

Critical Path Analysis

1 (2 marks)

Total float is the amount by which the activity can be delayed without delaying the whole project

B1

Total float = latest finish – earliest start – duration

B1

Answer: LF – ES – duration

2 marks

2 (2 marks)

Float = $20 - 9 - 6 = 5$

M1

Float is not 0, so the activity is not critical

A1

Answer: 5, not critical

2 marks

3 (2 marks)

C starts after the later of A and B finish: $\max(4, 7) = 7$

M1

Minimum time = $7 + 3 = 10$

A1

Answer: 7; 10

2 marks

4 (2 marks)

Minimum time = $5 + 8 + 4 = 17$ days

B1

Every activity is critical in a single chain, so the float is 0

B1

Answer: 17 days; 0

2 marks

5 (2 marks)

Activities on a path must be done one after another, so the project takes at least as long as every path

E1

The longest path therefore sets the shortest possible time for the project

E1

Answer: Every path must be completed in sequence

2 marks

6 (2 marks)

A, C and E

B1

They have total float 0, so any delay to them delays the project

E1

Answer: A, C, E

2 marks

7 (2 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 4; B: 0, 6; C: 4, 9; D: 6, 9; E: 6, 13; F: 9, 13; G: 13, 18; H: 18, 21

M1

Minimum project time = largest EF = 21

A1

Answer: 21

2 marks

8 (4 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 6; B: 0, 4; C: 6, 9; D: 6, 11; E: 4, 12; F: 11, 13;

M1

G: 12, 18; H: 18, 22

Minimum project time = largest EF = 22

A1

Latest finish (LF) and latest start (LS): H: 22, 18; G: 18, 12; F: 18, 16; E: 12, 4; D: 12, 7; C: 16, 13; B: 4, 0; A: 7, 1

M1

Critical activities (float 0): B, E, G, H; critical path B-E-G-H

A1

Answer: Critical: B, E, G, H; time 22

4 marks

9 (4 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 5; B: 0, 3; C: 5, 11; D: 3, 7; E: 7, 9; F: 11, 18; G: 7, 10; H: 18, 23

M1

Minimum project time = largest EF = 23

A1

Latest finish (LF) and latest start (LS): H: 23, 18; G: 18, 15; F: 18, 11; E: 11, 9; D: 9, 5; C: 11, 5; B: 5, 2; A: 5, 0

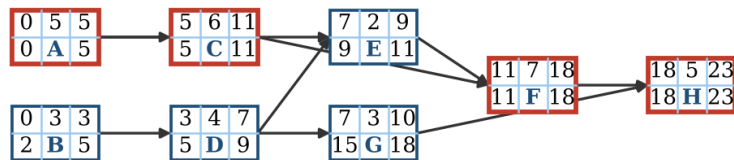
M1

Critical activities (float 0): A, C, F, H; critical path A-C-F-H

A1

Answer: Critical: A, C, F, H; time 23

4 marks



10 (3 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 2; B: 2, 5; C: 2, 6; D: 5, 10; E: 6, 12; F: 10, 13; G: 12, 16; H: 16, 18

M1

Minimum project time = largest EF = 18

A1

Latest finish (LF) and latest start (LS): H: 18, 16; G: 16, 12; F: 16, 13; E: 12, 6; D: 13, 8; C: 6, 2; B: 6, 3; A: 2, 0

M1

Answer: Time 18; latest starts A 0, B 3, C 2, D 8, E 6, F 13, G 12, H 16

3 marks

11 (4 marks)

(a) Float = $14 - 3 - 5 = 6$

M1 A1

Answer: 6

2 marks

(b) Float = $16 - 7 - 9 = 0$

M1

Critical

A1

Answer: 0, critical

2 marks

12 (3 marks)

Float of H = $22 - 14 - 3 = 5$

M1

G ends at 14, H could then start as late as 19; a 4-day delay to G moves H's start to 18 which is within H's float, so the project is not delayed

M1 A1

Answer: No

3 marks

13 (3 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 3; B: 0, 5; C: 3, 7; D: 5, 11; E: 5, 7; F: 11, 16; G: 11, 15; H: 16, 19; I: 16, 22; J: 22, 24

M1

Minimum project time = largest EF = 24

A1

Activities with earliest start 11: F, G

B1

Answer: 24

3 marks

14 (5 marks)

(a) Earliest start (ES) and earliest finish (EF): A: 0, 3; B: 0, 5; C: 3, 7; D: 5, 11; E: 5, 7; F: 11, 16; G: 11, 15; H: 16, 19; I: 16, 22; J: 22, 24

M1

Minimum project time = largest EF = 24

A1

Latest finish (LF) and latest start (LS): J: 24, 22; I: 22, 16; H: 22, 19; G: 16, 12; F: 16, 11; E: 12, 10; D: 11, 5; C: 11, 7; B: 5, 0; A: 5, 2

M1

Answer: Time 24 days

3 marks

(b) Critical activities (float 0): B, D, F, I, J; critical path B-D-F-I-J

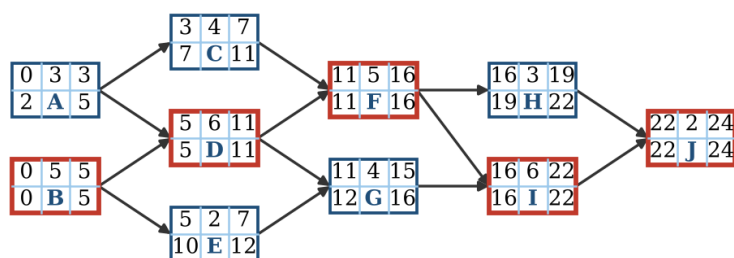
A1

Total float = LS - ES: A 2, B 0, C 4, D 0, E 5, F 0, G 1, H 3, I 0, J 0

A1

Answer: Critical: B, D, F, I, J

2 marks



15 (5 marks)

(a) Earliest start (ES) and earliest finish (EF): A: 0, 4; B: 0, 6; C: 4, 9; D: 6, 9; E: 6, 13; F: 9, 13; G: 13, 18; H: 18, 21

M1

Minimum project time = largest EF = 21

A1

Critical activities (float 0): B, E, G, H; critical path B-E-G-H

A1

Answer: 21; B-E-G-H

3 marks

(b) Float of D = 4

M1

$2 \leq 4$, so the project is not delayed

A1

Answer: Not affected

2 marks

16 (5 marks)

(a) Using the forward pass: minimum time 23

M1

Critical path A-C-F-H

A1

Answer: 23

2 marks

(b) D now finishes at 12 and A finishes at 5, so E starts at 12 and finishes at 14

M1

F starts at $\max(11, 14) = 14$ and finishes at 21, so H starts at 21

A1

New minimum time 26, critical path B-D-E-F-H

A1

Answer: 26

3 marks

17 (5 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 6; B: 0, 4; C: 6, 9; D: 6, 11; E: 4, 12; F: 11, 13;

M1

G: 12, 18; H: 18, 22

Minimum project time = largest EF = 22

A1

Latest finish (LF) and latest start (LS): H: 22, 18; G: 18, 12; F: 18, 16; E: 12, 4; D: 12, 7; C: 16, 13; B: 4, 0; A: 7, 1

M1

Total float = LS – ES: A 1, B 0, C 7, D 1, E 0, F 5, G 0, H 0

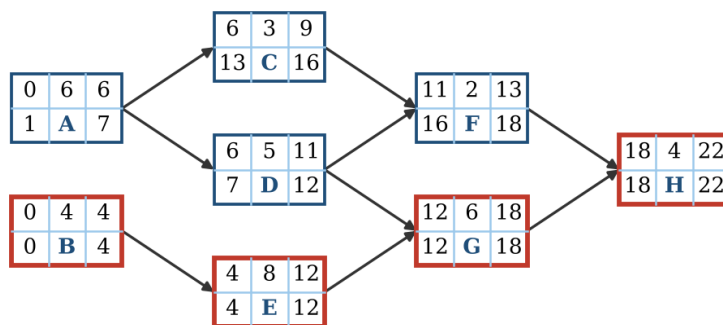
A1

Critical activities (float 0): B, E, G, H; critical path B-E-G-H

A1

Answer: Critical: B, E, G, H

5 marks

**18 (6 marks)**

(a) Earliest start (ES) and earliest finish (EF): A: 0, 2; B: 2, 5; C: 2, 6; D: 5, 10; E: 6, 12; F: 10, 13; G: 12, 16; H: 16, 18

M1

Minimum project time = largest EF = 18

A1

Latest finish (LF) and latest start (LS): H: 18, 16; G: 16, 12; F: 16, 13; E: 12, 6; D: 13, 8; C: 6, 2; B: 6, 3; A: 2, 0

M1

Total float = LS – ES: A 0, B 1, C 0, D 3, E 0, F 3, G 0, H 0

A1

Answer: ES/LS found; floats A 0, B 1, C 0, D 3, E 0, F 3, G 0, H 0

4 marks

(b) Correct floats: B 1, C 0, D 3, E 0, F 3, G 0

M1

Errors: B (should be 1), C (should be 0), D (should be 3)

A1

Answer: Compare with the correct floats

2 marks

19 (5 marks)

Float X = $20 - 8 - 5 = 7$; float Y = $25 - 13 - 4 = 8$

M1 A1

Path X then Y: float = $25 - 8 - (5 + 4) = 8$

B1

If X uses its float the earliest start of Y moves later, so some of Y's float is used up: the two floats are shared along the path, not independent

E1 E1

Answer: 7, 8 and 8; floats are shared

5 marks

20 (5 marks)

Earliest start (ES) and earliest finish (EF): A: 0, 2; B: 2, 5; C: 2, 6; D: 5, 10; E: 6, 12; F: 10, 13;

M1

G: 12, 16; H: 16, 18

Minimum project time = largest EF = 18

A1

Latest finish (LF) and latest start (LS): H: 18, 16; G: 16, 12; F: 16, 13; E: 12, 6; D: 13, 8; C: 6, 2;

M1

B: 6, 3; A: 2, 0

Critical activities (float 0): A, C, E, G, H; critical path A-C-E-G-H

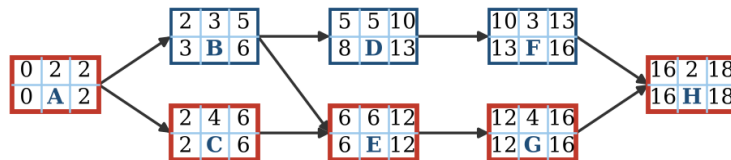
A1

Latest finish of F = 16

A1

Answer: 18; LF of F = 16

5 marks

**21 (7 marks)**

(a) Earliest start (ES) and earliest finish (EF): A: 0, 3; B: 0, 5; C: 3, 7; D: 5, 11; E: 5, 7; F: 11, 16;

M1

G: 11, 15; H: 16, 19; I: 16, 22; J: 22, 24

Minimum project time = largest EF = 24

A1

Critical activities (float 0): B, D, F, I, J; critical path B-D-F-I-J

A1

Answer: 24

3 marks

(b) With D = 4: new minimum time 22

M1 A1

Reduction = $24 - 22 = 2$ days; critical path(s) B-D-F-I-J

A1

Answer: 2 days

3 marks

(c) The longest path is unchanged because the critical path does not contain that activity

E1

Answer: Critical path unchanged

1 mark

22 (6 marks)

(a) Path B-D-G-H = $6 + 3 + 5 + 3 = 17 < \text{B-E-G-H} = 6 + 7 + 5 + 3 = 21$

M1

The minimum completion time is 21 and D is not on a longest path

A1

Answer: 21

2 marks

(b) B-D-G-H = $6 + x + 5 + 3 = 14 + x$ and B-E-G-H = 21

M1

$14 + x \geq 21$ gives $x \geq 7$, so the smallest integer is 7

A1

Answer: 7

2 marks

(c) B-D-G-H = $6 + 10 + 5 + 3 = 24$, A-D-G-H = $4 + 10 + 5 + 3 = 22$ and B-E-G-H = 21, so the longest path is B-D-G-H

M1

Minimum time = 24

A1

Answer: 24

2 marks

23 (6 marks)

(a) Earliest start (ES) and earliest finish (EF): A: 0, 5; B: 0, 3; C: 5, 11; D: 3, 7; E: 7, 9; F: 11, 18;

M1

G: 7, 10; H: 18, 23

Minimum project time = largest EF = 23

A1

Latest finish of G = latest start of H = 18, so float of G = $18 - 7 - 3 = 8$

A1

Answer: 23; 8

3 marks

(b) G takes $14 - 3 = 11$ days longer than planned, and its float is 8

M1

Delay to the project = $11 - 8 = 3$ days

A1

New minimum time $23 + 3 = 26$ days

A1

Answer: 26 days, a delay of 3 days

3 marks

24 (7 marks)

(a) Earliest start (ES) and earliest finish (EF): A: 0, 4; B: 0, 6; C: 4, 9; D: 6, 9; E: 6, 13; F: 9, 13;

M1

G: 13, 18; H: 18, 21

Minimum project time = largest EF = 21

A1

Critical activities (float 0): B, E, G, H; critical path B-E-G-H

A1

Answer: 21 days; B, E, G, H

3 marks

(b) F is not critical (float 5), so reducing it does not shorten the project	E1
Only the critical activities B, E and G (£250, £300, £200 per day) can shorten the project; G is the cheapest	M1
Reduce G by 2 days (the most allowed): cost $2 \times £200 = £400$, and B-E-G-H becomes $6 + 7 + 3 + 3 = 19$ days	A1
Check the other paths: B-D-G-H = $6 + 3 + 3 + 3 = 15$ and A-C-F-H = 16, so B-E-G-H is still the longest and the time is 19 days	A1
Answer: Reduce G by 2 days, £400	4 marks
25 (5 marks)	
(a) Earliest start (ES) and earliest finish (EF): A: 0, 6; B: 0, 4; C: 6, 9; D: 6, 11; E: 4, 12; F: 11, 13; G: 12, 18; H: 18, 22	M1
Minimum project time = largest EF = 22	A1
Answer: A 0, B 0, C 6, D 6, E 4, F 11, G 12, H 18	2 marks
(b) Errors: F (should be 11), G (should be 12), H (should be 18)	M1
The student ignored the other predecessor: F must wait for both C and D, and G for both D and E	E1
Correct minimum time 22, not 21	A1
Answer: F, G and H are wrong; 22	3 marks
26 (9 marks)	
(a) Longest path B-D-F-I-J = $5 + 6 + 5 + 6 + 2 = 24$	M1
B-E-G-I-J = $5 + 5 + 4 + 6 + 2 = 22 < 24$, so E is not critical and the minimum time is 24	A1
Answer: 24	2 marks
(b) B-E-G-I-J = $17 + x$	M1
$17 + x \leq 24$ gives $x \leq 7$	A1
Answer: $x \leq 7$	2 marks
(c) Minimum time = $17 + x$	B1
E and G become critical when $17 + x = 24$, that is $x = 7$	M1 A1
Answer: $17 + x$; $x = 7$	3 marks
(d) F is on B-D-F-I-J of length 24 but the longest path is now $17 + x$, so float = $(17 + x) - 24 = x - 7$	M1 A1
Answer: $x - 7$	2 marks
27 (6 marks)	
(a) If P and R are critical they are on a longest path; with Q the only link between them, Q lies on that longest path	E1
Then Q would have float 0, contradicting $f > 0$	E1
Answer: Q would be critical	2 marks
(b) Path P-Q-R is longer than P-S-R by $7 - 4 = 3$	M1
So Q is critical and the float on S is 3	A1
Answer: Q critical; S float 3	2 marks
(c) S's float is 3, the delay of 5 exceeds this by 2	M1
The project is delayed by 2	A1
Answer: 2	2 marks
28 (8 marks)	
(a) Earliest start (ES) and earliest finish (EF): A: 0, 3; B: 0, 5; C: 3, 7; D: 5, 11; E: 5, 7; F: 11, 16; G: 11, 15; H: 16, 19; I: 16, 22; J: 22, 24	M1
Minimum project time = largest EF = 24	A1
Critical activities (float 0): B, D, F, I, J; critical path B-D-F-I-J	A1
Answer: 24	3 marks
(b) K starts at EF of D = 11 and finishes at 19; J can start at max(EF of H, I, K)	M1
Path B-D-K-J = $5 + 6 + 8 + 2 = 21 < 24$, so nothing changes: K has float 3	M1 A1
Answer: No change; time 24	3 marks
(c) K starts at 11 and J must start by 22	M1
Greatest duration = $22 - 11 = 11$	A1
Answer: 11	2 marks

29 (7 marks)

(a) Earliest start (ES) and earliest finish (EF): A: 0, 7; B: 0, 7; C: 7, 13; D: 7, 9; E: 7, 13; F: 13, 16; G: 16, 21 M1

Minimum project time = largest EF = 21 A1

Critical paths: A-C-F-G and B-E-F-G A1 A1

Answer: 21 4 marks

(b) The activity must be on every critical path: F and G are on both (B-E-F-G and A-C-F-G) M1 A1

An activity on only one critical path (A, C, B, E) leaves the other path unchanged E1

Answer: F or G 3 marks

30 (6 marks)

(a) Free float of X = (earliest ES of its successors) – EF of X M1

A: successors C (ES 3) and D (ES 5), EF of A = 3: free float 0 A1

C: successor F (ES 11), EF of C = 7: free float 4; E: successor G (ES 11), EF of E = 7: free float 4 A1 A1

Answer: A 0, C 4, E 4 4 marks

(b) Delaying an activity by its free float does not move any successor; so the project is certainly E1

unaffected, hence total float is at least the free float

Total float allows successors to move too E1

Answer: Free float $\leq$ total float 2 marks

Activity Networks

1 (2 marks)

A dummy shows a precedence relationship that cannot be shown otherwise (or gives each activity a unique pair of start and end events) B1

It has zero duration and uses no resources B1

Answer: Shows dependence; zero duration 2 marks

2 (2 marks)

Float = late time at end – early time at start – duration M1

= 19 – 11 – 5 = 3 A1

Answer: 3 2 marks

3 (2 marks)

Draw dummy activities from events 2 and 3 to event 4 (or make both P and Q end at event 4 if they can share an end event) B1

The early time of event 4 is then $\max(5, 7) = 7$ B1

Answer: Use dummies to event 4; early time 7 2 marks

4 (2 marks)

F starts at event 4, which is reached by C (via event 2) and D (via event 3) M1

Preceding F: C, D directly, and hence also A and B A1

Answer: C and D (so A and B too) 2 marks

5 (2 marks)

The float of the activity: the time the activity can be delayed without delaying the project B1

The bar plus the float ends at the latest finish time B1

Answer: Float 2 marks

6 (2 marks)

Lower bound = $52 \div 13 = 4$ M1 A1

Answer: 4 2 marks

7 (5 marks)

Early event times (forward pass): 1: 0, 2: 6, 3: 6, 4: 9, 5: 13, 6: 16

M1 A1

Late event times (backward pass): 1: 0, 2: 6, 3: 6, 4: 12, 5: 13, 6: 16

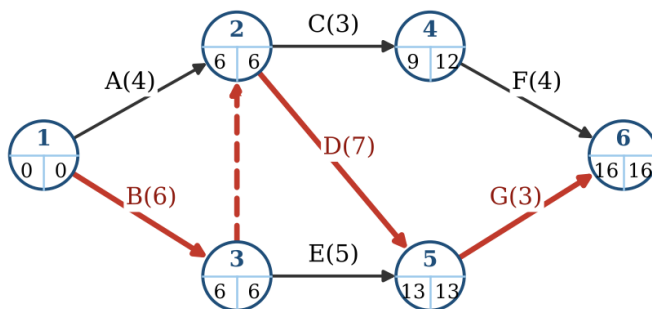
M1 A1

Minimum project time = early time at the last event = 16

B1

Answer: 16

5 marks

**8 (5 marks)**

Early event times (forward pass): 1: 0, 2: 5, 3: 3, 4: 11, 5: 11, 6: 15, 7: 20

M1 A1

Late event times (backward pass): 1: 0, 2: 5, 3: 4, 4: 11, 5: 11, 6: 15, 7: 20

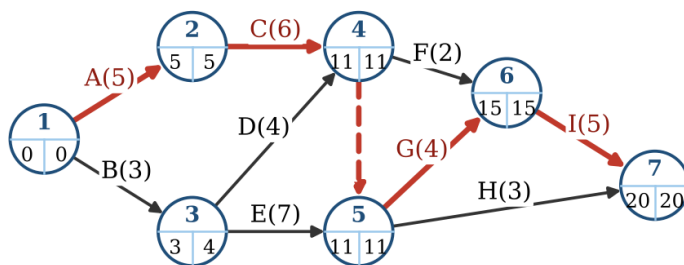
M1 A1

Critical activities (early = late at both ends, and duration matches): A, C, G, I

A1

Answer: Critical: A, C, G, I; time 20

5 marks

**9 (5 marks)**

Early event times (forward pass): 1: 0, 2: 3, 3: 5, 4: 7, 5: 11, 6: 15

M1 A1

Late event times (backward pass): 1: 0, 2: 8, 3: 5, 4: 12, 5: 11, 6: 15

M1 A1

Float of C = late at 4 – early at 2 – 4 = 12 – 3 – 4 = 5

A1

Answer: 15; 5

5 marks

10 (3 marks)

(a) Predecessors: A: -; B: -; C: A; D: B; E: B; F: C, D; G: E

M1 A1

Answer: A: -; B: -; C: A; D: B; E: B; F: C, D; G: E

2 marks

(b) Early time at the last event = 15

B1

Answer: 15

1 mark

11 (3 marks)

Earliest starts: A 0, B 0, C 6, D 6, E 6, F 9, G 13

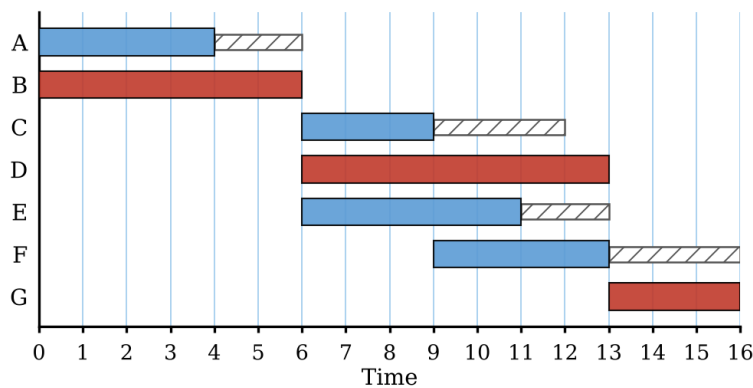
M1 A1

Active at t = 5: B

A1

Answer: 1

3 marks

**12 (3 marks)**

B can finish up to 1 day later without delaying the project (late 4 – early 3 = 1), so B has float 1
 Float of D = late at 4 – early at 3 – 4 = 11 – 3 – 4 = 4

Answer: B: 1; D: 4

B1
M1 A1
3 marks

13 (3 marks)

A: event 1 to 2, B: event 1 to 3, C: event 2 to 4

A single dummy from 3 to 2 would make C follow B as well, which is not allowed

Instead give D its own start event 5, with dummies 2 → 5 and 3 → 5, and then D from 5 to 6; now D follows A and B and C follows only A

Answer: Use dummies 2 → 5 and 3 → 5 so that only D follows both

B1
M1
A1

3 marks

14 (7 marks)

(a) F depends only on C and D, but G and H depend on C, D and E

A dummy from the end of C and D to the start of G and H shows E's extra dependency without giving F the same predecessors

Answer: F and G/H have different predecessor sets

(b) Early event times (forward pass): 1: 0, 2: 5, 3: 3, 4: 11, 5: 11, 6: 15, 7: 20

Late event times (backward pass): 1: 0, 2: 5, 3: 4, 4: 11, 5: 11, 6: 15, 7: 20

Minimum time = 20

Answer: 20

B1
E1

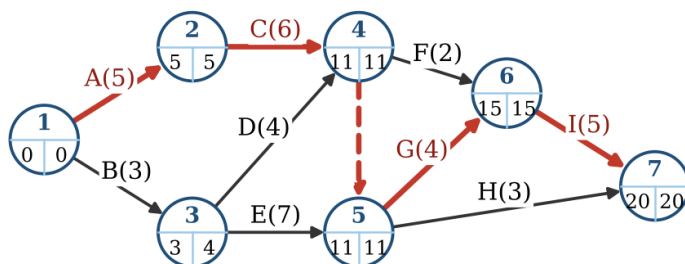
2 marks

M1 A1

M1 A1

B1

5 marks

**15 (7 marks)**

(a) Predecessors: A: -; B: -; C: -; D: A; E: B, C; F: C; G: D, E; H: F

Answer: A: -; B: -; C: -; D: A; E: B, C; F: C; G: D, E; H: F

(b) Early event times (forward pass): 1: 0, 2: 2, 3: 4, 4: 3, 5: 7, 6: 9, 7: 11

Late event times (backward pass): 1: 0, 2: 2, 3: 5, 4: 3, 5: 7, 6: 9, 7: 11

Answer: 11

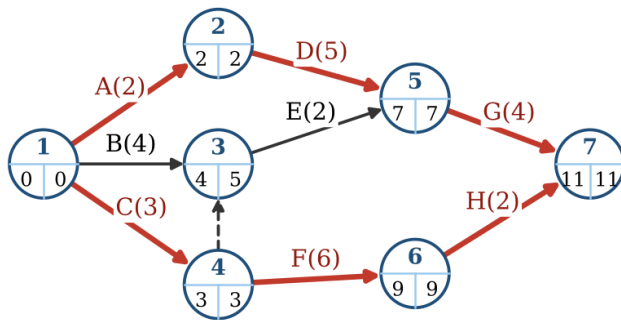
M1 A1 A1

3 marks

M1 A1

M1 A1

4 marks



16 (5 marks)

(a) Earliest start times: A 0, B 0, C 6, D 6, E 6, F 9, G 13

M1

Bars: A from 0 to 4, B from 0 to 6, C from 6 to 9, D from 6 to 13, E from 6 to 11, F from 9 to 13, G from 13 to 16

A1

Answer: A 0 to 4; B 0 to 6; C 6 to 9; D 6 to 13; E 6 to 11; F 9 to 13; G 13 to 16

2 marks

(b) Workers needed in each unit of time from 0: 2, 2, 2, 2, 1, 1, 3, 3, 3, 3, 3, 2, 2, 1, 1, 1

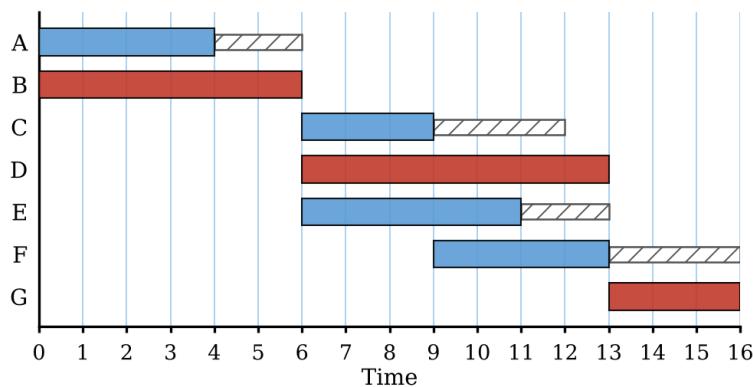
M1 A1

Greatest number = 3

A1

Answer: 3

3 marks



17 (6 marks)

(a) From the early event times of the start events: A 0, B 0, C 5, D 3, E 3, F 11, G 11, H 11, I 15

M1 A1

Answer: A 0, B 0, C 5, D 3, E 3, F 11, G 11, H 11, I 15

2 marks

(b) At $t = 8$ (between 8 and 9): active activities C, E = 2 workers

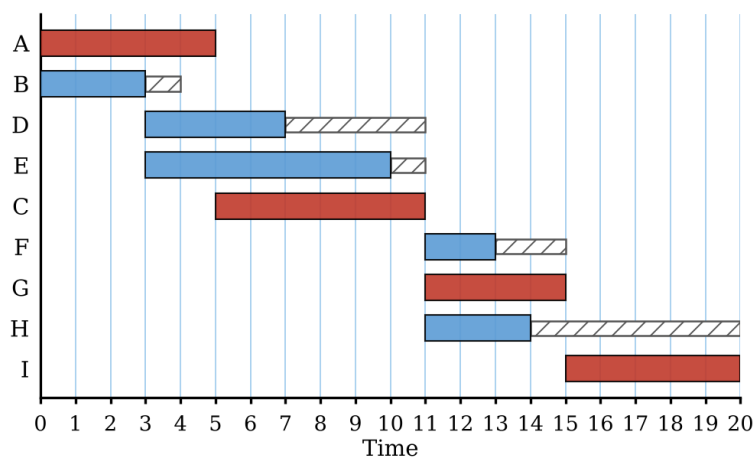
M1 A1

Total work = 39, minimum time = 20, so at least $\lceil 39 \div 20 \rceil = 2$ workers

M1 A1

Answer: 2 at $t = 8$; at least 2

4 marks



18 (2 marks)

A non-critical activity can start later than its earliest start without delaying the project, up to its float

E1

Delaying it to a time of low demand lowers the peak in the histogram, as long as no precedences are broken

E1

Answer: Delay non-critical activities to smooth demand

2 marks

19 (7 marks)

(a) Early event times (forward pass): 1: 0, 2: 3, 3: 5, 4: 7, 5: 11, 6: 15

M1 A1

Late event times (backward pass): 1: 0, 2: 8, 3: 5, 4: 12, 5: 11, 6: 15

M1 A1

Answer: 1: 0, 2: 3, 3: 5, 4: 7, 5: 11, 6: 15

4 marks

(b) The student's early time at event 5, 6 is wrong

M1

Event 5: early time = 5 + 6 = 11 (activity E takes 6 days), not 9; event 6 is then 11 + 4 = 15 from G, which is later than 7 + 3 = 10 from F

M1

Correct minimum time = 15

A1

Answer: Events 5 and 6 are wrong; 15

3 marks

20 (6 marks)

Early event times (forward pass): 1: 0, 2: 3, 3: 5, 4: 7, 5: 11, 6: 15

M1 A1

Late event times (backward pass): 1: 0, 2: 8, 3: 5, 4: 12, 5: 11, 6: 15

M1 A1

Critical activities: B, E, G

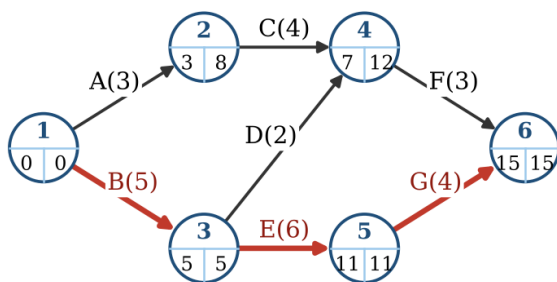
A1

Floats: A 5, C 5, D 5, F 5

A1

Answer: 15; critical B, E, G

6 marks

**21 (8 marks)**

(a) Minimum project time 20

B1

Total of durations = 39

B1

Answer: 20; 39

2 marks

(b) Workers $\geq \text{total} \div \text{time} = 39 \div 20 = 1.95$

M1

So at least 2 workers

A1

Answer: 2

2 marks

(c) The critical activities A, C, G, I run one after another with no float, so one worker is fully occupied with them until the last of them ends

M1

G cannot start until C, D and E are finished, and G starts at 11; so before time 11 the second worker must do B, D and E, which take $3 + 4 + 7 = 14 > 11$

M1 A1

Hence two workers need longer than 20 days

A1

Answer: Two workers are not enough

4 marks

22 (11 marks)

(a) Early event times (forward pass): 1: 0, 2: 2, 3: 4, 4: 3, 5: 7, 6: 9, 7: 11

M1 A1

Late event times (backward pass): 1: 0, 2: 2, 3: 5, 4: 3, 5: 7, 6: 9, 7: 11

M1 A1

Critical: A, C, D, F, G, H; minimum time 11

A1

Answer: 11

5 marks

(b) Workers per unit time: 3, 3, 3, 3, 3, 2, 2, 2, 2, 2

M1 A1

Greatest 3

A1

Answer: 3

3 marks

(c) A and C are critical, so they run from 0 to 2 and 0 to 3

M1

B has float 1 so it starts at 0 or 1 and is still in progress throughout the interval from 1 to 2
During that interval A, B and C are all in progress, so three workers are needed

Answer: Three activities overlap

M1

A1

3 marks

23 (3 marks)

(a) Float of C = 3

M1

Delay of 6 exceeds float by 3, so the project is delayed by 3 days

A1

Answer: 3 days

2 marks

(b) $16 + 3 = 19$

A1

Answer: 19

1 mark

24 (5 marks)

(a) Late at 4 = 11; late at 3 = $\min(\text{late at 4} - D, \text{late at 5} - E) = \min(11 - 4, 11 - 7) = 4$

M1 A1

Answer: 4

2 marks

(b) Early time is the earliest the event can occur given preceding activities; late time is the latest it can occur without delaying the project; they differ when activities leading to it have float

E1

Answer: Float

1 mark

(c) B has early start 0 and its end event 3 has early 3 and late 4, so slack 1: B is not critical

M1 A1

Answer: Not critical

2 marks

25 (5 marks)

(a) Paths: A-C-F = $3 + 4 + 3 = 10$; B-D-F = $5 + 2 + 3 = 10$; B-E-G = $5 + 2 + 2 = 9$

M1

Minimum time = 10 days; float of E = $10 - 9 = 1$

A1

Answer: 10 days; 1

2 marks

(b) B-E-G = $5 + x + 2 = 7 + x$; the other paths are A-C-F = 10 and B-D-F = 10

M1

E is critical when $7 + x \geq 10$, so $x \geq 3$

A1

Answer: $x \geq 3$

2 marks

(c) B-E-G = $7 + 8 = 15$, which is now the longest path

A1

Answer: 15

1 mark

26 (5 marks)

(a) Two workers can do at most $2 \times t$ of work in time t , so $t \geq 32 \div 2 = 16.0$

M1

The critical path needs 19, so $t \geq 19$

A1

Answer: 19

2 marks

(b) Schedule (priority to activities with smallest latest start): A: start 0, worker 1, B: start 0, worker 2, C: start 5, worker 2, D: start 4, worker 1, E: start 3, worker 2, F: start 10, worker 1, G: start 10, worker 2, H: start 14, worker 1

M1 A1

Finish time 19

A1

Answer: 19

3 marks

27 (9 marks)

(a) Earliest start (ES) and earliest finish (EF): A: 0, 4; B: 0, 3; C: 4, 9; D: 4, 10; E: 3, 5; F: 10, 14; G: 10, 13; H: 14, 19

M1

Minimum project time = largest EF = 19

A1

Critical activities (float 0): A, D, F, H; critical path A-D-F-H

A1

Answer: 19

3 marks

(b) Total = 32, so workers $\geq 32 \div 19 = 1.68$, at least 2

M1 A1

Answer: 2

2 marks

(c) Schedule with 2 workers: A starts at 0 (worker 1), B starts at 0 (worker 2), C starts at 5

M1 A1 A1

(worker 2), D starts at 4 (worker 1), E starts at 3 (worker 2), F starts at 10 (worker 1), G starts at 10 (worker 2), H starts at 14 (worker 1)

Finish time = 19

A1

Answer: 2 workers

4 marks

28 (5 marks)

(a) With k workers in time T the total work possible is at most kT

M1

This must be at least W , so $k \geq W \div T$, and k is an integer

E1 E1

Answer: $k \geq \lceil W \div T \rceil$

3 marks

- (b) The bound counts only the total work and ignores precedence: activities that must run at the same time may need more workers than the average suggests E1
 Example: activity X lasts 1 day and four other activities of 1 day each can start only when X is finished: $T = 2$, $W = 5$, so $\lceil W \div T \rceil = 3$, but all four must run together in day 2, so 4 workers are needed E1
Answer: Precedences can force more workers 2 marks

29 (9 marks)

- (a) Earliest-start histogram counts: 2, 2, 2, 3, 3, 3, 3, 2, 2, 2, 1, 3, 3, 2, 1, 1, 1, 1, 1, 1 M1 A1
 Latest-start histogram counts: 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 1, 1, 2, 2, 1, 1, 2, 2, 2 M1 A1
 Greatest: 3 and 3 A1
Answer: 3 and 3 5 marks
 (b) Moving a non-critical activity within its float changes which activities it overlaps, so the peak in the histogram can change without delaying the project E1
 Both histograms have a peak of 3. The critical activities A, C, G and I keep one worker busy for all 20 days, and the other worker would have to do B, D and E ($3 + 4 + 7 = 14$ days of work) before G starts at 11 M1 A1
 This is impossible, so 3 workers are needed whatever the starting times chosen A1
Answer: No; 3 workers are needed 4 marks

30 (7 marks)

- (a) D follows both A and B, but C follows only A and E follows only B, so the end events of A and B cannot simply both be the start event of D E1
 F follows C and D, while G follows D and E: D's end event must lead to both F and G, but C must not precede G and E must not precede F, so dummy activities are needed E1
Answer: Overlapping predecessor sets need dummies 2 marks
 (b) Earliest start (ES) and earliest finish (EF): A: 0, 3; B: 0, 4; C: 3, 5; D: 4, 9; E: 4, 7; F: 9, 13; G: 9, 11; H: 13, 16 M1
 Minimum project time = largest EF = 16 A1
 Critical activities (float 0): B, D, F, H; critical path B-D-F-H A1
Answer: 16 3 marks
 (c) Total float = LS – ES: A 1, B 0, C 4, D 0, E 4, F 0, G 2, H 0 A1 A1
Answer: A 1, C 4, E 4, G 2 2 marks

Linear Programming

1 (2 marks)

- $x = 0$: $y = 9$, so (0, 9) B1
 $y = 0$: $x = 6$, so (6, 0) B1
Answer: (0, 9) and (6, 0) 2 marks

2 (2 marks)

- $6 + 10 = 16 > 14$ M1
 So (6, 5) is not in the wanted region (it is in the unwanted region) A1
Answer: No, since $16 > 14$ 2 marks

3 (2 marks)

- Oven time used = $3x + 5y$ M1
 $3x + 5y \leq 120$ A1
Answer: $3x + 5y \leq 120$ 2 marks

4 (2 marks)

- $P = 40x + 90y$ B1 B1
Answer: $P = 40x + 90y$ 2 marks

5 (2 marks)Subtract: $2x = 8$, so $x = 4$

M1

 $y = 6$, giving (4, 6)

A1

Answer: (4, 6)

2 marks

6 (2 marks)

The objective function is linear, so on any edge it changes steadily from one end to the other

E1

So its greatest value on the region occurs at an end of an edge, that is, at a vertex

E1

Answer: Linear functions are extreme at vertices

2 marks

7 (4 marks)

Vertices of the feasible region: (0, 0), (12, 0), (8, 6), (0, 10)

M1 A1

 $P = 5x + 4y$ at each vertex: (0, 0): 0, (12, 0): 60, (8, 6): 64, (0, 10): 40

M1

Greatest is P at (8, 6), where $P = 64$

A1

Answer: 64 at (8, 6)

4 marks

8 (4 marks)

Vertices of the feasible region: (0, 0), (8, 0), (6, 4), (0, 6)

M1 A1

 $P = 3x + 4y$ at each vertex: (0, 0): 0, (8, 0): 24, (6, 4): 34, (0, 6): 24

M1

Greatest is P at (6, 4), where $P = 34$

A1

Answer: 34 at (6, 4)

4 marks

9 (4 marks)Vertices: (0, 8), (2, 4), (10, 0) (the intersection of the lines: $2x + y = 8$ and $x + 2y = 10$ gives (2, 4))

M1 A1

 $C: (0, 8) \rightarrow 16, (2, 4) \rightarrow 14, (10, 0) \rightarrow 30$

M1

Minimum $C = 14$ at (2, 4)

A1

Answer: 14 at (2, 4)

4 marks

10 (6 marks)(a) $x + y \leq 40, x \geq 10, y \leq 2x, x, y \geq 0$

B1 B1

Objective: maximise $P = 300x + 200y$

B1

Answer: $x + y \leq 40, x \geq 10, y \leq 2x; P = 300x + 200y$

3 marks

(b) Vertices: (10, 0), (40, 0), (40/3, 80/3) and (10, 20)

M1

 $P: 3000, 12000, 300 \times 40/3 + 200 \times 80/3 = 9333.33, 3000 + 4000 = 7000$

M1

Maximum at (40, 0) with $P = 12000$

A1

Answer: $x = 40, y = 0, £12000$

3 marks

11 (4 marks)Objective lines $2x + 3y = k$ have gradient $-\frac{2}{3}$

B1

Slide the line outwards from the origin until it last touches R: this is at the vertex (6, 6)

M1 A1

 $P = 30$

A1

Answer: 30 at (6, 6); gradient $-2/3$

4 marks

12 (4 marks)Vertices of the feasible region: (0, 3), (3, 0), (7, 0), $(\frac{26}{5}, \frac{27}{5})$, (0, 8)

M1 A1

 $C = 4x + 1y$ at each vertex: (0, 3): 3, (3, 0): 12, (7, 0): 28, $(\frac{26}{5}, \frac{27}{5})$: $\frac{131}{5}$, (0, 8): 8

M1

Least is C at (0, 3), where $C = 3$

A1

Answer: 3 at (0, 3)

4 marks

13 (5 marks) $2x + 3y \leq 60$ and $x + y \leq 25, x, y \geq 0$

B1 B1

Vertices: (0, 0), (25, 0), (15, 10), (0, 20); maximise $x + y$: 0, 25, 25, 20

M1 A1

The maximum 25 occurs at both (25, 0) and (15, 10), and at every point between them, since $x + y = 25$ is also a constraint

E1

Answer: 25 units per day

5 marks

14 (7 marks)**(a)** $4x + 3y \leq 240$, $x + y \leq 70$, $y \geq 10$, $x \geq 0$ **B1 B1**Maximise $P = 30x + 25y$; R is the feasible region where all constraints hold**B1****Answer: $4x + 3y \leq 240$, $x + y \leq 70$, $y \geq 10$, $x \geq 0$; maximise $P = 30x + 25y$** **3 marks****(b)** Vertices of the feasible region: $(0, 10)$, $(\frac{105}{2}, 10)$, $(30, 40)$, $(0, 70)$ **M1 A1** $P = 30x + 25y$ at each vertex: $(0, 10)$: 250, $(\frac{105}{2}, 10)$: 1825, $(30, 40)$: 1900, $(0, 70)$: 1750**M1**Greatest is P at $(30, 40)$, where $P = 1900$ **A1****Answer: 30 tables and 40 chairs, £1900****4 marks****15 (6 marks)**Constraints: $2x + 3y \geq 30$, $3x + y \geq 15$, $x \leq 12$, $y \leq 10$, $x, y \geq 0$; minimise $C = 5x + 4y$ **B1 B1**Vertices of the feasible region: $(12, 2)$, $(12, 10)$, $(\frac{5}{3}, 10)$, $(\frac{15}{7}, \frac{60}{7})$ **M1 A1** $C = 5x + 4y$ at each vertex: $(12, 2)$: 68, $(12, 10)$: 100, $(\frac{5}{3}, 10)$: $\frac{145}{3}$, $(\frac{15}{7}, \frac{60}{7})$: 45**M1**Least is C at $(\frac{15}{7}, \frac{60}{7})$, where $C = 45$ **A1****Answer: $\frac{15}{7}$ kg of A and $\frac{60}{7}$ kg of B, £45****6 marks****16 (6 marks)****(a)** Vertices: $(3, 3)$, $(6, 2)$, $(2, 6)$ **M1 A1 A1****Answer: $(3, 3)$, $(6, 2)$, $(2, 6)$** **3 marks****(b)** Values: $(3, 3) \rightarrow 9$, $(6, 2) \rightarrow 14$, $(2, 6) \rightarrow 10$ **M1**

Maximum 14, minimum 9

A1 A1**Answer: max 14, min 9****3 marks****17 (7 marks)****(a)** Cutting: $2x + y \leq 100$; assembly: $x + y \leq 70$; finishing: $x + 3y \leq 120$ **B1 B1 B1****Answer: as stated****3 marks****(b)** Vertices of the feasible region: $(0, 0)$, $(50, 0)$, $(36, 28)$, $(0, 40)$ **M1 A1** $P = 4x + 3y$ at each vertex: $(0, 0)$: 0, $(50, 0)$: 200, $(36, 28)$: 228, $(0, 40)$: 120**M1**Greatest is P at $(36, 28)$, where $P = 228$ **A1****Answer: 36 standard, 28 deluxe, £228****4 marks****18 (6 marks)****(a)** Vertices of the feasible region: $(0, 0)$, $(6, 0)$, $(3, 6)$, $(0, 9)$ **M1 A1** $P = 1x + 1y$ at each vertex: $(0, 0)$: 0, $(6, 0)$: 6, $(3, 6)$: 9, $(0, 9)$: 9**M1**Greatest is P at $(3, 6)$, where $P = 9$ **A1****Answer: 9****4 marks****(b)** Vertices $(0, 0)$, $(6, 0)$, $(3, 6)$, $(0, 9)$: $P = 0, 18, 15, 9$ **M1**Maximum 18 at $(6, 0)$ **A1****Answer: 18 at $(6, 0)$** **2 marks****19 (5 marks)** P : 0, 24, 26, 12 so maximum 26 at $(6, 4)$ **M1 A1** $kx + 2y$ at $(6, 4) = 6k + 8$ must be $\geq 8k$ (at $(8, 0)$): $k \leq 4$ **M1**and ≥ 12 (at $(0, 6)$): $6k \geq 4$, $k \geq \frac{2}{3}$ **M1** $\frac{2}{3} \leq k \leq 4$ **A1****Answer: $\frac{2}{3} \leq k \leq 4$** **5 marks**

20 (5 marks)

(a) Intersection of $y = x + 3$ with $x + y \leq 12$: (4.5, 7.5); of $y = x - 2$ with $x + y = 12$: (7, 5); $y = x - 2$ meets $y = 0$ at (2, 0); $y = x + 3$ meets $x = 0$ at (0, 3)

Vertices: (0, 0), (2, 0), (7, 5), (4.5, 7.5), (0, 3)

Answer: (0,0),(2,0),(7,5),(4.5,7.5),(0,3)

M1 A1

3 marks

(b) Values: 0, 4, 29, 31.5, 9

M1

Maximum 31.5 at (4.5, 7.5)

A1

Answer: 31.5

2 marks

21 (8 marks)

(a) Vertices of the feasible region: (0, 0), (5, 0), $(\frac{18}{5}, \frac{28}{5})$, (0, 8)

M1 A1

$P = 5x + 4y$ at each vertex: (0, 0): 0, (5, 0): 25, $(\frac{18}{5}, \frac{28}{5})$: $\frac{202}{5}$, (0, 8): 32

M1

Greatest is P at $(\frac{18}{5}, \frac{28}{5})$, where $P = \frac{202}{5}$

A1

Answer: $\frac{202}{5}$ at $(\frac{18}{5}, \frac{28}{5})$

4 marks

(b) The vertex $(\frac{18}{5}, \frac{28}{5})$ is not an integer point; test integer points near it

M1

(3, 6): $6 + 18 = 24 \leq 24$ and $12 + 6 = 18 \leq 20$, $P = 39$; (4, 4): $P = 36$; (3, 5): $P = 35$; (2, 6): $P = 34$; (4, 5) is not feasible since $16 + 5 = 21 > 20$

M1 A1

Maximum $P = 39$ at (3, 6)

A1

Answer: 39 at (3, 6)

4 marks

22 (5 marks)

(a) Vertices: (0, 0), (10, 0), $(\frac{17}{2}, \frac{9}{2})$, (6, 7), (0, 10)

M1 A1 A1

Answer: (0, 0), (10, 0), $(\frac{17}{2}, \frac{9}{2})$, (6, 7), (0, 10)

3 marks

(b) $P = 5x + 6y$ at each vertex: (0, 0): 0, (10, 0): 50, $(\frac{17}{2}, \frac{9}{2})$: $\frac{139}{2}$, (6, 7): 72, (0, 10): 60

M1

Greatest is P at (6, 7), where $P = 72$

A1

Answer: 72 at (6, 7)

2 marks

23 (5 marks)

(a) Vertices: (0, 10), (12, 6), (10, 10), (0, 20)

M1 A1 A1

Answer: (0, 10), (12, 6), (10, 10), (0, 20)

3 marks

(b) $C = 3x + 2y$ at each vertex: (0, 10): 20, (12, 6): 48, (10, 10): 50, (0, 20): 40

M1

Least is C at (0, 10), where $C = 20$

A1

Answer: 20 at (0, 10)

2 marks

24 (4 marks)

(a) $x + y = 8$ and $2x + y = 12$ give $x = 4$, $y = 4$

M1

It satisfies both constraints with equality, so it is a vertex

A1

Answer: (4, 4)

2 marks

(b) $P = 5x + 2y$ at each vertex: (0, 0): 0, (6, 0): 30, (4, 4): 28, (0, 8): 16

M1

Greatest is P at (6, 0), where $P = 30$

A1

Answer: 30 at (6, 0)

2 marks

25 (6 marks)

(a) Vertices of the feasible region: (0, 8), $(\frac{44}{3}, 8)$, (10, 15), (0, 25)

M1 A1

$P = 7x + 5y$ at each vertex: (0, 8): 40, $(\frac{44}{3}, 8)$: $\frac{428}{3}$, (10, 15): 145, (0, 25): 125

M1

Greatest is P at (10, 15), where $P = 145$

A1

Answer: 10 of A, 15 of B, £145

4 marks

(b) $P = 4x + 5y$ at each vertex: (0, 8): 40, $(\frac{44}{3}, 8)$: $\frac{296}{3}$, (10, 15): 115, (0, 25): 125

M1

Greatest is P at (0, 25), where $P = 125$

A1

Answer: 125 at (0, 25)**2 marks****26 (5 marks)****(a)** Vertices: (30, 0), (40, 0), $(\frac{100}{3}, \frac{25}{3})$, (10, 20)**M1 A1 A1****Answer: (30, 0), (40, 0), $(\frac{100}{3}, \frac{25}{3})$, (10, 20)****3 marks****(b)** $C = 3x + 2y$ at each vertex: (30, 0): 90, (40, 0): 120, $(\frac{100}{3}, \frac{25}{3})$: $\frac{350}{3}$, (10, 20): 70**M1**Least is C at (10, 20), where $C = 70$ **A1****Answer: 70 at (10, 20)****2 marks****27 (11 marks)****(a)** Vertices: (0, 0), (10, 0), (9, 3), (8, 4), (0, 8)**M1 A1 A1****Answer: (0, 0), (10, 0), (9, 3), (8, 4), (0, 8)****3 marks****(b)** $k = 2$: values (0, 0): 0, (10, 0): 20, (9, 3): 21, (8, 4): 20, (0, 8): 8**M1 A1**

Maximum 21 at (9, 3)

A1**Answer: 21 at (9, 3)****3 marks****(c)** At (9, 3), $P = 9k + 3$. Compare with the neighbouring vertex (10, 0): $9k + 3 \geq 10k$ gives $k \leq 3$ **M1 A1**Compare with the neighbouring vertex (8, 4): $9k + 3 \geq 8k + 4$ gives $k \geq 1$ **M1 A1** $1 \leq k \leq 3$ (at $k = 1$ or $k = 3$ the maximum is shared with a neighbouring vertex)**A1****Answer: $1 \leq k \leq 3$** **5 marks****28 (9 marks)****(a)** Cutting: $2x + 3y \leq 120$; finishing: $x + 2y \leq 70$ **B1**

Vertices: (0, 0), (60, 0), (30, 20), (0, 35)

M1 A1**Answer: (0, 0), (60, 0), (30, 20), (0, 35)****3 marks****(b)** $P = 5x + 8y$ at each vertex: (0, 0): 0, (60, 0): 300, (30, 20): 310, (0, 35): 280**M1**Greatest is P at (30, 20), where $P = 310$ **A1****Answer: £310 at (30, 20)****2 marks****(c)** The edges meeting at (30, 20) have gradients $-\frac{2}{3}$ (cutting) and $-\frac{1}{2}$ (finishing)**B1**The objective line $ax + by$ has gradient $-\frac{a}{b}$; (30, 20) is optimal when $\frac{1}{2} \leq \frac{a}{b} \leq \frac{2}{3}$ **M1 A1**That is $\frac{1}{2} \leq a \div b \leq \frac{2}{3}$ **A1****Answer: $\frac{1}{2} \leq a/b \leq \frac{2}{3}$** **4 marks****29 (9 marks)****(a)** Vertices of the feasible region: (0, 0), (10, 0), (6, 4), (0, 7)**M1 A1** $P = 2x + 5y$ at each vertex: (0, 0): 0, (10, 0): 20, (6, 4): 32, (0, 7): 35**M1**Greatest is P at (0, 7), where $P = 35$ **A1****Answer: 35 at (0, 7)****4 marks****(b)** The objective function is a weighted sum ($2x + 5y$), not distance; the optimal vertex is where the objective line, moved away from the origin, last touches the region**E1****Answer: Use the objective line, not distance****1 mark****(c)** Neighbouring vertices of (6, 4): (0, 7) and (10, 0)**M1**Value at (6, 4) = $6c + 20 \geq 35$ (at (0, 7)) gives $c \geq 2.5$; $\geq 10c$ gives $c \leq 5$ **M1 A1** $2.5 \leq c \leq 5$ **A1****Answer: $2.5 \leq c \leq 5$** **4 marks****30 (9 marks)****(a)** Vertices: (0, 0), (4, 0), (3, 3), (0, 4)**M1 A1 A1**The three lines meet at (3, 3): $3 + 3 = 6$, $3 + 9 = 12$, $9 + 3 = 12$, so they are concurrent, and $x + y$ **E1** ≤ 6 is redundant for this region**Answer: Concurrent at (3, 3); $x + y \leq 6$ redundant****4 marks****(b)** P values at the vertices: (0, 0): 0, (4, 0): 4, (3, 3): 6, (0, 4): 4**M1**

Maximum 6 at (3, 3), and the objective line $x + y = 6$ coincides with the constraint $x + y = 6$; here only the point (3, 3) of it is in R

A1 E1

Answer: 6 at (3, 3)

3 marks

(c) Integer points in R with $x + y = 6$: only (3, 3) is on the line within the region, since (2, 4) gives $2 + 12 = 14 > 12$

M1 A1

Answer: (3, 3) only

2 marks

Optimisation with Constraints

1 (2 marks)

$3x + 2y + r = 18$, with $r \geq 0$

B1

r is the amount of the resource (18 units) that is unused

B1

Answer: $3x + 2y + r = 18$

2 marks

2 (1 mark)

$P - 5x - 4y = 0$

B1

Answer: $P - 5x - 4y = 0$

1 mark

3 (2 marks)

The y column

B1

It has the most negative entry (-7) in the objective row, so increasing y raises P fastest

E1

Answer: y column

2 marks

4 (2 marks)

Ratios: $12 \div 2 = 6$ and $20 \div 5 = 4$

M1

The smallest is 4, so the second row is the pivot row

A1

Answer: Second row

2 marks

5 (2 marks)

The entering variable increases until the first basic variable reaches zero

E1

Choosing any larger ratio would make a basic variable negative, which is infeasible

E1

Answer: Keeps all variables ≥ 0

2 marks

6 (2 marks)

4 slack variables (one per constraint)

B1

Columns: P, x, y, z , 4 slack = 8

B1

Answer: 4 slack variables; 8 columns

2 marks

7 (3 marks)

$3x + 2y + r = 36$, $x + 2y + s = 20$, $P - 5x - 4y = 0$

M1

Tableau (columns P, x, y, r, s , value; rows r, s, P):

$$\begin{pmatrix} 0 & 3 & 2 & 1 & 0 & 36 \\ 0 & 1 & 2 & 0 & 1 & 20 \\ 1 & -5 & -4 & 0 & 0 & 0 \end{pmatrix}$$

A1 A1

Answer: Tableau (columns P, x, y, r, s , value):

$$\begin{pmatrix} 0 & 3 & 2 & 1 & 0 & 36 \\ 0 & 1 & 2 & 0 & 1 & 20 \\ 1 & -5 & -4 & 0 & 0 & 0 \end{pmatrix}$$

3 marks

8 (4 marks)Pivot column x (-5); ratios $36 \div 3 = 12$ and $20 \div 1 = 20$; pivot row r, pivot 3

M1 A1

$$\text{New tableau: } \left(\begin{array}{cccccc} 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 12 \\ 0 & 0 & \frac{4}{3} & \frac{-1}{3} & 1 & 8 \\ 1 & 0 & \frac{-2}{3} & \frac{5}{3} & 0 & 60 \end{array} \right)$$

A1 A1

$$\text{Answer: New tableau: } \left(\begin{array}{cccccc} 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 12 \\ 0 & 0 & \frac{4}{3} & \frac{-1}{3} & 1 & 8 \\ 1 & 0 & \frac{-2}{3} & \frac{5}{3} & 0 & 60 \end{array} \right)$$

4 marks

9 (3 marks)

Pivot column y (most negative in the P row); ratios x: 18, s: 6; pivot row s

M1

$$\text{Tableau 2: } \left(\begin{array}{cccccc} 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & 8 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{3}{4} & 6 \\ 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 64 \end{array} \right)$$

A1

No negative entries in the P row, so optimal: $x = 8$, $y = 6$, $P = 64$

A1

Answer: $x = 8$, $y = 6$, $P = 64$

3 marks

10 (4 marks)

Pivot column y (most negative in the P row); ratios r: 10, s: 6, t: 16; pivot row s

M1 A1

$$\text{Tableau 1: } \left(\begin{array}{cccccc} 0 & \frac{2}{3} & 0 & 1 & \frac{-1}{3} & 0 & 4 \\ 0 & \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 6 \\ 0 & \frac{5}{3} & 0 & 0 & \frac{-1}{3} & 1 & 10 \\ 1 & \frac{-5}{3} & 0 & 0 & \frac{4}{3} & 0 & 24 \end{array} \right)$$

A1

The new tableau gives $x = 0$, $y = 6$, $P = 24$

B1

Answer: $x = 0$, $y = 6$, $P = 24$

4 marks

11 (3 marks) $x = 8$, $y = 6$, $P = 64$ from the value column of the basic variables and the objective row

M1 A1

 $r = s = 0$ (non-basic), so both constraints are binding

B1

Answer: $x = 8$, $y = 6$, $P = 64$; both binding

3 marks

12 (2 marks)

There is a negative entry (-2) in the y column of the objective row

M1

Not optimal; the next pivot column is y

A1

Answer: Not optimal; y column

2 marks

13 (3 marks)

$$\text{Tableau (columns P, x, y, r, s, t, value): } \left(\begin{array}{cccccc} 0 & 1 & 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 3 & 0 & 1 & 0 & 24 \\ 0 & 3 & 1 & 0 & 0 & 1 & 24 \\ 1 & -2 & -3 & 0 & 0 & 0 & 0 \end{array} \right)$$

M1 A1

The most negative objective entry is -3 (y column); ratios 12, 8, 24 give pivot row s, pivot 3

A1

Answer: Pivot on y in row s

3 marks

14 (8 marks)

Initial tableau (columns P, x, y, r, s, t, value; rows r, s, t, P):

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 2 & 0 & 1 & 0 & 16 \\ 0 & 3 & 1 & 0 & 0 & 1 & 30 \\ 1 & -6 & -5 & 0 & 0 & 0 & 0 \end{pmatrix}$$
M1 A1

Pivot column x (most negative in the P row); ratios r: 12, s: 16, t: 10; pivot row t

M1 A1

Tableau 1:

$$\begin{pmatrix} 0 & 0 & \frac{2}{3} & 1 & 0 & \frac{-1}{3} & 2 \\ 0 & 0 & \frac{5}{3} & 0 & 1 & \frac{-1}{3} & 6 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 10 \\ 1 & 0 & -3 & 0 & 0 & 2 & 60 \end{pmatrix}$$
A1

Pivot column y (most negative in the P row); ratios r: 3, s: $\frac{18}{5}$, x: 30; pivot row r

M1

Tableau 2:

$$\begin{pmatrix} 0 & 0 & 1 & \frac{3}{2} & 0 & \frac{-1}{2} & 3 \\ 0 & 0 & 0 & \frac{-5}{2} & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{-1}{2} & 0 & \frac{1}{2} & 9 \\ 1 & 0 & 0 & \frac{9}{2} & 0 & \frac{1}{2} & 69 \end{pmatrix}$$
A1

No negative entries in the P row, so optimal: y = 3, s = 1, x = 9, P = 69

A1

Answer: x = 9, y = 3, P = 69

8 marks**15 (8 marks)**

Initial tableau (columns P, x, y, r, s, t, value; rows r, s, t, P):

$$\begin{pmatrix} 0 & 4 & 3 & 1 & 0 & 0 & 240 \\ 0 & 1 & 1 & 0 & 1 & 0 & 70 \\ 0 & 0 & 1 & 0 & 0 & 1 & 60 \\ 1 & -30 & -25 & 0 & 0 & 0 & 0 \end{pmatrix}$$
M1 A1

Pivot column x (most negative in the P row); ratios r: 60, s: 70; pivot row r

M1 A1

Tableau 1:

$$\begin{pmatrix} 0 & 1 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 60 \\ 0 & 0 & \frac{1}{4} & \frac{-1}{4} & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 1 & 60 \\ 1 & 0 & \frac{-5}{2} & \frac{15}{2} & 0 & 0 & 1800 \end{pmatrix}$$
A1

Pivot column y (most negative in the P row); ratios x: 80, s: 40, t: 60; pivot row s

M1

Tableau 2:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & -3 & 0 & 30 \\ 0 & 0 & 1 & -1 & 4 & 0 & 40 \\ 0 & 0 & 0 & 1 & -4 & 1 & 20 \\ 1 & 0 & 0 & 5 & 10 & 0 & 1900 \end{pmatrix}$$
A1

No negative entries in the P row, so optimal: x = 30, y = 40, t = 20, P = 1900

A1

Answer: 30 tables, 40 chairs, profit £1900

8 marks

16 (10 marks)

(a) Initial tableau (columns P, x, y, r, s, value; rows r, s, P):
$$\begin{pmatrix} 0 & 3 & 2 & 1 & 0 & 60 \\ 0 & 1 & 1 & 0 & 1 & 25 \\ 1 & -7 & -5 & 0 & 0 & 0 \end{pmatrix}$$
 M1 A1

Pivot column x (most negative in the P row); ratios r: 20, s: 25; pivot row r M1 A1

Tableau 1:
$$\begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 20 \\ 0 & 0 & \frac{1}{3} & \frac{-1}{3} & 1 & 5 \\ 1 & 0 & \frac{-1}{3} & \frac{7}{3} & 0 & 140 \end{pmatrix}$$
 A1

Pivot column y (most negative in the P row); ratios x: 30, s: 15; pivot row s M1

Tableau 2:
$$\begin{pmatrix} 0 & 1 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & -1 & 3 & 15 \\ 1 & 0 & 0 & 2 & 1 & 145 \end{pmatrix}$$
 A1

No negative entries in the P row, so optimal: $x = 10$, $y = 15$, $P = 145$ A1

Answer: $x = 10$, $y = 15$, $P = 145$ 8 marks

(b) Slack variables: $r = 0$, $s = 0$ M1

A slack of 0 means that the constraint is binding: the resource is fully used A1

Answer: $r = 0$, $s = 0$ 2 marks

17 (5 marks)

(a) 3 constraints, so slack variables r, s, t: $x + y + z + r = 20$, $2x + y + s = 30$, $y + 3z + t = 36$, $P = 5x - 4y - 2z = 0$ M1

Tableau (columns P, x, y, z, r, s, t, value; rows r, s, t, P):
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 36 \\ 1 & -5 & -4 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$
 A1 A1

Answer: Tableau (columns P, x, y, z, r, s, t, value):
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 36 \\ 1 & -5 & -4 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$
 3 marks

(b) Most negative entry -5 in the x column; ratios $20 \div 1 = 20$ and $30 \div 2 = 15$ (third row has 0) M1

Pivot row s, pivot element 2 A1

Answer: Pivot on x in row s 2 marks

18 (3 marks)

Each iteration swaps one basic variable for another, which corresponds to moving to an adjacent vertex E1

The objective function increases (or stays the same) at each move E1

No negative entries means that increasing any non-basic variable would reduce P, so no adjacent vertex is better, and P is maximal E1

Answer: Adjacent vertices; no improvement possible 3 marks

19 (7 marks)

(a) Basic variables y, s, z take the values in the value column; non-basic variables are 0 M1

$x = 0$, $y = 9$, $z = 12$, $P = 81$ A1

Answer: $x = 0$, $y = 9$, $z = 12$, $P = 81$ 2 marks

(b) r and t are non-basic, so $r = t = 0$; the first and third constraints are fully used M1

$s = 28$ is basic, so 28 units of the second resource (limit 40) are unused A1

Answer: Constraints 1 and 3 are binding; 28 units of resource 2 unused 2 marks

(c) The entry under r in the objective row is $\frac{12}{5}$, so P rises by $\frac{12}{5}$ per extra unit of the first resource M1 A1

Increase for 5 extra units $= 5 \times \frac{12}{5} = 12$ A1

Answer: $\frac{12}{5}$ per unit; 12

3 marks

20 (8 marks)

Initial tableau (columns P, x, y, r, s, value; rows r, s, P):

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 14 \\ 0 & 3 & 1 & 0 & 1 & 18 \\ 1 & -4 & -5 & 0 & 0 & 0 \end{pmatrix}$$

M1 A1

Pivot column y (most negative in the P row); ratios r: 7, s: 18; pivot row r

M1 A1

Tableau 1:

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 7 \\ 0 & \frac{5}{2} & 0 & \frac{-1}{2} & 1 & 11 \\ 1 & \frac{-3}{2} & 0 & \frac{5}{2} & 0 & 35 \end{pmatrix}$$

A1

Pivot column x (most negative in the P row); ratios y: 14, s: $\frac{22}{5}$; pivot row s

M1

Tableau 2:

$$\begin{pmatrix} 0 & 0 & 1 & \frac{3}{5} & \frac{-1}{5} & \frac{24}{5} \\ 0 & 1 & 0 & \frac{-1}{5} & \frac{2}{5} & \frac{22}{5} \\ 1 & 0 & 0 & \frac{11}{5} & \frac{3}{5} & \frac{208}{5} \end{pmatrix}$$

A1

No negative entries in the P row, so optimal: $y = \frac{24}{5}$, $x = \frac{22}{5}$, $P = \frac{208}{5}$

A1

Answer: $x = \frac{22}{5}$, $y = \frac{24}{5}$, $P = \frac{208}{5}$

8 marks

21 (10 marks)

Initial tableau (columns P, x, y, z, r, s, t, value; rows r, s, t, P):

M1 A1

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 36 \\ 1 & -5 & -4 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot column x (most negative in the P row); ratios r: 20, s: 15; pivot row s

M1 A1

Tableau 1:

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 1 & 1 & \frac{-1}{2} & 0 & 5 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 15 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 36 \\ 1 & 0 & \frac{-3}{2} & -2 & 0 & \frac{5}{2} & 0 & 75 \end{pmatrix}$$

A1

Pivot column z (most negative in the P row); ratios r: 5, t: 12; pivot row r

M1

Tableau 2:

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 1 & 1 & \frac{-1}{2} & 0 & 5 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 15 \\ 0 & 0 & \frac{-1}{2} & 0 & -3 & \frac{3}{2} & 1 & 21 \\ 1 & 0 & \frac{-1}{2} & 0 & 2 & \frac{3}{2} & 0 & 85 \end{pmatrix}$$

A1

Pivot column y (most negative in the P row); ratios z: 10, x: 30; pivot row z

M1

Tableau 3:

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 2 & -1 & 0 & 10 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 26 \\ 1 & 0 & 0 & 1 & 3 & 1 & 0 & 90 \end{pmatrix}$$

A1

No negative entries in the P row, so optimal: $y = 10$, $x = 10$, $t = 26$, $P = 90$

A1

Answer: $x = 10$, $y = 10$, $z = 0$, $P = 90$

10 marks

22 (8 marks)

Initial tableau (columns P, x, y, z, r, s, t, value; rows r, s, t, P):

M1 A1

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 0 & 0 & 30 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 1 & 3 & 0 & 0 & 1 & 45 \\ 1 & -2 & -5 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot column y (most negative in the P row); ratios r: 15, t: 45; pivot row r

M1 A1

Tableau 1:
$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 15 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & \frac{1}{2} & 0 & \frac{5}{2} & \frac{-1}{2} & 0 & 1 & 30 \\ 1 & \frac{1}{2} & 0 & \frac{-1}{2} & \frac{5}{2} & 0 & 0 & 75 \end{pmatrix}$$

A1

Pivot column z (most negative in the P row); ratios y: 30, s: 40, t: 12; pivot row t

M1

Tableau 2:
$$\begin{pmatrix} 0 & \frac{2}{5} & 1 & 0 & \frac{3}{5} & 0 & \frac{-1}{5} & 9 \\ 0 & \frac{9}{5} & 0 & 0 & \frac{1}{5} & 1 & \frac{-2}{5} & 28 \\ 0 & \frac{1}{5} & 0 & 1 & \frac{-1}{5} & 0 & \frac{2}{5} & 12 \\ 1 & \frac{3}{5} & 0 & 0 & \frac{12}{5} & 0 & \frac{1}{5} & 81 \end{pmatrix}$$

A1

No negative entries in the P row, so optimal: y = 9, s = 28, z = 12, P = 81

A1

Answer: x = 0, y = 9, z = 12, P = 81

8 marks

23 (12 marks)

(a) Initial tableau (columns P, x, y, r, s, value; rows r, s, P):
$$\begin{pmatrix} 0 & 2 & 3 & 1 & 0 & 24 \\ 0 & 4 & 1 & 0 & 1 & 20 \\ 1 & -5 & -4 & 0 & 0 & 0 \end{pmatrix}$$

M1 A1

Pivot column x (most negative in the P row); ratios r: 12, s: 5; pivot row s

M1 A1

Tableau 1:
$$\begin{pmatrix} 0 & 0 & \frac{5}{2} & 1 & \frac{-1}{2} & 14 \\ 0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 5 \\ 1 & 0 & \frac{-11}{4} & 0 & \frac{5}{4} & 25 \end{pmatrix}$$

A1

Pivot column y (most negative in the P row); ratios r: $\frac{28}{5}$, x: 20; pivot row r

M1

Tableau 2:
$$\begin{pmatrix} 0 & 0 & 1 & \frac{2}{5} & \frac{-1}{5} & \frac{28}{5} \\ 0 & 1 & 0 & \frac{-1}{10} & \frac{3}{10} & \frac{18}{5} \\ 1 & 0 & 0 & \frac{11}{10} & \frac{7}{10} & \frac{202}{5} \end{pmatrix}$$

A1

No negative entries in the P row, so optimal: y = $\frac{28}{5}$, x = $\frac{18}{5}$, P = $\frac{202}{5}$

A1

Answer: x = $\frac{18}{5}$, y = $\frac{28}{5}$, P = $\frac{202}{5}$

8 marks

(b) Rounding to (4, 6) violates $4x + y \leq 20$? $16 + 6 = 22 > 20$, so it is infeasible; rounding down (3, 5) gives P = 35 which is not best

E1

Check nearby feasible integer points: (3, 6): $6 + 18 = 24$, $12 + 6 = 18$, P = 39; (4, 4): P = 36

M1 A1

Optimal integer solution (3, 6) with P = 39

A1

Answer: (3, 6), P = 39

4 marks

24 (5 marks)**(a)** Intersection of $3x + 2y = 37$ and $x + 2y = 20$: $2x = 17$, $x = 8.5$, $y = 5.75$ **M1 A1**

$$P = 5 \times 8.5 + 4 \times 5.75 = \frac{131}{2}$$

A1**Answer:** $\frac{131}{2}$ **3 marks****(b)** Increase = $\frac{131}{2} - 64 = \frac{3}{2}$ **M1**Valid while the optimal vertex stays at the intersection of the two constraint lines, that is for $20 \leq b \leq 60$ **A1****Answer:** $\frac{3}{2}$ per unit**2 marks****25 (5 marks)****(a)** Initial tableau (columns P, x, y, r, value): $\begin{pmatrix} 0 & 1 & -1 & 1 & 3 \\ 1 & -2 & -1 & 0 & 0 \end{pmatrix}$; pivot on x (-2), ratio $3 \div 1 = 3$ **M1 A1**After the iteration: $\begin{pmatrix} 0 & 1 & -1 & 1 & 3 \\ 1 & 0 & -3 & 2 & 6 \end{pmatrix}$; the y column of the objective row is -3 but the y column entry in the constraint row is -1 , which is not positive, so no ratio exists**M1 A1****Answer: Unbounded****4 marks****(b)** The feasible region $x - y \leq 3$, $x, y \geq 0$ extends to infinity along $y = x - 3$ and increasing y, in which direction $2x + y$ grows without limit**E1****Answer: Region is unbounded in a direction of increasing P****1 mark****26 (8 marks)****(a)** Objective row: $P + x - 3y = 0$ **B1**Initial tableau (columns P, x, y, r, s, value; rows r, s, P): $\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 12 \\ 0 & 3 & 1 & 0 & 1 & 15 \\ 1 & 1 & -3 & 0 & 0 & 0 \end{pmatrix}$ **M1 A1****Answer: $P + x - 3y = 0$; tableau as in the solution****3 marks****(b)** Pivot column y (most negative in the P row); ratios r: 6, s: 15; pivot row r**M1 A1**Tableau 1: $\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 6 \\ 0 & \frac{5}{2} & 0 & \frac{-1}{2} & 1 & 9 \\ 1 & \frac{5}{2} & 0 & \frac{3}{2} & 0 & 18 \end{pmatrix}$ **A1**No negative entries in the P row, so optimal: $y = 6$, $s = 9$, $P = 18$ **A1****Answer: $x = 0$, $y = 6$, $P = 18$** **4 marks****(c)** $C = -P = -18$, at $x = 0$, $y = 6$ **B1****Answer: $C = -18$ at $(0, 6)$** **1 mark**

27 (9 marks)

Initial tableau (columns P, x, y, z, r, s, t, value; rows r, s, t, P):

M1 A1

$$\begin{pmatrix} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ 1 & -5 & -4 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot column x (most negative in the P row); ratios r: $\frac{5}{2}$, s: $\frac{11}{4}$, t: $\frac{8}{3}$; pivot row r**M1 A1**

Tableau 1:
$$\begin{pmatrix} 0 & 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{-3}{2} & 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{7}{2} & \frac{-1}{2} & \frac{5}{2} & 0 & 0 & \frac{25}{2} \end{pmatrix}$$

A1

Pivot column z (most negative in the P row); ratios x: 5, t: 1; pivot row t

M1

Tableau 2:
$$\begin{pmatrix} 0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 & 1 & 13 \end{pmatrix}$$

A1

No negative entries in the P row, so optimal: x = 2, s = 1, z = 1, P = 13

A1

Slack variables at the optimum: r = 0, s = 1, t = 0; a zero slack means that constraint is fully used

A1**Answer: x = 2, y = 0, z = 1, P = 13****9 marks**

28 (10 marks)

Initial tableau (columns P, x, y, z, r, s, t, value; rows r, s, t, P):

M1 A1

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 3 & 0 & 1 & 0 & 5 \\ 0 & 2 & 1 & 3 & 0 & 0 & 1 & 7 \\ 1 & -3 & -2 & -4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot column z (most negative in the P row); ratios r: 2, s: $\frac{5}{3}$, t: $\frac{7}{3}$; pivot row s

M1 A1

Tableau 1:

$$\begin{pmatrix} 0 & \frac{-1}{3} & 1 & 0 & 1 & \frac{-2}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 2 \\ 1 & \frac{-1}{3} & -2 & 0 & 0 & \frac{4}{3} & 0 & \frac{20}{3} \end{pmatrix}$$

A1

Pivot column y (most negative in the P row); ratios r: $\frac{2}{3}$, t: 2; pivot row r

M1

Tableau 2:

$$\begin{pmatrix} 0 & \frac{-1}{3} & 1 & 0 & 1 & \frac{-2}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{5}{3} \\ 0 & \frac{1}{3} & 0 & 0 & -1 & \frac{-1}{3} & 1 & \frac{4}{3} \\ 1 & -1 & 0 & 0 & 2 & 0 & 0 & 8 \end{pmatrix}$$

A1

Pivot column x (most negative in the P row); ratios z: $\frac{5}{2}$, t: 4; pivot row z

M1

Tableau 3:

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{-1}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & -1 & \frac{-1}{2} & 1 & \frac{1}{2} \\ 1 & 0 & 0 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & \frac{21}{2} \end{pmatrix}$$

A1

No negative entries in the P row, so optimal: $y = \frac{3}{2}$, $x = \frac{5}{2}$, $t = \frac{1}{2}$, $P = \frac{21}{2}$

A1

Answer: $x = \frac{5}{2}$, $y = \frac{3}{2}$, $z = 0$, $P = \frac{21}{2}$

10 marks

29 (7 marks)

(a) Neighbouring vertices of (8, 6): (12, 0) and (0, 10)

B1

At (8, 6): $8p + 24$; at (12, 0): $12p$ so $8p + 24 \geq 12p$ gives $p \leq 6$

M1 A1

At (0, 10): 40 so $8p + 24 \geq 40$ gives $p \geq 2$

M1

 $2 \leq p \leq 6$

A1

Answer: $2 \leq p \leq 6$

5 marks

(b) At $p = 6$ the objective is equal at (8, 6) and (12, 0), so every point on the edge between them is optimal (alternative optima)

E1

Similarly $p = 2$ gives equality with (0, 10)

E1

Answer: Alternative optimal solutions

2 marks

30 (6 marks)(a) Row operation: new row = row $- \frac{a}{p} \times$ pivot row

M1

Value becomes $v - \frac{a u}{p}$

A1

Answer: $v - \frac{a u}{p}$

2 marks

(b) If $a \leq 0$ then $v - \frac{a u}{p} \geq v \geq 0$ since $u \geq 0$ and $p > 0$

M1 A1

If $a > 0$ the ratio test gives $\frac{u}{p} \leq \frac{v}{a}$, so $a\frac{u}{p} \leq v$ and $v - \frac{a u}{p} \geq 0$

M1 A1**Answer: Non-negative****4 marks**