85), P(X \geq 10) = 1 - P(X \leq 9) \\
 &= 0.736
 \end{aligned}$$

M1

A1

Answer: 0.736

2 marks

$$\text{(b)} \quad P(X = 12) = 0.85^{12} = 0.142$$

A1

Answer: 0.142

1 mark

11 (2 marks)

Binomial needs a fixed probability of success on each trial and independent trials

M1

Cards are not replaced, so after each card the probability of a heart changes (the trials are not independent)

E1

Answer: Trials not independent; probability changes

2 marks

12 (5 marks)

$$\text{(a)} \quad P(X < 2) = P(X \leq 1) = 0.3758$$

M1 A1

Answer: 0.3758

2 marks

$$\text{(b)} \quad P(X = 2) = 0.3020$$

B1

$$\begin{aligned}
 P(\text{exactly one equals } 2) &= 2 \times 0.3020 \times (1 - 0.3020) \\
 &= 0.422
 \end{aligned}$$

M1

A1

Answer: 0.422

3 marks

13 (7 marks)

$$\text{(a)} \quad \text{Var}(X) = 18p(1 - p) = 4 \text{ so } 18p^2 - 18p + 4 = 0, \text{ i.e. } 9p^2 - 9p + 2 = 0$$

M1 A1

$$(3p - 1)(3p - 2) = 0 \text{ so } p = \frac{1}{3} \text{ or } p = \frac{2}{3}$$

M1 A1

$$\text{Answer: } p = \frac{1}{3} \text{ or } \frac{2}{3}$$

4 marks

$$\text{(b)} \quad \text{Mean} = 18p > 9 \text{ so } p > 0.5, \text{ hence } p = \frac{2}{3}$$

B1

$$P(X = 12) = C(18, 12)\left(\frac{2}{3}\right)^{12}\left(\frac{1}{3}\right)^6 = 0.196$$

M1 A1

Answer: 0.196

3 marks

14 (7 marks)

$$\text{(a)} \quad X \sim B(15, 0.08)$$

B1

$$P(X = 0) = 0.92^{15} = 0.286$$

A1

Answer: 0.286

2 marks

$$\text{(b)} \quad P(X \leq 2) \text{ from calculator}$$

M1

$$= 0.887$$

A1

Answer: 0.887

2 marks

$$\text{(c)} \quad 1 - 0.92^n > 0.95 \text{ so } 0.92^n < 0.05$$

M1

$$n > \frac{\ln 0.05}{\ln 0.92} = 35.93\dots$$

M1

$$n = 36$$

A1

Answer: n = 36

3 marks

15 (6 marks)

$$\text{(a)} \quad X \text{ takes the values } 0 \text{ to } 6, \text{ so } n = 6$$

B1

The distribution is symmetrical about 3, so $p = 0.5$

B1

$$\text{Check: } P(X = 0) = 0.5^6 = 0.0156 = 0.016$$

B1

Answer: n = 6, p = 0.5

3 marks

$$(b) P(X \geq 5) = P(X = 5) + P(X = 6) = \frac{6 + 1}{64}$$

M1

$$= \frac{7}{64} = 0.109$$

A1

$$\text{Answer: } \frac{7}{64}$$

2 marks

$$(c) \text{Var}(X) = np(1 - p) = 6(0.5)(0.5) = 1.5$$

A1

$$\text{Answer: } 1.5$$

1 mark

16 (7 marks)

$$(a) X \sim B(20, 0.2)$$

B1

$$\text{Answer: } X \sim B(20, 0.2)$$

1 mark

$$(b) M = 4X - (20 - X) = 5X - 20$$

M1

$$E(M) = 5E(X) - 20 = 5 \times 4 - 20 = 0$$

M1 A1

$$\text{Answer: shown}$$

3 marks

$$(c) \text{Var}(X) = 20 \times 0.2 \times 0.8 = 3.2$$

B1

$$\text{Var}(M) = 5^2 \times 3.2 = 80$$

M1 A1

$$\text{Answer: } 80$$

3 marks

17 (4 marks)

$$P(X \leq 3) = 0.03324 \text{ and } P(X \leq 4) = 0.09047$$

M1 A1

The first is ≤ 0.05 and the second is > 0.05

M1

$$\text{So } c = 3 \text{ and } P(X \leq 3) = 0.03324$$

A1

$$\text{Answer: } c = 3, P = 0.03324$$

4 marks

18 (4 marks)

$$P(X \geq 3 | X \geq 1) = \frac{P(X \geq 3)}{P(X \geq 1)}$$

M1

$$P(X \geq 3) = 1 - P(X \leq 2) = 0.3215$$

M1 A1

$$P(X \geq 1) = 1 - 0.75^8 = 0.8999; \text{ ratio} = 0.357$$

A1

$$\text{Answer: } 0.357$$

4 marks

19 (5 marks)

$$P(\text{a box has no cracked eggs}) = 0.92^{10} = 0.4344$$

M1 A1

$$\text{Number of such boxes } Y \sim B(5, 0.4344)$$

M1

$$P(Y = 2) = C(5, 2)(0.4344)^2(1 - 0.4344)^3 = 0.341$$

M1 A1

$$\text{Answer: } 0.341$$

5 marks

20 (5 marks)

$$(1 - p)^5 = 0.07776$$

M1

$$1 - p = 0.07776^{1/5} = 0.6, \text{ so } p = 0.4$$

M1 A1

$$P(X = 3) = C(5, 3)(0.4)^3(0.6)^2 = 0.230$$

M1 A1

$$\text{Answer: } p = 0.4, P(X = 3) = 0.230$$

5 marks

21 (9 marks)

$$(a) P(X = 1) = 12p(1 - p)^{11} \text{ and } P(X = 0) = (1 - p)^{12}$$

M1 A1

$$12p(1 - p)^{11} = 3(1 - p)^{12}$$

M1

$$\text{Dividing by } (1 - p)^{11}: 12p = 3(1 - p), \text{ so } 15p = 3 \text{ and } p = 0.2$$

A1

$$\text{Answer: } p = 0.2$$

4 marks

$$(b) P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$$

M1

$$= 1 - 0.8^{12} - 12(0.2)(0.8)^{11} = 1 - 0.06872 - 0.2062$$

M1

$$= 0.725$$

A1

$$\text{Answer: } 0.725$$

3 marks

$$(c) P(X = 1) = 0.206, P(X = 2) = 0.283, P(X = 3) = 0.236$$

M1

The largest is at $r = 2$, so the mode is 2

A1

$$\text{Answer: } 2$$

2 marks

22 (7 marks)**(a)** $G \sim B(6, 0.2)$

B1

Pass needs $4 + G \geq 7$, so $G \geq 3$

M1

$$P(G \geq 3) = 1 - P(G \leq 2) = 0.09888$$

A1

Answer: 0.0989

3 marks

(b) Exactly 7 means $G = 3$: $P(G = 3) = C(6, 3)(0.2)^3(0.8)^3 = 0.08192$

M1 A1

$$P(G = 3 | G \geq 3) = \frac{0.08192}{0.09888} = 0.828$$

A1

Answer: 0.828

3 marks

(c) Sam may not guess at random (some answers can be eliminated), so the probability is not constant at 0.2

E1

Answer: Guesses may not be random / p not constant

1 mark

23 (10 marks)**(a)** Attending patients $\sim B(20, 0.8)$: $P(\text{all attend}) = 0.8^{20} = 0.0115$

M1 A1

Non-attenders $N \sim B(20, 0.2)$: $P(N > 3) = 1 - P(N \leq 3) = 0.5886$

M1 A1

Answer: 0.0115 and 0.5886

4 marks

(b) Attenders $A \sim B(21, 0.8)$; need $A \leq 20$

M1

$$P(A \leq 20) = 1 - P(A = 21) = 1 - 0.8^{21} = 0.9908$$

M1 A1

Answer: 0.9908

3 marks

(c) $A \sim B(25, 0.8)$; a patient cannot be seen when $A \geq 21$

M1

$$P(A \geq 21) = P(A = 21) + \dots + P(A = 25)$$

M1

$$= 0.4207$$

A1

Answer: 0.4207

3 marks

24 (8 marks)**(a)** $P = P(X = 1) \times P(Y = 1) = 0.1977 \times 0.07118$

M1

$$= 0.0141$$

A1

Answer: 0.0141

2 marks

(b) $P(X + Y = 2) = P(0, 2) + P(1, 1) + P(2, 0)$

M1

$$= 0.05765(0.1678) + 0.1977(0.07118) + 0.2965(0.01384)$$

M1

$$= 0.0278$$

A1

Answer: 0.0278

3 marks

(c) $X + Y$ counts the successes in 20 independent trials each with probability 0.3

E1

$$P(B(20, 0.3) = 2) = C(20, 2)(0.3)^2(0.7)^{18} = 0.0278, \text{ which agrees}$$

M1 A1

Answer: 0.0278

3 marks

25 (7 marks)**(a)** $P(X = 0) + P(X = 1) = 0.97^n + n(0.03)(0.97)^{n-1}$

M1 A1

$$= 0.97^{n-1}(0.97 + 0.03n)$$

A1

Answer: shown

3 marks

(b) Trial: $n = 17$ gives 0.9091; $n = 18$ gives 0.8997

M1 A1 A1

So the largest n is 17

A1

Answer: $n = 17$

4 marks

26 (5 marks)

$$P(\text{first two trials succeed, remaining four fail}) = p^2(1 - p)^4$$

M1 A1

$$P(X = 2) = C(6, 2)p^2(1 - p)^4 = 15p^2(1 - p)^4$$

M1

$$\text{Conditional probability} = \frac{p^2(1 - p)^4}{15p^2(1 - p)^4} = \frac{1}{15}, \text{ independent of } p$$

M1 A1

Answer: $\frac{1}{15}$

5 marks

27 (11 marks)

$$(a) P(X = r + 1) = C(n, r + 1) p^{r+1} q^{n-r-1} \quad \text{B1}$$

$$\frac{C(n, r + 1)}{C(n, r)} = \frac{n!}{(r + 1)!(n - r - 1)!} \times \frac{r!(n - r)!}{n!} = \frac{n - r}{r + 1} \quad \text{M1}$$

$$\text{So the ratio is } \frac{n - r}{r + 1} \times \frac{p}{q} \quad \text{A1}$$

Answer: shown 3 marks

$$(b) \text{ The ratio exceeds 1 when } (20 - r)(0.35) > (r + 1)(0.65) \quad \text{M1}$$

$$7 - 0.35r > 0.65r + 0.65, \text{ so } r < 6.35 \quad \text{M1 A1}$$

The probabilities increase up to $r = 7$ and decrease after, so the mode is 7 A1

Answer: 7 4 marks

$$(c) \text{ Mode } m \text{ satisfies } (n + 1)p - 1 \leq m \leq (n + 1)p \quad \text{M1}$$

$$0.35(n + 1) - 1 \leq 11 \leq 0.35(n + 1) \quad \text{A1}$$

$$31.43 \leq n + 1 \leq 34.29, \text{ so } n + 1 = 32, 33 \text{ or } 34 \quad \text{M1}$$

$$n = 31, 32 \text{ or } 33 \quad \text{A1}$$

Answer: $n = 31, 32, 33$ 4 marks

28 (11 marks)

$$(a) P(X = 2) = C(n, 2) p^2 (1 - p)^{n-2}; \text{ let } g(p) = p^2(1 - p)^{n-2} \quad \text{M1}$$

$$g'(p) = 2p(1 - p)^{n-2} - (n - 2)p^2(1 - p)^{n-3} \quad \text{M1 A1}$$

$$g'(p) = p(1 - p)^{n-3}[2(1 - p) - (n - 2)p] = p(1 - p)^{n-3}(2 - np) \quad \text{M1}$$

This is zero ($0 < p < 1$) when $p = \frac{2}{n}$; $g' > 0$ before and $g' < 0$ after, so it is a maximum A1

$$\text{Answer: } p = \frac{2}{n} \quad \text{5 marks}$$

$$(b) p = 0.2: P(X = 2) = 45(0.2)^2(0.8)^8 \quad \text{M1}$$

$$= 0.302 \quad \text{A1}$$

Answer: 0.302 2 marks

$$(c) (q + p)^n = \sum P(X = r) = 1 \text{ and } (q - p)^n = \sum (-1)^r P(X = r) = P(\text{even}) - P(\text{odd}) \quad \text{M1 A1}$$

$$\text{Adding: } 1 + (q - p)^n = 2P(\text{even}); q - p = 1 - 2p \quad \text{M1}$$

$$n = 10, p = 0.3: P(\text{even}) = \frac{1}{2}(1 + 0.4^{10}) = 0.50005 \quad \text{A1}$$

$$\text{Answer: } \frac{1}{2}[1 + (1 - 2p)^n]; 0.50005 \quad \text{4 marks}$$

29 (13 marks)

$$(a) P(\text{accept}) = (1 - p)^{20} + 20p(1 - p)^{19} \quad \text{M1 A1}$$

$$p = 0.02: 0.9401 \quad \text{A1}$$

$$p = 0.10: 0.3917 \quad \text{A1}$$

Answer: 0.9401 and 0.3917 4 marks

$$(b) (1 - p)^{19}[(1 - p) + 20p] = (1 - p)^{19}(1 + 19p) \quad \text{M1 A1}$$

More defectives in the batch makes it less likely that the sample has at most one E1

Answer: shown 3 marks

$$(c) \text{ Solve } (1 - p)^{19}(1 + 19p) = 0.5 \quad \text{M1}$$

$$p = 0.08: 0.5169 \text{ (above 0.5)}, p = 0.085: 0.4836 \text{ (below 0.5)} \quad \text{M1 A1}$$

The root lies between 0.08 and 0.085, and testing $p = 0.0825$ gives 0.5001 so $p \approx 0.083$ A1

Answer: $p \approx 0.083$ 4 marks

$$(d) P(\text{reject}) = 1 - 0.9401 = 0.05990, \text{ so about 6.0\% of good batches are rejected} \quad \text{M1 A1}$$

Answer: about 6.0\% rejected 2 marks

30 (11 marks)

$$(a) \text{ State N: } D \sim B(10, 0.02), P(D \leq 1) = 0.9838 \quad \text{M1 A1}$$

$$\text{State F: } D \sim B(10, 0.2), P(D \leq 1) = 0.3758 \quad \text{A1}$$

Answer: 0.9838 and 0.3758 3 marks

$$(b) P(D \leq 1) = 0.9 \times 0.9838 + 0.1 \times 0.3758 \quad \text{M1}$$

$$= 0.9230 \quad \text{A1}$$

Answer: 0.9230 2 marks

$$(c) P(F | D \leq 1) = \frac{P(F \text{ and } D \leq 1)}{P(D \leq 1)}$$

M1

$$= \frac{0.1 \times 0.3758}{0.9230} = 0.0407$$

M1 A1

Answer: 0.0407

3 marks

$$(d) P(F | D \geq 2) = \frac{0.1(1 - 0.3758)}{1 - 0.9230} = 0.811$$

M1 A1

The prior probability of a fault was only 0.1, so a result of $D \geq 2$ raises it above 0.8: such a sample is strong evidence of a fault

E1

Answer: 0.811

3 marks

Poisson Distribution

1 (2 marks)

$$P(X = 2) = \frac{e^{-3} 3^2}{2!}$$

M1

$$= 0.224$$

A1

Answer: 0.224

2 marks

2 (2 marks)

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-2.5}(1 + 2.5)$$

M1

$$= 0.287$$

A1

Answer: 0.287

2 marks

3 (4 marks)

$$(a) P(X = 3) = e^{-2} \frac{2^3}{3!} = \frac{8}{6} e^{-2} = \frac{4}{3} e^{-2}$$

M1 A1

Answer: shown

2 marks

$$(b) P(X \leq 1) = e^{-2} + 2e^{-2} = 3e^{-2}$$

M1 A1

Answer: $3e^{-2}$

2 marks

4 (2 marks)

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1.2}$$

M1

$$= 0.699$$

A1

Answer: 0.699

2 marks

5 (2 marks)

For a Poisson distribution the mean and the variance both equal λ

M1

$$E(X) = 7, \text{Var}(X) = 7, \text{sd} = \sqrt{7} = 2.65$$

A1

Answer: 7, 7, 2.65

2 marks

6 (2 marks)

$$\text{Mean in 20 minutes} = 6 \times \frac{1}{3} = 2, \text{ so } X \sim \text{Po}(2)$$

M1

$$P(X = 0) = e^{-2} = 0.135$$

A1

Answer: 0.135

2 marks

7 (4 marks)

(a) Calls occur independently of each other

B1

Calls occur singly at a constant average rate

B1

Answer: Independent; singly; constant rate

2 marks

(b) $X \sim \text{Po}(4.5)$

B1

$$P(X = 6) = \frac{e^{-4.5} 4.5^6}{6!} = 0.128$$

A1

Answer: 0.128

2 marks

8 (4 marks)

(a) Mean = $12 \times \frac{10}{60} = 2$, so Po(2)

B1

Answer: Po(2)

1 mark

(b) $P(X \geq 3) = 1 - P(X \leq 2)$

M1

$= 1 - e^{-2}(1 + 2 + 2)$

M1

$= 0.323$

A1

Answer: 0.323

3 marks

9 (3 marks)

$P(3 \leq X \leq 7) = P(X \leq 7) - P(X \leq 2)$

M1

$= 0.8095 - 0.08838$

M1

$= 0.721$

A1

Answer: 0.721

3 marks

10 (6 marks)

(a) Orders in half a day $\sim \text{Po}(1.75)$

B1

$P(X = 2) = e^{-1.75} \frac{1.75^2}{2} = 0.266$

M1 A1

Answer: 0.266

3 marks

(b) Orders in a day $X \sim \text{Po}(3.5)$; the stock is not enough when $X \geq 6$

M1

$P(X \geq 6) = 1 - P(X \leq 5) = 0.142$

M1 A1

Answer: 0.142

3 marks

11 (2 marks)

Poisson requires events to occur at random, independently of each other

M1

Timetabled arrivals are regular (almost exactly 6 per hour), so the variance would be much smaller than the mean

E1

Answer: Not random: buses are regular

2 marks

12 (4 marks)

$e^{-\lambda} = 0.1$ so $\lambda = \ln 10 = 2.303$

M1 A1

$P(X = 2) = e^{-\lambda} \frac{\lambda^2}{2} = 0.1 \times \frac{(\ln 10)^2}{2}$

M1

$= 0.265$

A1

Answer: $\lambda = \ln 10$, $P(X = 2) = 0.265$

4 marks

13 (5 marks)

$e^{-\lambda} \frac{\lambda^5}{120} = e^{-\lambda} \frac{\lambda^6}{720}$

M1

Dividing by $e^{-\lambda} \lambda^5$: $\frac{1}{120} = \frac{\lambda}{720}$ so $\lambda = 6$

M1 A1

$P(3 < X \leq 8) = P(X \leq 8) - P(X \leq 3)$

M1

$= 0.8472 - 0.1512 = 0.696$

A1

Answer: $\lambda = 6$, $P = 0.696$

5 marks

14 (7 marks)

(a) Flaws in A: $\text{Po}(2 \times 0.8) = \text{Po}(1.6)$; in B: $\text{Po}(1.1)$

M1

Sum of independent Poissons: $\text{Po}(2.7)$

A1

Answer: Po(2.7)

2 marks

(b) $P(T \geq 4) = 1 - P(T \leq 3)$ with $T \sim \text{Po}(2.7)$

M1

$= 0.286$

A1

Answer: 0.286

2 marks

(c) $P(2 \text{ in A and } 1 \text{ in B}) = 0.2584 \times 0.3662 = 0.09463$

M1 A1

$P(T = 3) = 0.2205$, so the conditional probability = 0.429

A1

Answer: 0.429

3 marks

15 (7 marks)

(a) $P(X = 0) = e^{-\lambda} = 0.05$ M1

$\lambda = \ln 20 = 2.996 \approx 3$ A1

Answer: $\lambda \approx 3$ 2 marks

(b) $P(X \geq 6) = 1 - P(X \leq 5)$ M1

$= 1 - e^{-3}(1 + 3 + 4.5 + 4.5 + 3.375 + 2.025)$ M1

$= 0.0839$ A1

Answer: 0.0839 3 marks

(c) $P(X = 2) = e^{-3} \frac{9}{2}$, $P(X = 3) = e^{-3} \frac{27}{6} = e^{-3} \frac{9}{2}$ M1

The bars at $X = 2$ and $X = 3$ have the same height (joint modes) E1**Answer: Equal; joint tallest bars** 2 marks**16 (5 marks)**Mean in d days is $0.8d$, so $P(\text{at least one}) = 1 - e^{-0.8d}$ M1

$1 - e^{-0.8d} > 0.95$ gives $e^{-0.8d} < 0.05$ M1

$d > \frac{\ln 20}{0.8} = 3.745$ M1 A1

Smallest whole number is $d = 4$ A1**Answer: $d = 4$** 5 marks**17 (5 marks)**Failures in 5 days $X \sim \text{Po}(7.5)$ B1Running out with s spares means $X > s$, i.e. $X \geq s + 1$; need $P(X > s) < 0.01$ M1

$P(X > 14) = 0.01026$ (too big) and $P(X > 15) = 0.004608$ M1 A1

The smallest stock is 15 tubes A1

Answer: 15 tubes 5 marks**18 (5 marks)**(a) n is large and p is small E1Mean = $np = 3$, so approximate by $\text{Po}(3)$ B1**Answer: $\text{Po}(3)$** 2 marks

(b) Poisson: $P(X \leq 2) = 0.4232$ M1 A1

Exact: $P(X \leq 2) = 0.4215$; the two agree to 2 decimal places A1**Answer: approx 0.4232, exact 0.4215** 3 marks**19 (7 marks)**

(a) $\frac{e^{-\lambda} \lambda^{r+1}}{(r+1)!} \div \frac{e^{-\lambda} \lambda^r}{r!}$ M1

$= \frac{\lambda}{r+1}$ A1

Answer: shown 2 marks(b) $P(r+1) > P(r)$ when $r+1 < \lambda$; for $\lambda = 4.3$, probabilities rise while $r+1 < 4.3$, i.e. up to $r = 3 \rightarrow P(4)$ M1 A1

$P(5)/P(4) = \frac{4.3}{5} < 1$ so the mode is 4 A1

For $\lambda = 4$: $P(4)/P(3) = \frac{4}{4} = 1$, so $P(3) = P(4)$ and both are modes M1 A1**Answer: mode 4; modes 3 and 4 when $\lambda = 4$** 5 marks**20 (7 marks)**

(a) Mean = $\frac{0 + 22 + 54 + 63 + 52 + 30 + 12}{100} = 2.33$ M1 A1

Variance = $\frac{\sum fr^2}{100} - \text{mean}^2 = 7.490 - 5.429 = 2.06$ M1 A1

Answer: mean 2.33, variance 2.06 4 marks

(b) The mean and the variance are close in value, as they would be for a Poisson distribution E1

With $\lambda = 2.33$, $100 \times P(X = 2) = 100 \times 0.2641 = 26.4$ M1 A1

Answer: 26.4 days**3 marks****21 (8 marks)**

(a) $\frac{e^{-\lambda} \lambda^3}{6} = 1.5 e^{-\lambda} \lambda$

M1Divide by $e^{-\lambda} \lambda$ (non-zero): $\frac{\lambda^2}{6} = 1.5$ **M1**

$\lambda^2 = 9$ so $\lambda = 3$ ($\lambda > 0$)

A1**Answer: $\lambda = 3$** **3 marks**

(b) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - e^{-3}(1 + 3 + 4.5 + 4.5)$
 $= 0.353$

M1**A1****Answer: 0.353****2 marks**

(c) $Y = \text{number of observations} \geq 4: Y \sim B(5, 0.3528)$

M1 A1

$P(Y = 2) = 0.337$

A1**Answer: 0.337****3 marks****22 (8 marks)**

(a) $P(X \geq 1) = 1 - e^{-0.6}$
 $= 0.451$

M1**A1****Answer: 0.451****2 marks**

(b) Mean for 5 pages = 3, $Y \sim \text{Po}(3)$

B1

$P(Y \leq 2) = e^{-3}(1 + 3 + 4.5)$
 $= 0.423$

M1**A1****Answer: 0.423****3 marks**

(c) n pages: mean $0.6n$, $P(\text{no errors}) = e^{-0.6n}$

M1

$e^{-0.6n} < 0.01$ so $n > \frac{\ln 100}{0.6} = 7.67...$

M1

$n = 8$

A1**Answer: $n = 8$** **3 marks****23 (5 marks)**

$P(\text{no vehicle in a minute}) = e^{-1.5} = 0.2231$

M1 A1

Number of empty intervals $Y \sim B(10, 0.2231)$

M1

$P(Y = 3) = C(10, 3)(0.2231)^3(0.7769)^7 = 0.228$

M1 A1**Answer: 0.228****5 marks****24 (7 marks)**

(a) $\text{Var}(Y) = a^2 \text{Var}(X) = 4a^2 = 16$, so $a = 2$ ($a > 0$)

M1 A1

$E(Y) = 2 \times 4 + b = 16$ so $b = 8$

M1 A1**Answer: $a = 2$, $b = 8$** **4 marks**

(b) $Y = 2X + 8$ can only take the values 8, 10, 12, ...

M1

So $P(Y = 0) = 0$, but a Poisson distribution with mean 16 has $P(Y = 0) = e^{-16} > 0$

M1 A1**Answer: $P(Y = 0) = 0$, not e^{-16}** **3 marks****25 (9 marks)**

(a) Busy: mean 16, quiet: mean 9. Independent Poisson variables add

M1 A1

$T \sim \text{Po}(25)$

A1**Answer: $T \sim \text{Po}(25)$** **3 marks**

(b) $P(T \leq 20) = 0.1855$

B1

$P(T > 30) = 1 - P(T \leq 30) = 0.1367$

M1 A1**Answer: 0.1855, 0.1367****3 marks**

(c) Busy calls $B \sim \text{Po}(16)$, quiet $Q \sim \text{Po}(9)$

M1

$P(B = 14 \mid T = 22) = C(22, 14)\left(\frac{16}{25}\right)^{14}\left(\frac{9}{25}\right)^8 = 0.174$

M1 A1**Answer: 0.174****3 marks**

26 (10 marks)**(a)** $E(F) = \text{Var}(F) = 0.7$, so $E(C) = 50 \times 0.7 = \text{£}35$

M1 A1

 $\text{Var}(C) = 50^2 \times 0.7 = 1750$, $\text{sd} = \text{£}41.83$

M1 A1

Answer: £35, £41.83

4 marks

(b) $C > 100$ means $F > 2$, i.e. $F \geq 3$

M1

 $P(F \geq 3) = 1 - P(F \leq 2) = 1 - 0.9659 = 0.0341$

M1 A1

Answer: 0.0341

3 marks

(c) F over 6 months $\sim \text{Po}(4.2)$; cost < 200 means $F \leq 3$

M1 A1

 $P(F \leq 3) = 0.3954$

A1

Answer: 0.3954

3 marks

27 (11 marks)**(a)** $X + Y \sim \text{Po}(\lambda + \mu)$

B1

$$P(X = r \mid X + Y = n) = \frac{P(X = r) P(Y = n - r)}{P(X + Y = n)}$$

M1

$$= \frac{e^{-\lambda} \lambda^r}{r!} \times \frac{e^{-\mu} \mu^{n-r}}{(n-r)!} \div \frac{e^{-(\lambda+\mu)} (\lambda + \mu)^n}{n!}$$

M1 A1

$$= \frac{n!}{r!(n-r)!} \frac{\lambda^r \mu^{n-r}}{(\lambda + \mu)^n}, \text{ which is the stated form, } B(n, \frac{\lambda}{\lambda + \mu})$$

A1

Answer: $X \mid X + Y = n \sim B(n, \frac{\lambda}{\lambda + \mu})$

5 marks

(b) $X \mid X + Y = 5 \sim B(5, \frac{2}{5})$

M1

$$P = C(5, 2)(0.4)^2(0.6)^3$$

M1

$$= 0.3456$$

A1

Answer: 0.3456

3 marks

(c) n is large and p is small, so use $\text{Po}(np) = \text{Po}(3)$

B1 E1

$$P(W \leq 2) \approx e^{-3}(1 + 3 + 4.5) = 0.4232$$

A1

Answer: 0.4232 (exact 0.4209)

3 marks

28 (10 marks)**(a)** Mean in 20 minutes = 1

B1

$$P(\text{none}) = e^{-1} = 0.368$$

A1

Answer: 0.368

2 marks

(b) $T > t$ means no calls in the next t hours, whose number is $\text{Po}(3t)$

M1

$$P(T > t) = e^{-3t}$$

A1

Answer: e^{-3t}

2 marks

$$\text{(c)} P(\frac{1}{6} < T < \frac{1}{2}) = P(T > \frac{1}{6}) - P(T > \frac{1}{2})$$

M1

$$= e^{-0.5} - e^{-1.5} = 0.383$$

A1

Answer: 0.383

2 marks

$$\text{(d)} \text{Required} = \frac{P(2 \text{ in first 20 min}) \times P(0 \text{ in last 40 min})}{P(2 \text{ in the hour})}$$

M1

$$= \frac{e^{-1}(\frac{1^2}{2}) \times e^{-2}}{e^{-3}(\frac{3^2}{2})}$$

M1 A1

$$= \frac{1}{9}$$

A1

Answer: $\frac{1}{9}$

4 marks

29 (10 marks)

$$\text{(a)} \sum_{r=0}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!}$$

M1

$$= e^{-\lambda} \times e^{\lambda} = 1$$

A1

Answer: shown

2 marks

$$(b) E(X) = \sum_{r=1}^{\infty} r e^{-\lambda} \frac{\lambda^r}{r!} = \lambda e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$

M1 A1

The sum is e^{λ} , so $E(X) = \lambda$

A1

Answer: λ

3 marks

$$(c) E[X(X-1)] = \sum_{r=2}^{\infty} r(r-1) e^{-\lambda} \frac{\lambda^r}{r!} = \lambda^2 e^{-\lambda} \sum_{r=2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!}$$

M1 A1

$$= \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

A1

$$E(X^2) = E[X(X-1)] + E(X) = \lambda^2 + \lambda, \text{ so } \text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

M1 A1

Answer: $\text{Var}(X) = \lambda$

5 marks

30 (12 marks)

(a) Given $N = n$, $V \sim B(n, 0.3)$

M1

Sum over all $n \geq v$ of $P(N = n) \times P(V = v | N = n)$

M1 A1

Answer: shown

3 marks

$$(b) C(n, v) \frac{20^n}{n!} = \frac{20^n}{v!(n-v)!}, \text{ so } P(V = v) = e^{-20} \frac{(0.3 \times 20)^v}{v!} \sum_{n=v}^{\infty} \frac{(0.7 \times 20)^{n-v}}{(n-v)!}$$

M1 A1

$$\text{The sum is } e^{14}, \text{ so } P(V = v) = e^{-6} \frac{6^v}{v!}$$

M1 A1

This is the Poisson distribution with mean 6

A1

Answer: $V \sim \text{Po}(6)$

5 marks

$$(c) P(V \geq 4) = 1 - P(V \leq 3) = 1 - 0.1512 = 0.849$$

M1 A1

Answer: 0.849

2 marks

$$(d) \text{ Each of the 5 cars is not a van with probability 0.7, so } P = 0.7^5 = 0.168$$

M1

A1

Answer: 0.168

2 marks

Normal Distribution

1 (2 marks)

$$z = \frac{54 - 50}{4} = 1$$

M1

$$P(Z < 1) = 0.8413$$

A1

Answer: 0.8413

2 marks

2 (2 marks)

$$z = \frac{130 - 100}{15} = 2$$

M1

$$P(X > 130) = P(Z > 2) = 0.0228$$

A1

Answer: 0.0228

2 marks

3 (4 marks)

$$(a) P(X > 33) = P(Z > 1) = 0.159$$

M1 A1

Answer: 0.159

2 marks

$$(b) P(27 < X < 36) = P(-1 < Z < 2) = 0.819$$

M1

A1

Answer: 0.819

2 marks

4 (2 marks)

$$P = \Phi(2) - \Phi(-1) = 0.9772 - 0.1587 = 0.819$$

M1

A1

Answer: 0.819

2 marks

5 (3 marks)

Recorded mass 160 g means the true mass lies between 159.5 g and 160.5 g

M1

$$P(159.5 < X < 160.5) = P(1.1875 < Z < 1.3125)$$

M1

$$= 0.0228$$

A1

Answer: 0.0228

3 marks

6 (2 marks)

z-value for 0.8 is 0.8416 (calculator)

M1

$$a = 40 + 0.8416 \times 5 = 44.2$$

A1

Answer: 44.2

2 marks

7 (5 marks)

(a) z-values: $\frac{165 - 172}{8} = -0.875$ and $\frac{180 - 172}{8} = 1$

M1

$$P(-0.875 < Z < 1) = P(Z < 1) - P(Z < -0.875)$$

M1

$$= 0.8413 - 0.1908 = 0.6506$$

A1

Answer: 0.651

3 marks

(b) $P(H > 185) = P(Z > 1.625) = 0.05208$

M1

$$\text{Expected number} = 500 \times 0.05208 = 26.0, \text{ about } 26$$

A1

Answer: about 26

2 marks

8 (5 marks)

(a) $P(X < a) = 0.9$, so $\frac{a - 60}{10} = 1.2816$ (from calculator)

M1

$$a = 72.8$$

A1

Answer: a = 72.8

2 marks

(b) Upper z-value for 0.975 is 1.96

M1

$$60 \pm 1.96 \times 10$$

M1

$$(40.4, 79.6)$$

A1

Answer: (40.4, 79.6)

3 marks

9 (3 marks)Upper tail 0.1 so $z = 1.2816$ (calculator)

M1 A1

$$k = 8 + 1.2816 \times 2 = 10.6$$

A1

Answer: 10.6

3 marks

10 (4 marks)

(a) $z = \frac{1.46 - 1.5}{0.02} = -2$

M1

$$P(Z < -2) = 0.02275$$

A1

Answer: 0.02275

2 marks

(b) $2000 \times 0.02275 = 45.5$

M1

About 46 bags

A1

Answer: about 46

2 marks

11 (3 marks)z for $P(Z < z) = 0.25$ is -0.6745 (calculator)

M1

$$\text{Lower quartile} = 70 - 0.6745 \times 12 = 61.9$$

A1

$$\text{By symmetry upper quartile} = 70 + 0.6745 \times 12 = 78.1$$

A1

Answer: 61.9 and 78.1

3 marks

12 (5 marks)

(a) $P(12 < X < 18) = 1 - 0.2 - 0.2$

M1

$$= 0.6$$

A1

Answer: 0.6

2 marks

(b) $z = -0.8416$ for $P(Z < z) = 0.2$, so $\frac{12 - 15}{\sigma} = -0.8416$

M1 A1

$$\sigma = \frac{3}{0.8416} = 3.56$$

A1

Answer: 3.56**3 marks****13 (3 marks)** $z = 0.5244$ for an upper tail of 0.3**B1**

$$\frac{40 - \mu}{6} = 0.5244$$

M1

$$\mu = 40 - 3.146 = 36.9$$

A1**Answer: 36.9****3 marks****14 (6 marks)**

$$P(X < 22) = 0.1 \text{ so } \frac{22 - \mu}{\sigma} = -1.2816$$

M1 A1

$$P(X > 35) = 0.05 \text{ so } \frac{35 - \mu}{\sigma} = 1.6449$$

M1 A1

$$\text{Subtract: } \frac{13}{\sigma} = 2.9265, \text{ so } \sigma = 4.44$$

M1

$$\mu = 22 + 1.2816\sigma = 27.7$$

A1**Answer: $\sigma = 4.44$, $\mu = 27.7$** **6 marks****15 (6 marks)**

$$(a) z = \frac{65 - 60}{5} = 1$$

M1

$$P(X > 65) = 0.1587$$

A1**Answer: 0.1587****2 marks**

$$(b) Y \sim B(10, 0.1587)$$

M1

$$P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1)$$

M1

$$= 1 - 0.1777 - 0.3351$$

M1

$$= 0.487$$

A1**Answer: 0.487****4 marks****16 (6 marks)****(a)** By symmetry μ is halfway between 21 and 33, so $\mu = 27$ **M1 A1**

$$P(Y > 33) = 0.1587 \text{ gives } \frac{33 - 27}{\sigma} = 1.000 \text{ (calculator)}$$

M1

$$\sigma = 6$$

A1**Answer: $\mu = 27$, $\sigma = 6.00$** **4 marks**

$$(b) P(24 < Y < 35) = P(-0.5 < Z < 1.333)$$

M1

$$= 0.6002$$

A1**Answer: 0.6002****2 marks****17 (6 marks)****(a)** By symmetry of the normal distribution, $P(X < \mu - a) = P(X > \mu + a) = 0.15$ **B1****Answer: 0.15****1 mark**

$$(b) P = 1 - 0.15 - 0.15 = 0.7$$

M1 A1**Answer: 0.7****2 marks**

$$(c) P(X > \mu - a) = 1 - 0.15 = 0.85$$

B1

$$P(X < \mu + a \mid X > \mu - a) = \frac{P(\mu - a < X < \mu + a)}{P(X > \mu - a)} = \frac{0.7}{0.85}$$

M1

$$= \frac{14}{17}$$

A1**Answer: $\frac{14}{17}$** **3 marks****18 (6 marks)****(a)** λ is large, so the Poisson distribution is approximately symmetric and bell-shaped**E1** $Y \sim N(36, 36)$, that is mean 36 and variance 36**B1****Answer: $N(36, 6^2)$** **2 marks**

$$(b) P(X \leq 30) \approx P(Y < 30.5)$$

M1

$$z = \frac{30.5 - 36}{6} = -0.9167, \text{ so probability} = 0.1797$$

M1 A1

Exact Poisson value = 0.1806

A1

Answer: approx 0.1797, exact 0.1806

4 marks

19 (4 marks)

$$P(X > \mu + \sigma | X > \mu) = \frac{P(X > \mu + \sigma)}{P(X > \mu)}$$

M1

$$= \frac{P(Z > 1)}{0.5} = \frac{0.1587}{0.5}$$

M1 A1

$$= 0.317$$

A1

Answer: 0.317

4 marks

20 (4 marks)

$$P(T < 0) = P(Z < \frac{0 - 8}{4}) = P(Z < -2)$$

M1 A1

$$= 0.0228 \approx 2.3\%$$

A1

A waiting time cannot be negative, so the normal model is not a good fit in the lower tail

E1

Answer: 0.0228; model gives impossible values

4 marks

21 (8 marks)

(a) n is large and p is close enough to 0.5 (np = 32 and nq = 48 both greater than 5)

E1

Mean = np = 32

B1

Variance = npq = 80(0.4)(0.6) = 19.2, so $Y \sim N(32, 19.2)$

B1

Answer: N(32, 19.2)

3 marks

(b) $P(X \geq 38) \approx P(Y > 37.5)$

M1

$$z = \frac{37.5 - 32}{\sqrt{19.2}} = 1.255$$

M1

$$P(Z > 1.255) = 0.105$$

A1

Answer: 0.105

3 marks

(c) $P(29.5 < Y < 36.5)$: $z = -0.570$ and 1.027

M1

 $= 0.5636$, which agrees with the exact value 0.5617 to 2 decimal places

A1

Answer: 0.5636

2 marks

22 (9 marks)

$$(a) P(V < 500) = 0.02, \text{ so } \frac{500 - \mu}{0.8} = -2.0537$$

M1 A1

$$\mu = 500 + 2.0537 \times 0.8 = 501.64$$

A1

Answer: $\mu = 501.6$

3 marks

$$(b) z = \frac{503 - 501.643}{0.8} = 1.696$$

M1

$$P(V > 503) = 0.0449$$

A1

Answer: 0.0449

2 marks

(c) $\mu - 2.0537\sigma = 500$ and $\mu + 2.3263\sigma = 503$

M1 A1

Subtract: $4.3800\sigma = 3$, so $\sigma = 0.685$

M1

$$\mu = 500 + 2.0537\sigma = 501.41$$

A1

Answer: $\sigma = 0.685$, $\mu = 501.41$

4 marks

23 (8 marks)(a) Pass mark: $P(X > m) = 0.85$ so $z = -1.0364$, $m = 58 - 1.0364 \times 14 = 43.49$

M1 A1

Distinction: $z = 1.1750$, mark = $58 + 1.1750 \times 14 = 74.45$

M1 A1

Pass mark 43 and distinction mark 74 (to the nearest whole number)

A1

Answer: 43 and 74

5 marks

$$(b) P(\text{distinction} | \text{pass}) = \frac{0.12}{0.85}$$

M1 A1

$$= 0.141$$

A1

Answer: 0.141

3 marks

24 (8 marks)

$$(a) P(X < k) = P(Z < \frac{k - 80}{5}) \text{ and } P(Y > k) = P(Z > \frac{k - 50}{10}) = P(Z < -\frac{k - 50}{10})$$

M1 A1

$$\text{So } \frac{k - 80}{5} = -\frac{k - 50}{10}, \text{ giving } 2(k - 80) = -(k - 50)$$

M1

$$3k = 210 \text{ so } k = 70$$

A1

Answer: k = 70

4 marks

$$(b) P(Z < -2) = 0.02275$$

M1 A1

Answer: 0.02275

2 marks

$$(c) P(X < 70) = 0.02275, P(Y < 70) = P(Z < 2) = 0.9772$$

M1

$$P = 0.02223$$

A1

Answer: 0.02223

2 marks

25 (10 marks)

$$(a) P(1.95 < X < 2.08) = P(-1.25 < Z < 2)$$

M1

$$= 0.8716 - 0.1056 = 0.8716$$

M1 A1

Answer: 0.8716

3 marks

$$(b) \text{ The median } m \text{ satisfies } P(1.95 < X < m) = 0.5 \times 0.8716 = 0.4358$$

M1

$$P(X < m) = 0.1056 + 0.4358 = 0.5414$$

M1 A1

$$\frac{m - 2}{0.04} = 0.1041 \text{ so } m = 2.004 \text{ m, i.e. } 2.004 \text{ m}$$

A1

Answer: 2.004 m

4 marks

$$(c) P(X < Q_1) = 0.1056 + 0.25 \times 0.8716 = 0.3235$$

M1

$$Q_1 = 1.982 \text{ m, i.e. } 1.982 \text{ m}$$

M1 A1

Answer: 1.982 m

3 marks

26 (7 marks)

$$(a) Q_3 = \mu + 0.6745\sigma \text{ and } Q_1 = \mu - 0.6745\sigma$$

M1 A1

$$Q_3 - Q_1 = 1.349\sigma$$

A1

Answer: 1.349σ

3 marks

$$(b) 1.5 \times 1.349\sigma = 2.0235\sigma$$

M1

$$\text{Upper: } \mu + 0.6745\sigma + 2.0235\sigma = \mu + 2.698\sigma; \text{ lower: } \mu - 2.698\sigma$$

A1

Answer: μ ± 2.698σ

2 marks

$$(c) P(|Z| > 2.698) = 2P(Z > 2.698)$$

M1

$$= 0.00698$$

A1

Answer: 0.00698

2 marks

27 (11 marks)

$$(a) P(T > 40) = P(Z > 1) = 0.1587$$

B1

$$P(T > 35) = P(Z > \frac{1}{6}) = 0.4338$$

B1

$$\text{Ratio} = 0.366$$

A1

Answer: 0.366

3 marks

$$(b) Y \sim B(5, 0.1587)$$

M1

$$P(Y = 1) = 5(0.1587)(0.8413)^4 = 0.3975$$

M1 A1

$$P(Y = 1 | Y \geq 1) = \frac{0.3975}{1 - 0.8413^5} = 0.687$$

A1

Answer: 0.687

4 marks

$$(c) P(T > 30) = P(Z > -\frac{2}{3}) = 0.7475$$

B1

$$P(30 < T < t) = 0.5 \times 0.7475 = 0.3738$$

M1

$$P(T < t) = 0.2525 + 0.3738 = 0.6262$$

M1

$$t = 34 + 6 \times 0.3219 = 35.9$$

A1

Answer: t = 35.9

4 marks

28 (11 marks)

(a) $P(k < X < k + 4) = \int \text{over } (k, k+4) \text{ of } \varphi(x) \, dx = \Phi(k + 4) - \Phi(k)$, where Φ is the cumulative distribution function **M1**

$$\frac{d}{dk} [\Phi(k + 4) - \Phi(k)] = \varphi(k + 4) - \varphi(k) \quad \text{M1 A1}$$

This is zero when $\varphi(k + 4) = \varphi(k)$; as φ is symmetrical and decreasing away from μ , this needs $k + 4 - \mu = \mu - k$ **M1**

So $k = \mu - 2$; for k below this the derivative is positive and above it negative, so it is a maximum **E1**

Answer: $k = \mu - 2$ **5 marks**

(b) Greatest when $k = 48$; $P(48 < X < 52)$ **M1**

$$= P(-0.667 < Z < 0.667) = 0.4950 \quad \text{A1}$$

Answer: 0.495 **2 marks**

(c) By part (a) the shortest interval is symmetrical about the mean **E1**

$$P(Z < z) = 0.95 \text{ gives } z = 1.6449 \quad \text{M1}$$

$$\text{Width} = 2 \times 1.6449 \times 3 \quad \text{M1}$$

$$= 9.87 \quad \text{A1}$$

Answer: 9.87 **4 marks**

29 (14 marks)

(a) Girls: $P(X > 175) = P(Z > 1.667) = 0.04779$ **M1 A1**

Boys: $P(X > 175) = P(Z > -0.2857) = 0.6125$ **A1**

$$P = 0.55(0.04779) + 0.45(0.6125) = 0.3019 \quad \text{M1 A1}$$

Answer: 0.3019 **5 marks**

$$(b) P(\text{boy} | > 175) = \frac{0.45 \times 0.6125}{0.3019} \quad \text{M1 A1}$$

$$= 0.913 \quad \text{A1}$$

Answer: 0.913 **3 marks**

(c) Let $G(x) = 0.55P(\text{girl} < x) + 0.45P(\text{boy} < x)$; the median m satisfies $G(m) = 0.5$ **M1**

$G(169.7) = 0.4976$ and $G(169.8) = 0.5018$, so m lies between 169.7 and 169.8 **M1 A1**

$G(169.75) = 0.4997 < 0.5$, so m lies between 169.75 and 169.8 and $m = 169.8$ cm to 1 d.p. **A1**

The 90th percentile h satisfies $0.55P(\text{girl} > h) + 0.45P(\text{boy} > h) = 0.1$; testing $h = 182.4$ gives **M1**

$$0.1001 \quad \text{A1}$$

So $h = 182.4$ cm **A1**

Answer: median 169.8 cm, $h = 182.4$ cm **6 marks**

30 (14 marks)

(a) $np = 20$ and $npq = 10$ **B1**

X is approximately $N(20, 10)$ **B1**

Answer: $N(20, 10)$ **2 marks**

(b) Exact: 0.2148 **B1**

$$(i) P(Y < 17) = P(Z < -0.9487) = 0.1714 \quad \text{M1 A1}$$

$$(ii) P(Y < 17.5) = P(Z < -0.7906) = 0.2146 \quad \text{M1 A1}$$

Answer: exact 0.2148, 0.1714, 0.2146 **5 marks**

(c) X takes integer values, so the value 17 occupies the interval 16.5 to 17.5 on the continuous scale **E1**

$P(X \leq 17)$ therefore corresponds to $P(Y < 17.5)$, which includes the whole bar at 17 **E1**

Answer: integer 17 covers 16.5 to 17.5 **2 marks**

$$(d) P(X = 20) \approx P(19.5 < Y < 20.5) = 0.1256 \quad \text{M1 A1}$$

$$\text{Exact value} = 0.1254 \quad \text{B1}$$

$$\text{Percentage error} = \frac{|0.1256 - 0.1254|}{0.1254} \times 100 = 0.21\% \quad \text{M1 A1}$$

Answer: 0.1256, error 0.21% **5 marks**

Continuous Random Variables

1 (2 marks)

$$\int_0^2 kx \, dx = 1$$

M1

$$\left[\frac{kx^2}{2}\right]_0^2 = 2k = 1, \text{ so } k = \frac{1}{2}$$

A1

$$\text{Answer: } k = \frac{1}{2}$$

2 marks

2 (3 marks)

$$P(X < 1) = \int_0^1 \frac{3}{8}x^2 \, dx$$

M1

$$= \left[\frac{x^3}{8}\right]_0^1$$

A1

$$= \frac{1}{8}$$

A1

$$\text{Answer: } \frac{1}{8}$$

3 marks

3 (4 marks)

(a) $f(0) = -1 < 0$, and a probability density cannot be negative

B1

Answer: No: $f(0) = -1$ is negative

1 mark

$$(b) \int_0^2 \frac{x}{3} \, dx = \left[\frac{x^2}{6}\right]_0^2 = \frac{2}{3} \neq 1$$

M1 A1

$$\text{Answer: No: total area is } \frac{2}{3}$$

2 marks

$$(c) f(x) \geq 0 \text{ and } \int_0^2 \frac{3x^2}{8} \, dx = \left[\frac{x^3}{8}\right]_0^2 = 1$$

B1

Answer: Yes

1 mark

4 (2 marks)

$$P(X < 2) = \int_0^2 \frac{x}{8} \, dx = \left[\frac{x^2}{16}\right]_0^2$$

M1

$$= \frac{4}{16} = \frac{1}{4}$$

A1

$$\text{Answer: } \frac{1}{4}$$

2 marks

5 (3 marks)

$$f(x) = F'(x) = \frac{3x^2}{8} \text{ for } 0 \leq x \leq 2$$

M1 A1

$$P(X > 1) = 1 - F(1) = 1 - \frac{1}{8} = \frac{7}{8}$$

A1

$$\text{Answer: } f(x) = \frac{3x^2}{8}; \frac{7}{8}$$

3 marks

6 (3 marks)

$$P(3 < X < 5) = (5 - 3) \times \frac{1}{5} = \frac{2}{5}$$

M1 A1

$$E(X) = \text{midpoint of 2 and 7} = 4.5$$

A1

$$\text{Answer: } \frac{2}{5}; 4.5$$

3 marks

7 (5 marks)**(a)** The graph is a triangle with base 4 and height $2k$ **M1**

$$\text{Area} = \frac{1}{2} \times 4 \times 2k = 4k = 1$$

M1

$$k = \frac{1}{4}$$

A1

$$\text{Answer: } k = \frac{1}{4}$$

3 marks

$$\text{(b)} P(X < 1) = \frac{1}{2} \times 1 \times \frac{1}{4} = \frac{1}{8}$$

A1

$$\text{Answer: } \frac{1}{8}$$

1 mark**(c)** The distribution is symmetrical about $x = 2$, so $E(X) = 2$ **B1**

$$\text{Answer: } 2$$

1 mark**8 (4 marks)**

$$\text{(a)} f(x) = F'(x) = \frac{2x}{9} \text{ for } 0 \leq x \leq 3$$

B1

$$\text{Answer: } f(x) = \frac{2x}{9}, 0 \leq x \leq 3$$

1 mark

$$\text{(b)} F(2) - F(1) = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

A1

$$\text{Answer: } \frac{1}{3}$$

1 mark

$$\text{(c)} F(m) = \frac{1}{2}: \frac{m^2}{9} = \frac{1}{2}$$

M1

$$m = \sqrt{4.5} = 2.121$$

A1

$$\text{Answer: } m = 2.12$$

2 marks**9 (4 marks)**

$$\int_0^2 k(x+1) dx = k\left[\frac{x^2}{2} + x\right]_0^2 = 4k = 1, \text{ so } k = \frac{1}{4}$$

M1 A1

$$P(X > 1) = \int_1^2 \frac{x+1}{4} dx = \frac{1}{4}\left[\frac{x^2}{2} + x\right]_1^2 = \frac{1}{4}\left(4 - \frac{3}{2}\right) = \frac{5}{8}$$

M1 A1

$$\text{Answer: } k = \frac{1}{4}; P(X > 1) = \frac{5}{8}$$

4 marks**10 (5 marks)****(a)** The curve is symmetrical about $x = 1$, so $E(X) = 1$ **B1 E1**

$$\text{Answer: } 1$$

2 marks

$$\text{(b)} E(X^2) = \int_0^2 \frac{3}{4}(2x^3 - x^4) dx = \frac{3}{4}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left(8 - \frac{32}{5}\right) = \frac{6}{5}$$

M1 A1

$$\text{Var}(X) = \frac{6}{5} - 1 = \frac{1}{5}$$

A1

$$\text{Answer: } \frac{1}{5}$$

3 marks**11 (4 marks)**

$$F(m) = \int_0^m \frac{x}{2} dx = \frac{m^2}{4}$$

M1 A1

$$\text{Median: } \frac{m^2}{4} = \frac{1}{2} \text{ so } m^2 = 2$$

M1

$$m = \sqrt{2}$$

A1

$$\text{Answer: } \sqrt{2}$$

4 marks

12 (6 marks)

$$\int_0^3 k(3x^2 - x^3) dx = k\left[x^3 - \frac{x^4}{4}\right]_0^3 = \frac{27k}{4} = 1, \text{ so } k = \frac{4}{27}$$

M1 A1

$f'(x) = k(6x - 3x^2) = 0$ gives $x = 2$ (the maximum), so the mode is 2

M1 A1

$$P(X > 2) = \frac{4}{27}\left[x^3 - \frac{x^4}{4}\right]_2^3 = \frac{4}{27}\left(\frac{27}{4} - 4\right) = \frac{11}{27}$$

M1 A1

Answer: mode 2, $P(X > 2) = \frac{11}{27}$

6 marks

13 (6 marks)

(a) Area = $2a + \frac{1}{2} \times 2 \times a = 3a$

M1 A1

$3a = 1$ so $a = \frac{1}{3}$

A1

Answer: $\frac{1}{3}$

3 marks

(b) For $2 \leq x \leq 4$, $f(x) = \frac{4-x}{6}$

B1

$P(X > 2.5) = \frac{1}{2} \times 1.5 \times \frac{1.5}{6} = \frac{3}{16}$

M1 A1

Answer: $\frac{3}{16}$

3 marks

14 (7 marks)

(a) $\int_0^4 kx(4-x) dx = k \int_0^4 (4x - x^2) dx = 1$

M1

$k\left[2x^2 - \frac{x^3}{3}\right]_0^4 = k\left(32 - \frac{64}{3}\right) = \frac{32k}{3}$

M1

$\frac{32k}{3} = 1$, so $k = \frac{3}{32}$

A1

Answer: $k = \frac{3}{32}$

3 marks

(b) $E(X) = 2$ by symmetry about $x = 2$

B1

$E(X^2) = \frac{3}{32} \int_0^4 (4x^3 - x^4) dx = \frac{3}{32}\left[x^4 - \frac{x^5}{5}\right]_0^4$

M1

$= \frac{3}{32}(256 - 204.8) = 4.8$

A1

$\text{Var}(X) = 4.8 - 2^2 = 0.8$

A1

Answer: $E(X) = 2$, $\text{Var}(X) = 0.8$

4 marks

15 (7 marks)

(a) $F(x) = \int_1^x 3t^{-4} dt$

M1

$= [-t^{-3}]_1^x$

A1

$= 1 - x^{-3}$ for $x \geq 1$ (and 0 for $x < 1$)

A1

Answer: $F(x) = 1 - x^{-3}$, $x \geq 1$

3 marks

(b) $1 - m^{-3} = \frac{1}{2}$ so $m^3 = 2$

M1

$m = \sqrt[3]{2} = 1.260$

A1

Answer: $\sqrt[3]{2} = 1.26$

2 marks

(c) $E(X) = \int_1^\infty 3x^{-3} dx = \left[-\frac{3}{2}x^{-2}\right]_1^\infty$

M1

$= \frac{3}{2}$

A1

Answer: $\frac{3}{2}$

2 marks

16 (6 marks)

$$\int_1^4 \frac{k}{x^2} dx = k \left[-\frac{1}{x} \right]_1^4 = \frac{3k}{4} = 1, \text{ so } k = \frac{4}{3} \quad \text{M1 A1}$$

$$F(x) = \frac{4}{3} \left(1 - \frac{1}{x} \right); F(m) = \frac{1}{2} \text{ gives } 1 - \frac{1}{m} = \frac{3}{8}, \text{ so } m = \frac{8}{5} \quad \text{M1 A1}$$

$$E(X) = \int_1^4 \frac{4}{3x} dx = \frac{4}{3} \ln 4 = 1.85 \quad \text{M1 A1}$$

$$\text{Answer: } k = \frac{4}{3}, \text{ median } \frac{8}{5}, E(X) = \frac{4}{3} \ln 4 = 1.85 \quad \text{6 marks}$$

17 (8 marks)

$$(a) F(2) = \frac{8+2}{10} = 1 \quad \text{B1}$$

$$f(x) = F'(x) = \frac{3x^2 + 1}{10} \text{ for } 0 \leq x \leq 2 \quad \text{M1 A1}$$

$$\text{Answer: } f(x) = \frac{3x^2 + 1}{10} \quad \text{3 marks}$$

$$(b) F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right) = \frac{\frac{27}{8} + \frac{3}{2}}{10} - \frac{\frac{1}{8} + \frac{1}{2}}{10} = \frac{17}{40} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{17}{40} \quad \text{2 marks}$$

$$(c) F(m) = \frac{1}{2} \text{ gives } m^3 + m = 5, \text{ so } m^3 + m - 5 = 0 \quad \text{M1 A1}$$

$$\text{Solving numerically (calculator), } m = 1.52 \quad \text{A1}$$

$$\text{Answer: } m = 1.52 \quad \text{3 marks}$$

18 (6 marks)

$$E(X) = \int_0^2 \frac{3x^3}{8} dx = \frac{3}{8} \left[\frac{x^4}{4} \right]_0^2 = \frac{3}{2} \quad \text{M1 A1}$$

$$E(X^2) = \int_0^2 \frac{3x^4}{8} dx = \frac{3}{8} \times \frac{32}{5} = \frac{12}{5} \quad \text{M1 A1}$$

$$\text{Var}(X) = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}, \text{ standard deviation} = 0.387 \quad \text{M1 A1}$$

$$\text{Answer: } \frac{3}{2}, \frac{3}{20}, 0.387 \quad \text{6 marks}$$

19 (6 marks)

$$(a) \text{ The graph has its maximum at } x = 0, \text{ so the mode is } 0 \quad \text{B1}$$

$$\text{Answer: } 0 \quad \text{1 mark}$$

$$(b) F(m) = 2m - m^2 = \frac{1}{2} \text{ gives } m = 1 - \frac{1}{\sqrt{2}} = 0.293 \quad \text{M1 A1}$$

$$E(X) = \int_0^1 2x(1-x) dx = \left[x^2 - \frac{2x^3}{3} \right]_0^1 = \frac{1}{3} \quad \text{M1 A1}$$

$$\text{Answer: median } 0.293, \text{ mean } \frac{1}{3} \quad \text{4 marks}$$

$$(c) \text{ Mode} < \text{median} < \text{mean, with a long tail to the right: positive skew} \quad \text{E1}$$

$$\text{Answer: positive skew} \quad \text{1 mark}$$

20 (7 marks)

$$(a) P(X > 3) = \int_3^4 \frac{3x^2}{64} dx = \left[\frac{x^3}{64} \right]_3^4 = 1 - \frac{27}{64} = \frac{37}{64} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{37}{64} \quad \text{2 marks}$$

$$(b) \text{ The machine is still working after 3 years only if both components are, with probability } \left(\frac{37}{64}\right)^2 = \frac{1369}{4096}$$

$$P(\text{stops within 3 years}) = 1 - \frac{1369}{4096} = \frac{2727}{4096}$$

A1

Answer: $\frac{2727}{4096} = 0.666$

3 marks

(c) $F(x) = \frac{x^3}{64}$; $F(m) = \frac{1}{2}$ gives $m^3 = 32$

M1

$$m = 32^{1/3} = 3.17 \text{ years}$$

A1

Answer: 3.17 years

2 marks

21 (9 marks)

(a) Triangle of base 6 and height $2k$: $\frac{1}{2} \times 6 \times 2k = 6k = 1$, so $k = \frac{1}{6}$

M1

$$a = 2k = \frac{1}{3}$$

A1

Answer: $k = \frac{1}{6}$, $a = \frac{1}{3}$

2 marks

(b) $F(2) = \frac{1}{2} \times 2 \times \frac{1}{3} = \frac{1}{3} < \frac{1}{2}$, so the median m lies in $2 < m < 6$

M1 A1

Area to the right of m is a triangle: height $\frac{6-m}{12}$, base $6-m$, area $\frac{(6-m)^2}{24}$

M1

$$\frac{(6-m)^2}{24} = \frac{1}{2} \text{ so } (6-m)^2 = 12, m = 6 - 2\sqrt{3} = 2.536$$

A1

Answer: $m = 6 - 2\sqrt{3} = 2.54$

4 marks

(c) $E(X) = \int_0^2 \frac{x^2}{6} dx + \int_2^6 \frac{x(6-x)}{12} dx$

M1

$$\text{First integral} = \left[\frac{x^3}{18} \right]_0^2 = \frac{4}{9}$$

M1

$$\text{Second integral} = \frac{1}{12} \left[3x^2 - \frac{x^3}{3} \right]_2^6 = \frac{1}{12} \left(36 - \frac{28}{3} \right) = \frac{20}{9}; \text{ total } \frac{4}{9} + \frac{20}{9} = \frac{8}{3}$$

A1

Answer: $\frac{8}{3}$

3 marks

22 (8 marks)

(a) $\int_0^1 (a + bx) dx = a + \frac{b}{2} = 1$

M1 A1

$$E(X) = \int_0^1 (ax + bx^2) dx = \frac{a}{2} + \frac{b}{3} = 0.6$$

M1 A1

From the first, $a = 1 - \frac{b}{2}$; so $\frac{1}{2} - \frac{b}{4} + \frac{b}{3} = 0.6$, $\frac{b}{12} = 0.1$, $b = 1.2$ and $a = 0.4$

A1

Answer: $a = 0.4$, $b = 1.2$

5 marks

(b) $P(X > 0.5) = \int_{0.5}^1 (0.4 + 1.2x) dx = [0.4x + 0.6x^2]_{0.5}^1$

M1

$$= 1 - 0.35 = 0.65$$

A1

Answer: 0.65

2 marks

(c) $f(x) = 0.4 + 1.2x$ is increasing, so the mode is 1

B1

Answer: 1

1 mark

23 (9 marks)

(a) $\int_0^1 (ax^2 + bx) dx = \frac{a}{3} + \frac{b}{2} = 1$ so $2a + 3b = 6$

M1 A1

$$\int_0^1 (ax^3 + bx^2) dx = \frac{a}{4} + \frac{b}{3} = 0.7 \text{ so } 3a + 4b = 8.4$$

M1 A1

Solving: $a = 1.2$ and $b = 1.2$

M1 A1

Answer: $a = b = 1.2$

6 marks

(b) $P(X > 0.5) = \int_{0.5}^1 1.2(x^2 + x) dx = 1.2 \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{0.5}^1$

M1

$$= 1.2(0.8333 - 0.1667) = 0.800 = \frac{4}{5}$$

M1 A1

Answer: $\frac{4}{5}$

3 marks

24 (11 marks)

(a) For $0 \leq x \leq 2$: $F(x) = \int_0^x \frac{t}{8} dt = \frac{x^2}{16}$, and $F(2) = \frac{1}{4}$ M1 A1

For $2 < x \leq 5$: $F(x) = \frac{1}{4} + \int_2^x \frac{1}{4} dt = \frac{x-1}{4}$ M1 A1

$F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 5$ B1

Answer: $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 2$; $\frac{x-1}{4}$ for $2 < x \leq 5$; 0 for $x < 0$; 1 for $x > 5$ 5 marks

(b) $F(2) = \frac{1}{4}$, so $Q_1 = 2$ M1 A1

$\frac{x-1}{4} = \frac{1}{2}$ gives the median $x = 3$, and $\frac{x-1}{4} = \frac{3}{4}$ gives $Q_3 = 4$, so $IQR = 2$ M1 A1

$E(X) = \int_0^2 \frac{x^2}{8} dx + \int_2^5 \frac{x}{4} dx = \frac{1}{3} + \frac{21}{8} = \frac{71}{24}$ M1 A1

Answer: median 3, IQR 2, $E(X) = \frac{71}{24}$ 6 marks

25 (10 marks)

(a) $k[-\cos x]_0^\pi = k(1 + 1) = 2k = 1$ M1 A1

Answer: $k = \frac{1}{2}$ 2 marks

(b) $P(X < \frac{\pi}{3}) = \frac{1}{2}[-\cos x]_0^{\pi/3} = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$ M1 A1

By symmetry about $x = \frac{\pi}{2}$, the median is $\frac{\pi}{2}$ B1

Answer: $\frac{1}{4}$; $\frac{\pi}{2}$ 3 marks

(c) $E(X) = \frac{\pi}{2}$ by symmetry B1

$E(X^2) = \frac{1}{2} \int_0^\pi x^2 \sin x dx$; by parts twice, $\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x$ M1 A1

So $E(X^2) = \frac{1}{2}[(\pi^2 - 2) - 2] = \frac{\pi^2}{2} - 2$ M1

$\text{Var}(X) = \frac{\pi^2}{2} - 2 - \frac{\pi^2}{4} = \frac{\pi^2}{4} - 2$ A1

Answer: $\frac{\pi^2}{4} - 2$ 5 marks

26 (8 marks)

(a) $E(X^2) = \int_0^2 \frac{3x^4}{8} dx = \frac{12}{5}$ M1 A1

$E(C) = 20 + 15 \times \frac{12}{5} = \text{£}56$ M1 A1

Answer: £56 4 marks

(b) $C > 50$ means $15X^2 > 30$, so $X > \sqrt{2}$ M1 A1

$P(X > \sqrt{2}) = 1 - \frac{(\sqrt{2})^3}{8} = 1 - \frac{\sqrt{2}}{4}$ M1

$= 0.646$ A1

Answer: 0.646 4 marks

27 (12 marks)

(a) $F(x) = \int_1^x \frac{1}{t \ln 5} dt = \frac{1}{\ln 5} [\ln t]_1^x$ M1

$= \frac{\ln x}{\ln 5}$ A1

$F(5) = 1$ and F is increasing, so f is a valid pdf E1

Answer: $F(x) = \frac{\ln x}{\ln 5}$ 3 marks

$$(b) \frac{\ln m}{\ln 5} = \frac{1}{2} \text{ so } \ln m = \frac{1}{2} \ln 5$$

M1

$$m = \sqrt{5}$$

A1

$$\text{Answer: } \sqrt{5}$$

2 marks

$$(c) E(X) = \int_1^5 x \cdot \frac{1}{x \ln 5} dx$$

M1

$$= \frac{1}{\ln 5} [x]_1^5 = \frac{4}{\ln 5}$$

M1

$$= 2.49$$

A1

$$\text{Answer: } \frac{4}{\ln 5} = 2.49$$

3 marks

$$(d) P(Y \leq y) = P(\ln X \leq y) = P(X \leq e^y)$$

M1

$$= F(e^y) = \frac{y}{\ln 5}$$

M1 A1

This is the cumulative distribution function of the uniform distribution on $[0, \ln 5]$; the pdf is $\frac{1}{\ln 5}$

E1

Answer: Uniform on $[0, \ln 5]$

4 marks

28 (10 marks)

$$(a) F(x) = \int_0^x (n+1)t^n dt$$

M1

$$= [t^{n+1}]_0^x = x^{n+1}$$

A1

$$\text{Answer: } x^{n+1}$$

2 marks

$$(b) m^{n+1} = \frac{1}{2}$$

M1

$$m = \left(\frac{1}{2}\right)^{1/(n+1)}$$

A1

$$\text{Answer: } m = \left(\frac{1}{2}\right)^{1/(n+1)}$$

2 marks

$$(c) E(X) = \int_0^1 (n+1)x^{n+1} dx$$

M1

$$= (n+1) \left[\frac{x^{n+2}}{n+2} \right]_0^1$$

M1

$$= \frac{n+1}{n+2}$$

A1

$$\text{Answer: } \frac{n+1}{n+2}$$

3 marks

$$(d) \left(\frac{1}{2}\right)^{1/(n+1)} > 0.95; \text{ take logs: } \frac{1}{(n+1)} \ln 0.5 > \ln 0.95$$

M1

Dividing by $\ln 0.5 < 0$ reverses the inequality: $\frac{1}{(n+1)} < \frac{\ln 0.95}{\ln 0.5}$, so $n+1 > \frac{\ln 0.5}{\ln 0.95} = 13.51$

M1

$$n+1 \geq 14, \text{ so } n = 13$$

A1

Answer: $n = 13$

3 marks

29 (15 marks)

$$(a) G(y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = F(y^{1/3})$$

M1 A1

$$F(x) = \frac{x^3}{8}, \text{ so } G(y) = \frac{y}{8}$$

M1 A1

G is linear, so Y is uniformly distributed on $[0, 8]$

A1

Answer: Y is uniform on $[0, 8]$

5 marks

$$(b) E(Y) = 4 \text{ and } \text{Var}(Y) = \frac{8^2}{12} = \frac{16}{3}$$

B1 B1

$$E(X^3) = E(Y) = 4$$

B1

$$E(X^6) = E(Y^2) = \frac{16}{3} + 16 = \frac{64}{3}$$

M1 A1

$$\text{Answer: } 4 \text{ and } \frac{64}{3}$$

5 marks

$$(c) 8U \text{ is uniform on } [0, 8], \text{ so } Y = 8U$$

M1

$$X = Y^{1/3} = (8U)^{1/3} = 2U^{1/3}, \text{ so } g(U) = 2U^{1/3}$$

M1 A1

Answer: $g(U) = 2U^{1/3}$

3 marks

(d) $2U^{1/3} \leq 1$ means $U \leq \frac{1}{8}$, with probability $\frac{1}{8}$

M1

 $F(1) = \frac{1}{8}$, which agrees

A1

Answer: $\frac{1}{8}$

2 marks

30 (15 marks)**(a)** $E(X) = \int_0^b x f(x) dx = [xF(x)]_0^b - \int_0^b F(x) dx$

M1 A1

 $= b - \int_0^b F(x) dx$, using $F(b) = 1$

M1

 $= \int_0^b [1 - F(x)] dx$, since $\int_0^b 1 dx = b$

A1

Answer: shown

4 marks

(b) $F(x) = \frac{x^3}{27}$

B1

 $E(X) = \int_0^3 (1 - \frac{x^3}{27}) dx = [x - \frac{x^4}{108}]_0^3 = 3 - \frac{3}{4} = \frac{9}{4}$

M1 A1

Answer: $\frac{9}{4}$

3 marks

(c) $E(X) = \int_0^3 \frac{x^3}{9} dx = [\frac{x^4}{36}]_0^3 = \frac{81}{36} = \frac{9}{4}$

M1 A1

Answer: $\frac{9}{4}$

2 marks

(d) Parts: $\int_0^b x^2 f(x) dx = [x^2 F(x)]_0^b - \int_0^b 2xF(x) dx = b^2 - \int_0^b 2xF(x) dx$, and $b^2 = \int_0^b 2x dx$

M1 A1

 $E(X^2) = \int_0^3 2x(1 - \frac{x^3}{27}) dx = [x^2 - \frac{2x^5}{135}]_0^3 = 9 - \frac{18}{5} = \frac{27}{5}$

M1 A1

 $\text{Var}(X) = \frac{27}{5} - (\frac{9}{4})^2 = \frac{27}{80}$

M1 A1

Answer: $\frac{27}{80}$

6 marks

Exponential Distribution

1 (2 marks) $P(T > 2) = \int_2^\infty 0.5e^{-0.5t} dt = [-e^{-0.5t}]_2^\infty$

M1

 $= e^{-1} = 0.368$

A1

Answer: 0.368

2 marks

2 (3 marks)**(a)** $E(T) = \frac{1}{\lambda} = \frac{1}{4}$

B1

Answer: $\frac{1}{4}$

1 mark

(b) $P(T \leq t) = 1 - e^{-4t}$

M1

 $P(T \leq 0.5) = 1 - e^{-2} = 0.865$

A1

Answer: 0.865

2 marks

3 (3 marks)**(a)** $f(t) = F'(t) = 0.8e^{-0.8t}$ for $t \geq 0$

M1 A1

Answer: $f(t) = 0.8e^{-0.8t}$

2 marks

(b) $T \sim \text{Exp}(0.8)$, so $E(T) = \frac{1}{0.8} = 1.25$

B1

Answer: 1.25

1 mark

4 (3 marks)

$$E(T) = \frac{1}{\lambda} = 4$$

B1

$$\text{Var}(T) = \frac{1}{\lambda^2} = 16, \text{ standard deviation} = 4$$

B1 B1

Answer: 4, 16, 4

3 marks

5 (2 marks)

$$\lambda = \frac{1}{5} = 0.2$$

B1

$$P(T > 5) = e^{-0.2 \times 5} = e^{-1} = 0.368$$

A1

Answer: $\lambda = 0.2$, 0.368

2 marks

6 (2 marks)

$$F(t) = \int_0^t 3e^{-3u} \, du = [-e^{-3u}]_0^t$$

M1

$$= 1 - e^{-3t} \text{ for } t \geq 0$$

A1

Answer: $F(t) = 1 - e^{-3t}$

2 marks

7 (4 marks)

$$(a) \lambda = \frac{1}{8}, P(T > t) = e^{-\lambda t}$$

M1

$$P(T > 10) = e^{-1.25} = 0.287$$

A1

Answer: 0.287

2 marks

$$(b) P(5 < T < 12) = e^{-5/8} - e^{-12/8}$$

M1

$$= 0.5353 - 0.2231 = 0.312$$

A1

Answer: 0.312

2 marks

8 (4 marks)

$$(a) 1 - e^{-0.2m} = 0.5 \text{ so } e^{-0.2m} = 0.5$$

M1

$$m = \frac{\ln 2}{0.2} = 3.47 \text{ years}$$

A1

Answer: 3.47 years

2 marks

$$(b) e^{-0.2x} = 0.1$$

M1

$$x = \frac{\ln 10}{0.2} = 11.5 \text{ years}$$

A1

Answer: 11.5 years

2 marks

9 (4 marks)

$$e^{-3\lambda} = 2e^{-6\lambda}$$

M1

$$\text{Dividing by } e^{-6\lambda}: e^{3\lambda} = 2$$

M1

$$\lambda = \frac{\ln 2}{3} = 0.231 \text{ and } E(T) = \frac{1}{\lambda} = \frac{3}{\ln 2} = 4.33$$

A1 A1

$$\text{Answer: } \lambda = \frac{\ln 2}{3}, E(T) = \frac{3}{\ln 2} = 4.33$$

4 marks

10 (4 marks)

$$F(t) = 1 - e^{-0.1t}$$

B1

$$\text{Median: } e^{-0.1m} = 0.5 \text{ so } m = \frac{\ln 2}{0.1} = 6.93$$

M1 A1

$$\text{Upper quartile: } e^{-0.1q} = 0.25, q = \frac{\ln 4}{0.1} = 13.9$$

A1

Answer: median 6.93, upper quartile 13.9

4 marks

11 (3 marks)

$$e^{-2\lambda} = 0.3$$

M1

$$\lambda = \frac{-\ln 0.3}{2} = 0.602$$

A1

$$\text{Mean} = \frac{1}{\lambda} = 1.66$$

A1

Answer: $\lambda = 0.602$, mean 1.66

3 marks

12 (3 marks)

$$P(T > 8) = e^{-8/5} = 0.2019$$

M1 A1

$$\text{Expected number} = 200 \times 0.2019 = 40.4, \text{ about 40 bulbs}$$

A1

Answer: about 40

3 marks

13 (3 marks)

$$e^{-0.5t} = 0.01$$

M1

$$t = \frac{\ln 100}{0.5} = 9.210$$

M1 A1

Answer: 9.210

3 marks

14 (7 marks)

$$(a) P(T > 3) = e^{-3\lambda} = 0.6$$

M1

$$-3\lambda = \ln 0.6$$

M1

$$\lambda = \frac{-\ln 0.6}{3} = 0.1703$$

A1

Answer: $\lambda = 0.170$

3 marks

$$(b) \text{Mean} = \frac{1}{\lambda} = 5.87 \text{ hours}$$

A1

Answer: 5.87 hours

1 mark

$$(c) P(T > 6 | T > 3) = \frac{P(T > 6)}{P(T > 3)} = \frac{e^{-6\lambda}}{e^{-3\lambda}}$$

M1

$$= e^{-3\lambda} = 0.6$$

A1

This equals $P(T > 3)$: the distribution is memoryless

E1

Answer: 0.6; memoryless

3 marks

15 (7 marks)

$$(a) P(T > x) = e^{-\lambda x}$$

B1

$$P(T > s + t | T > s) = \frac{P(T > s + t)}{P(T > s)}$$

M1

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

M1

$$= e^{-\lambda t} = P(T > t)$$

A1

Answer: shown

4 marks

(b) By part (a), the age already reached is irrelevant

E1

$$\lambda = \frac{1}{8000}; P(T > 2000) = e^{-2000/8000}$$

M1

$$= e^{-0.25} = 0.779$$

A1

Answer: 0.779

3 marks

16 (8 marks)

$$(a) P(T > 10) = e^{-10\lambda} = 0.25$$

M1

$$10\lambda = \ln 4 \text{ so } \lambda = \frac{\ln 4}{10} = 0.139$$

A1

$$\text{Answer: } \lambda = \frac{\ln 4}{10} = 0.139$$

2 marks

$$(b) P(T < 5) = 1 - e^{-5\lambda} = 1 - e^{-\frac{\ln 4}{2}} = 1 - \frac{1}{2} = 0.5$$

M1 A1

$$\text{Expected number} = 40 \times 0.5 = 20$$

A1

Answer: 20

3 marks

(c) Served means $T < 10$, with probability 0.75

B1

$$P(T < 5 \mid T < 10) = \frac{0.5}{0.75} = \frac{2}{3}$$

M1 A1

Answer: $\frac{2}{3}$

3 marks

17 (6 marks)

(a) $T \sim \text{Exp}(12)$, with mean $\frac{1}{12}$ hour (5 minutes)

B1

Answer: $\text{Exp}(12)$

1 mark

(b) 10 minutes = $\frac{1}{6}$ hour; $P(T > \frac{1}{6}) = e^{-12/6} = e^{-2}$

M1

= 0.135

A1

Answer: 0.135

2 marks

(c) Number $N \sim \text{Po}(6)$

M1

$$P(N \geq 2) = 1 - P(N \leq 1) = 0.9826$$

M1 A1

Answer: 0.9826

3 marks

18 (5 marks)

$$P(T > 30) = e^{-30/20} = e^{-1.5} = 0.2231$$

M1 A1

Number lasting longer $Y \sim B(8, 0.2231)$

M1

$$P(Y = 3) = C(8, 3)(0.2231)^3(0.7769)^5 = 0.176$$

M1 A1

Answer: 0.176

5 marks

19 (5 marks)

(a) Mean of the model = $\frac{1}{\lambda} = 3.2$

M1

$$\lambda = \frac{1}{3.2} = 0.3125$$

A1

Answer: $\lambda = 0.3125$

2 marks

(b) Expected number = $100e^{-6 \times 0.3125} = 100e^{-1.875} = 15.3$

M1 A1

This is close to the observed 17, so the model fits well on this evidence

E1

Answer: 15.3, good fit

3 marks

20 (6 marks)

$$(a) \int_0^{\infty} ke^{-0.4t} dt = k \left[-\frac{e^{-0.4t}}{0.4} \right]_0^{\infty} = \frac{k}{0.4}$$

M1 A1

$$\frac{k}{0.4} = 1 \text{ so } k = 0.4$$

A1

Answer: $k = 0.4$

3 marks

(b) $T \sim \text{Exp}(0.4)$

B1

$$P(T > 3) = e^{-1.2} = 0.301$$

M1 A1

Answer: 0.301

3 marks

21 (8 marks)

$$(a) E(T) = \int_0^{\infty} \lambda t e^{-\lambda t} dt$$

$$= [-t e^{-\lambda t}]_0^{\infty} + \int_0^{\infty} e^{-\lambda t} dt$$

M1 A1

The first term is 0 at both limits ($e^{-\lambda t}$ dominates t)

B1

$$= \left[-\frac{1}{\lambda} e^{-\lambda t} \right]_0^{\infty} = \frac{1}{\lambda}$$

A1

Answer: $\frac{1}{\lambda}$

4 marks

$$(b) \text{Var}(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

M1

$$\text{sd} = \frac{1}{\lambda} = E(T)$$

A1

Answer: shown

2 marks

(c) Mean = sd = 5, so the value is 10, and $\lambda = 0.2$

M1

$$P(T > 10) = e^{-2} = 0.135$$

A1

Answer: 0.135

2 marks

22 (8 marks)

(a) $S > t$ means $T_A > t$ and $T_B > t$

M1

By independence $P(S > t) = e^{-0.1t} \times e^{-0.2t}$

M1

$$= e^{-0.3t}$$

A1

Answer: $e^{-0.3t}$

3 marks

(b) $S \sim \text{Exp}(0.3)$

M1

$$\text{Mean} = \frac{1}{0.3} = 3.33 \text{ hours}$$

A1

Answer: 3.33 hours

2 marks

(c) $P(\text{fails by } 10) = P(T_A \leq 10) P(T_B \leq 10) = (1 - e^{-1})(1 - e^{-2})$

M1 A1

$$P(\text{lasts} > 10) = 1 - (1 - e^{-1})(1 - e^{-2}) = e^{-1} + e^{-2} - e^{-3} = 0.453$$

A1

Answer: 0.453

3 marks

23 (9 marks)

$$(a) P(T_A < T_B) = \int_0^{\infty} f_A(t)P(T_B > t) dt$$

M1 A1

$$= \int_0^{\infty} \lambda e^{-\lambda t} \times e^{-\mu t} dt$$

A1

Answer: shown

3 marks

$$(b) = \lambda \int_0^{\infty} e^{-(\lambda+\mu)t} dt = \lambda \left[-\frac{e^{-(\lambda+\mu)t}}{\lambda+\mu} \right]_0^{\infty}$$

M1 A1

$$= \frac{\lambda}{\lambda+\mu}$$

A1

Answer: shown

3 marks

$$(c) P(A \text{ first}) = \frac{0.3}{0.5} = 0.6$$

B1

Time of first failure $\sim \text{Exp}(0.5)$, so $P(> 4) = e^{-2} = 0.135$

M1 A1

Answer: 0.6 and 0.135

3 marks

24 (10 marks)

$$(a) P(1 < T < 3) = e^{-\lambda} - e^{-3\lambda}$$

M1 A1

With $u = e^{-\lambda}$: $u - u^3 = 0.3$, so $u^3 - u + 0.3 = 0$

A1

Answer: shown

3 marks

(b) Solving numerically (calculator), $u = 0.3389$ or $u = 0.7865$

M1 A1

$$\lambda = -\ln u = 0.240 \text{ or } 1.08$$

M1 A1

Answer: $\lambda = 0.240$ or 1.08

4 marks

(c) Mean $\frac{1}{\lambda} > 2$ requires $\lambda < 0.5$, so $\lambda = 0.240$

M1 A1

$$\text{Mean} = 4.16$$

A1

Answer: $\lambda = 0.240$, mean 4.16

3 marks

25 (7 marks)

(a) $N \sim \text{Po}(10)$ since the rate is 0.01 per hour

M1 A1

$$\text{Expected cost} = 10 \times \text{£}250 = \text{£}2500$$

A1

Answer: Po(10), £2500

3 marks

(b) $P(N > 12) = 1 - P(N \leq 12)$

M1

$$= 1 - 0.7916 = 0.208$$

A1

Answer: 0.208

2 marks

(c) $P(T > 150 | T > 50) = P(T > 100)$ by the memoryless property

M1

$$= e^{-1} = 0.368$$

A1

Answer: 0.368

2 marks

26 (12 marks)

$$(a) E(M) = \int_0^c t f(t) dt + c \times P(T \geq c)$$

M1 A1

$$P(T \geq c) = e^{-\lambda c}$$

A1

Answer: shown

3 marks

$$(b) \int_0^c t \lambda e^{-\lambda t} dt = [-te^{-\lambda t}]_0^c + \int_0^c e^{-\lambda t} dt$$

M1 A1

$$= -ce^{-\lambda c} + \frac{1 - e^{-\lambda c}}{\lambda}$$

A1

$$\text{Adding } ce^{-\lambda c} \text{ gives } E(M) = \frac{1 - e^{-\lambda c}}{\lambda}$$

A1

Answer: shown

4 marks

$$(c) E(M) = \frac{1 - e^{-1}}{0.02} = 31.6 \text{ hours}$$

M1 A1

$$P(\text{replaced}) = e^{-1} = 0.368$$

A1

Answer: 31.6 hours, 0.368

3 marks

(d) $M \leq T$ always, with $M < T$ when $T \geq c$, so $E(M) < E(T)$

E1

$$\text{As } c \rightarrow \infty, e^{-\lambda c} \rightarrow 0 \text{ and } E(M) \rightarrow \frac{1}{\lambda} = E(T)$$

E1

Answer: $E(M) \rightarrow 50$

2 marks

27 (10 marks)(a) $T > t$ means no calls in $(0, t)$, probability e^{-6t}

M1

So $F(t) = 1 - e^{-6t}$; $T \sim \text{Exp}(6)$

A1

Answer: Exp(6)

2 marks

(b) $U > t$ means at most one call in $(0, t)$

E1

$$N \sim \text{Po}(6t): P(N \leq 1) = e^{-6t}(1 + 6t)$$

M1

Answer: $P(N \leq 1)$ for $N \sim \text{Po}(6t)$

2 marks

$$(c) P(U \leq 0.5) = 1 - e^{-3}(1 + 3)$$

M1

$$= 0.801$$

A1

Answer: 0.801

2 marks

$$(d) F_U(t) = 1 - e^{-6t} - 6te^{-6t}$$

B1

$$f_U(t) = F_U'(t) = 6e^{-6t} - 6e^{-6t} + 36te^{-6t}$$

M1 A1

$$= 36te^{-6t}, t \geq 0; \text{ the integral is } F_U(\infty) - F_U(0) = 1 - 0 = 1$$

E1

Answer: $f(t) = 36te^{-6t}$

4 marks

28 (12 marks)

$$(a) \lambda = \frac{1}{40}$$

B1

$$P(T > 60) = e^{-1.5} = 0.2231$$

A1

Answer: 0.2231

2 marks

(b) $Y \sim B(5, 0.2231)$

B1

$$P(Y \geq 2) = 1 - P(Y \leq 1)$$

M1

$$= 0.311$$

A1

Answer: 0.311

3 marks

$$(c) 1 - (1 - 0.2231)^n > 0.99, \text{ so } 0.7769^n < 0.01$$

M1

$$n > \frac{\ln 0.01}{\ln 0.7769} = 18.2...$$

M1

$$n = 19$$

A1

Answer: $n = 19$

3 marks

(d) Memoryless: required = $P(T < 30)$ (further life under 30 hours)

M1

$$= 1 - e^{-0.75} = 0.528$$

A1

Answer: 0.528

2 marks

$$(e) \text{Median: } e^{-m/40} = 0.5, \text{ so } m = 40 \ln 2$$

M1

$$= 27.7 < 40 = \text{mean (since } \ln 2 < 1)$$

A1

Answer: $27.7 < 40$ **2 marks****29 (11 marks)**

$$(a) P(N = n) = P(n \leq T < n + 1) = e^{-\lambda n} - e^{-\lambda(n+1)} \\ = e^{-\lambda n}(1 - e^{-\lambda}) = (1 - e^{-\lambda})(e^{-\lambda})^n$$

M1 A1**A1****Answer: shown****3 marks****(b)** The probabilities form a geometric progression in n **M1**Common ratio $q = e^{-\lambda}$ **A1****Answer: ratio $e^{-\lambda}$** **2 marks**

$$(c) E(N) = (1 - q) \sum_{n=0}^{\infty} nq^n = (1 - q)q(1 + 2q + 3q^2 + \dots)$$

M1

$$= (1 - q)q \times \frac{1}{(1 - q)^2} = \frac{q}{1 - q}$$

M1 A1**Answer: shown****3 marks**

$$(d) E(N) = \frac{0.6065}{0.3935} = 1.541, E(T) = 2$$

M1 A1 $N \leq T$ always, since the fractional part of T is discarded**E1****Answer: $E(N) = 1.541 < 2$** **3 marks****30 (13 marks)**

$$(a) P(T > t) = P(-\frac{1}{\lambda} \ln U > t) = P(\ln U < -\lambda t)$$

M1

$$= P(U < e^{-\lambda t}) = e^{-\lambda t} \text{ for } t \geq 0, \text{ since } U \text{ is uniform on } (0, 1)$$

M1 A1This is the survival function of the exponential distribution with parameter λ **E1****Answer: shown****4 marks**

$$(b) T = -4 \ln 0.3 = 4.82 \text{ and } T = -4 \ln 0.8 = 0.893$$

M1 A1**Answer: 4.82 and 0.893****2 marks**

$$(c) P(M > t) = P(T_1 > t, \dots, T_n > t)$$

M1

$$= (e^{-\lambda t})^n = e^{-n\lambda t}, \text{ which is the survival function of } \text{Exp}(n\lambda)$$

M1 A1**Answer: shown****3 marks****(d)** Each $T_i \sim \text{Exp}(0.01)$, so the first failure time $M \sim \text{Exp}(0.05)$ **M1**

$$E(M) = \frac{1}{0.05} = 20.0 \text{ hours}$$

A1

$$P(M > 30) = e^{-0.05 \times 30} = e^{-1.5} = 0.223$$

M1 A1**Answer: 20.0 hours; 0.223****4 marks****Combinations of Random Variables****1 (2 marks)**

$$E(X + Y) = 5 + 8 = 13$$

B1

$$\text{Var}(X + Y) = 2 + 3 = 5 \text{ (independent)}$$

B1**Answer: 13 and 5****2 marks****2 (3 marks)**

$$E(3X - 2Y) = 3(4) - 2(6) = 0$$

B1

$$\text{Var}(3X - 2Y) = 3^2 \text{Var}(X) + (-2)^2 \text{Var}(Y)$$

M1

$$= 9 + 8 = 17$$

A1**Answer: mean 0, variance 17****3 marks****3 (5 marks)**

$$E(X) = \text{Var}(X) = 4; E(Y) = 10 \times 0.3 = 3 \text{ and } \text{Var}(Y) = 10 \times 0.3 \times 0.7 = 2.1$$

B1 B1

$$E(X + Y) = 7$$

A1

$$\text{Var}(X + Y) = \text{Var}(X - Y) = 4 + 2.1 = 6.1$$

M1 A1**Answer: $E(X + Y) = 7$, $\text{Var}(X + Y) = 6.1$, $\text{Var}(X - Y) = 6.1$** **5 marks**

4 (4 marks)

$$X + Y \sim N(5 + 3, 4 + 1) = N(8, 5)$$

B1 B1

$$X - Y \sim N(5 - 3, 4 + 1) = N(2, 5)$$

B1 B1

Answer: N(8, 5) and N(2, 5)

4 marks

5 (3 marks)

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 25$$

M1 A1

$$\text{Standard deviation} = 5$$

A1

Answer: 25 and 5

3 marks

6 (2 marks)

$$E(S) = 4 \times 20 = 80$$

B1

$$\text{Var}(S) = 4 \times 9 = 36, \text{ so } S \sim N(80, 36)$$

B1

Answer: N(80, 36)

2 marks

7 (4 marks)

$$S \sim N(10 + 14, 4 + 9) = N(24, 13)$$

M1 A1

$$z = \frac{30 - 24}{\sqrt{13}} = 1.664$$

M1

$$P(S > 30) = 0.0480$$

A1

Answer: 0.0480

4 marks

8 (4 marks)Anna faster means $X < Y$, that is $Y - X > 0$

M1

$$Y - X \sim N(36 - 32, 9 + 4) = N(4, 13)$$

M1 A1

$$P(Y - X > 0) = P(Z > \frac{-4}{\sqrt{13}}) = 0.866$$

A1

Answer: 0.866

4 marks

9 (5 marks)

$$E(W) = 3 \times 6 - 2 \times 10 = -2$$

B1

$$\text{Var}(W) = 3^2 \times 5 + (-2)^2 \times 2 = 53, \text{ so } W \sim N(-2, 53)$$

M1 A1

$$P(W > 5) = P(Z > \frac{5 + 2}{\sqrt{53}}) = P(Z > 0.961) = 0.168$$

M1 A1

Answer: $W \sim N(-2, 53)$; 0.168

5 marks

10 (5 marks)

$$T = B_1 + B_2 + B_3 + C: E(T) = 7.5 + 0.4 = 7.9$$

M1 A1

$$\text{Var}(T) = 3(0.01) + 0.0025 = 0.0325$$

M1

$$P(T > 8) = P(Z > \frac{8 - 7.9}{\sqrt{0.0325}}) = P(Z > 0.5547) = 0.290$$

M1 A1

Answer: 0.290

5 marks

11 (8 marks)

$$(a) P(X > 12.5) = P(Z > 1.25) = 0.106$$

M1 A1

Answer: 0.106

2 marks

$$(b) X_1 + X_2 \sim N(20, 8)$$

M1 A1

$$P(> 25) = P(Z > \frac{5}{\sqrt{8}}) = 0.0385$$

A1

Answer: 0.0385

3 marks

$$(c) 2X \sim N(20, 16), P(2X > 25) = P(Z > 1.25) = 0.00621$$

M1 A1

Doubling one task amplifies its variation (variance 16), whereas two independent tasks partly cancel (variance 8)

E1

Answer: 0.00621

3 marks

12 (4 marks)

$$X + Y \sim \text{Po}(8), \text{ so } P(X + Y = 7) = e^{-8} \frac{8^7}{7!} = 0.140$$

M1 A1

$$E(X - Y) = 3 - 5 = -2$$

B1

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 3 + 5 = 8$$

B1

Answer: 0.140, -2, 8

4 marks

13 (4 marks)

$$X_2 - X_1 \sim N(0, 36 + 36) = N(0, 72)$$

M1 A1

$$P(X_2 - X_1 \geq 8) = P(Z \geq \frac{8}{\sqrt{72}}) = P(Z \geq 0.943)$$

M1

$$= 0.173$$

A1

Answer: 0.173

4 marks

14 (7 marks)

$$\text{(a) Mean} = 4(45) + 60 = 240$$

B1

$$\text{Variance} = 4(2^2) + 3^2 = 25$$

M1 A1

Answer: N(240, 25)

3 marks

$$\text{(b) } z = \frac{250 - 240}{5} = 2$$

M1

$$P = 0.0228$$

A1

Answer: 0.0228

2 marks

(c) That is the variance of 4X (one bar counted four times)

E1

Four different bars are independent, so the variances add: $4 \times 2^2 = 16$

E1

Answer: Variances of independent bars add

2 marks

15 (7 marks)

$$\text{(a) } E(X) = 0.2 + 1.0 + 0.9 = 2.1$$

B1

$$E(X^2) = 0.2 + 2.0 + 2.7 = 4.9$$

M1

$$\text{Var}(X) = 4.9 - 2.1^2 = 0.49$$

A1

Answer: E(X) = 2.1, Var(X) = 0.49

3 marks

(b) S takes the values 2, 3, 4, 5, 6

B1

$$P(S = 2) = 0.04, P(S = 3) = 2(0.2)(0.5) = 0.2, P(S = 4) = 2(0.2)(0.3) + 0.5^2 = 0.37$$

M1

$$P(S = 5) = 2(0.5)(0.3) = 0.3, P(S = 6) = 0.09$$

A1

Answer: 0.04, 0.2, 0.37, 0.3, 0.09 for S = 2 to 6

3 marks

(c) $\text{Var}(X_1 - X_2) = 2(0.49) = 0.98$, whereas $\text{Var}(2X) = 4(0.49) = 1.96$, twice as large

A1

Answer: 0.98 versus 1.96

1 mark

16 (7 marks)

$$\text{(a) } \text{Var}(W) = 2^2\sigma^2 + 5^2 \times 4 = 4\sigma^2 + 100$$

M1 A1

$$4\sigma^2 + 100 = 164 \text{ so } \sigma^2 = 16 \text{ and } \sigma = 4$$

A1

Answer: $\sigma = 4$

3 marks

$$\text{(b) } E(W) = 2 \times 20 - 5 \times 4 + 9 = 29$$

M1 A1

$$P(X < 29) = P(Z < \frac{29 - 20}{4}) = P(Z < 2.25) = 0.9878$$

M1 A1

Answer: E(W) = 29, P = 0.9878

4 marks

17 (6 marks)

$$E(\text{apples}) = 6 \times 120 = 720, E(\text{oranges}) = 4 \times 180 = 720$$

B1

$$\text{Var}(\text{apples}) = 6 \times 100 = 600, \text{Var}(\text{oranges}) = 4 \times 225 = 900$$

M1

$$D = A - O \sim N(0, 1500)$$

M1 A1

$$P(|D| < 30) = P(|Z| < \frac{30}{\sqrt{1500}}) = P(|Z| < 0.7746) = 0.561$$

M1 A1

Answer: 0.561

6 marks

18 (7 marks)

$$(a) H - D \sim N(0.05, 0.03^2 + 0.02^2) = N(0.05, 0.0013)$$

M1 A1

$$P(H - D > 0) = P(Z > \frac{-0.05}{\sqrt{0.0013}}) = 0.9172$$

M1 A1

Answer: 0.9172

4 marks

$$(b) \text{Number fitting} \sim B(10, 0.9172)$$

M1

$$P(\geq 9) = 10(0.9172)^9(0.08276) + (0.9172)^{10} = 0.802$$

M1 A1

Answer: 0.802

3 marks

19 (6 marks)

$$\bar{X} \sim N(50, \frac{64}{10}), \bar{Y} \sim N(48, \frac{36}{15}) = N(48, 2.4)$$

M1 A1

$$\bar{X} - \bar{Y} \sim N(2, 6.4 + 2.4) = N(2, 8.8)$$

M1 A1

$$P(\bar{X} - \bar{Y} < 0) = P(Z < \frac{-2}{\sqrt{8.8}}) = P(Z < -0.674) = 0.250$$

M1 A1

Answer: 0.250

6 marks

20 (7 marks)

$$E(XY) = E(X)E(Y) = 6 \text{ (independence)}$$

M1 A1

$$E(X^2) = 4 + 9 = 13, E(Y^2) = 1 + 4 = 5$$

B1

$$E[(X + Y)^2] = \text{Var}(X + Y) + [E(X + Y)]^2 = 5 + 25 = 30$$

M1 A1

$$\text{Var}(XY) = E(X^2Y^2) - [E(XY)]^2 = 13 \times 5 - 36 = 29$$

M1 A1

Answer: 6, 30, 29

7 marks

21 (8 marks)

$$(a) E(\bar{X}) = 1000$$

B1

$$\text{Var}(\bar{X}) = \frac{5 \times 8^2}{5^2} = \frac{64}{5} = 12.8$$

M1

$$\bar{X} \sim N(1000, 12.8)$$

A1

Answer: N(1000, 12.8)

3 marks

$$(b) z = \frac{995 - 1000}{\sqrt{12.8}} = -1.398$$

M1

$$P = 0.0811$$

A1

Answer: 0.0811

2 marks

$$(c) T \sim N(5000, 5 \times 64) = N(5000, 320)$$

M1 A1

$$z = \frac{30}{\sqrt{320}} = 1.677, P = 0.0468$$

A1

Answer: 0.0468

3 marks

22 (9 marks)

$$(a) \text{Total } L \sim N(8 \times 72, 8 \times 10^2) = N(576, 800)$$

M1 A1

$$z = \frac{600 - 576}{\sqrt{800}} = 0.8485$$

M1

$$P(L > 600) = 0.198$$

A1

Answer: 0.198

4 marks

$$(b) 8^2 \times 10^2 \text{ is the variance of } 8X, \text{ one passenger counted eight times}$$

E1

$$\text{Eight independent passengers: variances add, } 8 \times 100 = 800$$

E1

Answer: Variances add; 8 × 100 = 800

2 marks

$$(c) n \text{ passengers: } L \sim N(72n, 100n); \text{ need } P(L > 600) < 0.05, \text{ i.e. } \frac{600 - 72n}{10\sqrt{n}} > 1.645$$

M1

$$n = 7: \frac{600 - 504}{26.46} = 3.63 > 1.645 \text{ (satisfied); } n = 8: 0.8485 < 1.645 \text{ (fails)}$$

M1

Greatest number is 7

A1

Answer: 7

3 marks

23 (6 marks)

(a) $\bar{X} \sim N(\mu, \frac{36}{n})$

B1 B1

Answer: $N(\mu, \frac{36}{n})$

2 marks

(b) $P(|Z| < \frac{2\sqrt{n}}{6}) \geq 0.95$, so $\frac{2\sqrt{n}}{6} \geq 1.96$

M1 A1

$\sqrt{n} \geq 5.88$, so $n \geq 34.57$

M1

Smallest $n = 35$

A1

Answer: $n = 35$

4 marks

24 (10 marks)

(a) $E(F) = 3 + 1.5 \times 8 = 15$

B1

$\text{Var}(F) = 1.5^2 \times 4 = 9$, so $F \sim N(15, 9)$

B1

Answer: $N(15, 9)$

2 marks

(b) $\text{Total} \sim N(75, 45)$

M1 A1

$P(> 80) = P(Z > \frac{5}{\sqrt{45}}) = 0.228$

M1 A1

Answer: **0.228**

4 marks

(c) $F_1 - F_2 \sim N(0, 18)$

M1 A1

$P(> 6) = P(Z > \frac{6}{\sqrt{18}}) = P(Z > 1.414) = 0.0786$

M1 A1

Answer: **0.0786**

4 marks

25 (8 marks)(a) X has mean $np = 30$ and variance $npq = 12$, and is approximately normal since n is large and p is not close to 0 or 1; Y is approximately normal since its mean is large

E1 E1

$E(X - Y) = 30 - 16 = 14$ and $\text{Var}(X - Y) = 12 + 16 = 28$, so $X - Y \approx N(14, 28)$

B1 B1

Answer: $X - Y \approx N(14, 28)$

4 marks

(b) $X - Y$ takes whole-number values, so $P(X - Y \geq 18) \approx P(W > 17.5)$ where $W \sim N(14, 28)$

M1 A1

$= P(Z > \frac{17.5 - 14}{\sqrt{28}}) = P(Z > 0.661) = 0.254$

M1 A1

Answer: **0.254**

4 marks

26 (8 marks)

(a) $D = Y - (X_1 + X_2 + X_3) \sim N(95 - 90, 25 + 3 \times 16) = N(5, 73)$

M1 A1

$P(D > 0) = P(Z > \frac{-5}{\sqrt{73}}) = 0.721$

M1 A1

Answer: **0.721**

4 marks

(b) $D \sim N(\mu_Y - 90, 73)$ and $P(D > 0) = 0.1$

M1

$\frac{0 - (\mu_Y - 90)}{\sqrt{73}} = 1.2816$

M1 A1

$\mu_Y = 90 - 1.2816 \times 8.544 = 79.1$

A1

Answer: **79.1**

4 marks

27 (11 marks)

(a) $E(D) = 20 - \mu$

B1

$\text{Var}(D) = 25 + 9 = 34$, so $D \sim N(20 - \mu, 34)$

B1

Answer: $N(20 - \mu, 34)$

2 marks

(b) $P(D > 0) = 0.25$, so $P(D < 0) = 0.75$

M1

$\frac{0 - (20 - \mu)}{\sqrt{34}} = 0.6745$

M1 A1

$\mu = 20 + 0.6745 \sqrt{34} = 23.93$

A1

Answer: $\mu = 23.93$

4 marks

(c) $X + Y \sim N(20 + 23.93, 34) = N(43.93, 34)$

M1

$\frac{k - 43.93}{\sqrt{34}} = 1.2816$

M1

$$k = 51.41$$

Answer: $k = 51.41$

A1

3 marks

(d) $\text{Var}(3X) = 9 \times 25 = 225$, $\text{Var}(X_1 + X_2 + X_3) = 75$, difference 150

M1

$3X$ triples a single value so any spread is magnified; three separate values partly cancel

E1

Answer: 150

2 marks

28 (11 marks)

(a) $\text{Var}(\bar{X}) = \frac{1}{n^2} \text{Var}(X_1 + \dots + X_n)$

M1

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

A1

Answer: $\frac{\sigma^2}{n}$

2 marks

(b) $X_1 - \bar{X} = (1 - \frac{1}{n})X_1 - \frac{1}{n}(X_2 + \dots + X_n)$

M1

$$\text{Var} = (1 - \frac{1}{n})^2 \sigma^2 + (n - 1) \frac{1}{n^2} \sigma^2$$

M1 A1

$$= \frac{(n - 1)^2 + (n - 1)}{n^2} \sigma^2 = \frac{(n - 1)n}{n^2} \sigma^2 = \frac{(n - 1)\sigma^2}{n}$$

A1

Answer: shown

4 marks

(c) $X_1 - \bar{X}$ has mean 0 and variance $\frac{4 \times 16}{5} = 12.8$

M1

$$P(-3 < D < 3) = 2P(Z < \frac{3}{\sqrt{12.8}}) - 1$$

M1

$$= 2(0.7991) - 1 = 0.598$$

A1

Answer: 0.598

3 marks

(d) $\text{Var}(A - B) = \text{Var}(A) + \text{Var}(B)$ only for independent A and B

E1

X_1 is one of the values making up $\bar{X}$, so they are not independent

E1

Answer: X_1 and $\bar{X}$ are not independent

2 marks

29 (15 marks)

(a) $P(X = 0) = 0.3$, $P(X = 1) = 0.7$, $P(Y = 0) = 0.4$, $P(Y = 1) = 0.6$

M1 A1

$$P(X = 0)P(Y = 0) = 0.12 \neq 0.1 = P(X = 0, Y = 0)$$

M1 E1

Answer: not independent

4 marks

(b) $E(X) = 0.7$, $E(Y) = 0.6$

B1

$$\text{Var}(X) = 0.7 - 0.49 = 0.21, \text{Var}(Y) = 0.6 - 0.36 = 0.24$$

M1 A1

$$E(XY) = P(X = 1, Y = 1) = 0.4$$

B1

Answer: 0.7, 0.6, 0.21, 0.24, 0.4

4 marks

(c) $E(XY) - E(X)E(Y) = 0.4 - 0.42 = -0.02$

M1 A1

$$\text{Var}(X + Y) = 0.21 + 0.24 - 0.04 = 0.41$$

M1 A1

Answer: -0.02 and 0.41

4 marks

(d) $P(X + Y = 0, 1, 2) = 0.1, 0.5, 0.4$, so $E = 1.3$, $E(S^2) = 0.5 + 1.6 = 2.1$, $\text{Var} = 2.1 - 1.69 = 0.41$

M1 A1

The simple addition of variances needs independence, which fails here

E1

Answer: 0.41

3 marks

30 (15 marks)

(a) $\bar{X} \sim N(100, \frac{16}{5})$, standard error = $\frac{4}{\sqrt{5}} = 1.789$

M1 A1

$$\text{Limits: } 100 \pm 3(1.789) = 94.63 \text{ and } 105.4$$

M1 A1

Answer: 94.63 to 105.4

4 marks

(b) $P(|Z| > 3) = 2 \times 0.001350$

M1

$$= 0.00270$$

A1

Answer: 0.00270

2 marks

(c) $\bar{X} \sim N(102, 3.2)$

B1

$$P(\bar{X} > 105.4) = P(Z > 1.88) = 0.02992$$

M1 A1

$$P(\bar{X} < 94.63) = 0.000019106, \text{ so } P(\text{alarm}) = 0.02994$$

M1 A1

Answer: 0.02994

5 marks

(d) Expected number = $\frac{1}{0.02994} = 33.4$ hours

M1 A1

P(within 3 hours) = $1 - (1 - 0.02994)^3 = 0.0872$

M1 A1

Answer: 33.4 hours, 0.0872

4 marks