

Critical Path Analysis

- 1 In a critical path analysis, state what is meant by the 'total float' of an activity, and give the formula for calculating it. [2]
- 2 An activity has duration 6, earliest start time 9 and latest finish time 20. Find its total float and say whether it is critical. [2]
- 3 Activities A (duration 4) and B (duration 7) can start at the same time. Activity C (duration 3) can only start when both A and B are finished. Find the earliest start time of C and the minimum time to complete all three. [2]
- 4 A project has three activities done one after another with durations 5, 8 and 4 days. Find the minimum completion time and the total float of the middle activity. [2]
- 5 In a project the critical path is the longest path through the activity network. Explain why this longest path gives the minimum completion time. [2]
- 6 The total floats of activities in a project are A 0, B 3, C 0, D 5, E 0. State which activities are critical and explain how you know. [2]
- 7 A project is made up of the activities shown in the table. Carry out a forward pass to find the earliest start time of each activity and the minimum completion time of the project.

Activity	Duration	Immediately preceding
A	4	-
B	6	-
C	5	A
D	3	A, B
E	7	B
F	4	C, D
G	5	D, E
H	3	F, G

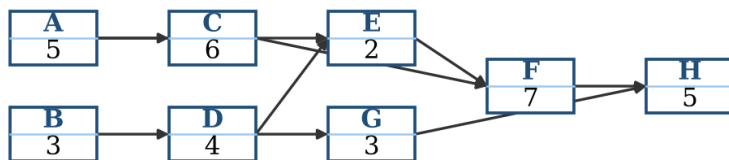
[2]

- 8 For the project in the table, carry out a forward pass and a backward pass, and use them to find the critical activities.

Activity	Duration	Immediately preceding
A	6	-
B	4	-
C	3	A
D	5	A
E	8	B
F	2	C, D
G	6	D, E
H	4	F, G

[4]

- 9 The activity network shows the activities of a project, with the duration of each below its name. Find the earliest start and latest start of each activity, and the critical activities.



[4]

- 10 Use a forward pass to find the minimum completion time of the project in the table, and a backward pass to find the latest start time of each activity.

Activity	Duration	Immediately preceding
A	2	-
B	3	A
C	4	A
D	5	B
E	6	B, C
F	3	D
G	4	E
H	2	F, G

[3]

- 11 A project has minimum completion time 24 days. For the activities shown, total floats are found from ES, duration and LF.
- (a) Activity P: ES 3, duration 5, LF 14. Find its total float. [2]
- (b) Activity Q: ES 7, duration 9, LF 16. Find its total float and state whether it is critical. [2]

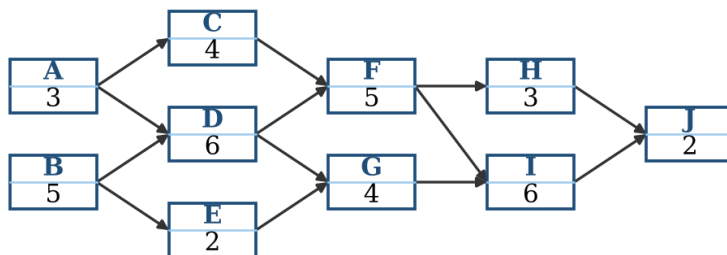
- 12 Activity H in a project has earliest start 14, duration 3 and latest finish 22. Activity G, which must be finished before H can start, has earliest finish 14. G is delayed by 4 days. Does this delay the project? Explain. [3]

- 13 Use a forward pass on the precedence table to find the minimum project time, and state which activities have the earliest start time 11.

Activity	Duration	Immediately preceding
A	3	-
B	5	-
C	4	A
D	6	A, B
E	2	B
F	5	C, D
G	4	D, E
H	3	F
I	6	F, G
J	2	H, I

[3]

- 14** The activity network shows a project. The number below each activity is its duration in days.



- (a) Carry out a forward pass and a backward pass to find the early and late times for each activity. [3]
- (b) State the critical activities and the total float of each non-critical activity. [2]

- 15** The table gives the activities of a project.

Activity	Duration	Immediately preceding
A	4	-
B	6	-
C	5	A
D	3	A, B
E	7	B
F	4	C, D
G	5	D, E
H	3	F, G

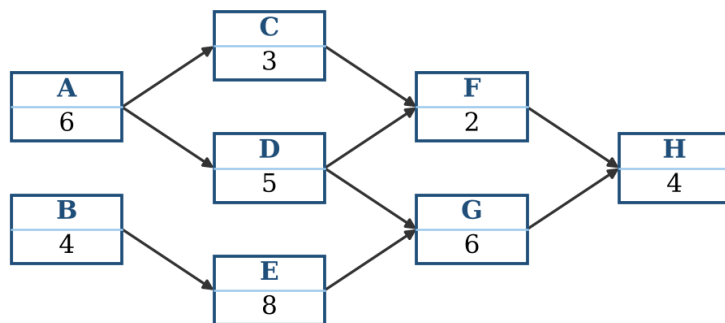
- (a) Find the minimum project time and the critical path. [3]
- (b) Activity D is delayed by 2 days. Explain whether the project completion time is affected. [2]

- 16** The project in the table has activities with the durations shown.

Activity	Duration	Immediately preceding
A	5	-
B	3	-
C	6	A
D	4	B
E	2	A, D
F	7	C, E
G	3	D
H	5	F, G

- (a) Find the minimum completion time and the critical path. [2]
- (b) Activity D now takes 9 days instead of 4. Find the new minimum completion time and critical path. [3]

- 17** Find the total float of every activity in the project shown and list the critical activities. Show your forward and backward passes.



[5]

- 18** For the project in the table, a student finds the total floats B 0, C 1, D 2, E 0, F 3, G 0.

Activity	Duration	Immediately preceding
A	2	-
B	3	A
C	4	A
D	5	B
E	6	B, C
F	3	D
G	4	E
H	2	F, G

- (a) Check the student's floats by finding the early and late times of the activities.
- (b) Identify any errors in the student's floats.

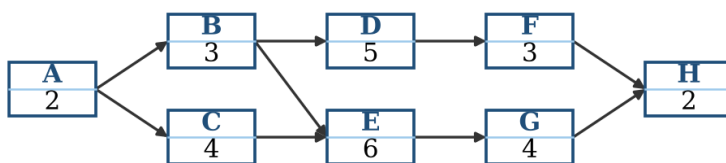
[4]

[2]

- 19** In a project, activity X has ES 8, duration 5 and LF 20, and activity Y, which must follow X, has ES 13, duration 4 and LF 25. Find the total float of each and of the path X then Y, and explain what the sharing of float means.

[5]

- 20** Find the minimum completion time and the critical path for the project shown, and find the latest time activity F can finish without delaying the project.



[5]

- 21** The project shown has the activities in the table.

Activity	Duration	Immediately preceding
A	3	-
B	5	-
C	4	A
D	6	A, B
E	2	B
F	5	C, D
G	4	D, E
H	3	F
I	6	F, G
J	2	H, I

- (a) Find the minimum project time and the critical path. [3]
- (b) By how many days can the project be shortened by reducing activity D from 6 days to 4 days? Find the new critical path(s). [3]
- (c) Explain why reducing the duration of an activity that is not critical never shortens the project. [1]

- 22** In the project in the table, the duration of activity D is x days.

Activity	Duration	Immediately preceding
A	4	-
B	6	-
C	5	A
D	x	A, B
E	7	B
F	4	C, D
G	5	D, E
H	3	F, G

- (a) Show that D is not critical when $x = 3$, and find the minimum completion time in that case. [2]
- (b) Find the smallest integer value of x for which D becomes critical. [2]
- (c) Find the minimum project time when $x = 10$. [2]

- 23** The project shown is planned to take the minimum time. During the project, activity G is delayed so that it takes 14 days instead of 3.

Activity	Duration	Immediately preceding
A	5	-
B	3	-
C	6	A
D	4	B
E	2	A, D
F	7	C, E
G	3	D
H	5	F, G

- (a) Find the original minimum time and the total float on G. [3]
 (b) Find the new minimum completion time with G = 14 days, and the delay to the project. [3]

- 24** The project in the table (activities A to H) can be shortened by paying to reduce the duration of activities. Reducing activity E by 1 day costs £300, G by 1 day costs £200, B by 1 day costs £250 and F by 1 day costs £100 (no activity may be reduced by more than 2 days).

Activity	Duration	Immediately preceding
A	4	-
B	6	-
C	5	A
D	3	A, B
E	7	B
F	4	C, D
G	5	D, E
H	3	F, G

- (a) Find the minimum project time and the critical activities. [3]
 (b) Explain why it is not worth reducing F, and find the cheapest way to reduce the project time by 2 days. [4]

- 25** A student carries out the forward pass for the project in the table and obtains earliest start times A 0, B 0, C 6, D 6, E 4, F 9, G 11, H 17. The student then states that the minimum project time is 21.

Activity	Duration	Immediately preceding
A	6	-
B	4	-
C	3	A
D	5	A
E	8	B
F	2	C, D
G	6	D, E
H	4	F, G

- (a) Find the earliest start times correctly. [2]
 (b) Identify the errors in the student's work and state the correct minimum project time. [3]

- 26** In the project in the table the duration of activity E is x days (x a positive integer).

Activity	Duration	Immediately preceding
A	3	-
B	5	-
C	4	A
D	6	A, B
E	x	B
F	5	C, D
G	4	D, E
H	3	F
I	6	F, G
J	2	H, I

- (a) Show that when $x = 5$, activity E is not critical and find the minimum time. [2]
 (b) Find the range of values of x for which the project time is 24. [2]
 (c) Find the minimum project time in terms of x when $x > 7$, and find the smallest x for which E and G are both critical. [3]
 (d) Find the total float on activity F as a function of x for $x \geq 7$. [2]

- 27** Activities P, Q, R must be done in that order on a path through a project; P and R are critical but Q is not, with total float f .

- (a) Explain why this is impossible if the path P, Q, R is the only path from P to R. [2]
 (b) Suppose instead that there is another path P, S, R in which S has duration 4 and Q has duration 7. Show that Q is critical and S has float 3, given P, Q, R, S have no other connections. [2]
 (c) If S and Q have floats 3 and 0 and S is delayed by 5, find the delay to the project. [2]

- 28** The project in the table is under way. A further activity K of duration 8 days is added, which can start when D is finished and must be finished before J can start.

Activity	Duration	Immediately preceding
A	3	-
B	5	-
C	4	A
D	6	A, B
E	2	B
F	5	C, D
G	4	D, E
H	3	F
I	6	F, G
J	2	H, I

- (a) Find the minimum project time and critical path before K is added. [3]
 (b) Add K and find the new minimum project time and critical path. [3]
 (c) Find the greatest duration of K for which the project time is unchanged. [2]

- 29** The project in the table has two critical paths.

Activity	Duration	Immediately preceding
A	7	-
B	7	-
C	6	A
D	2	A
E	6	B
F	3	C, D, E
G	5	F

- (a) Find the minimum completion time and both critical paths. [4]
 (b) The management wishes to reduce the project time by 1 day by shortening one activity by 1 day. Which activities will achieve this, and why can no other single activity? [3]

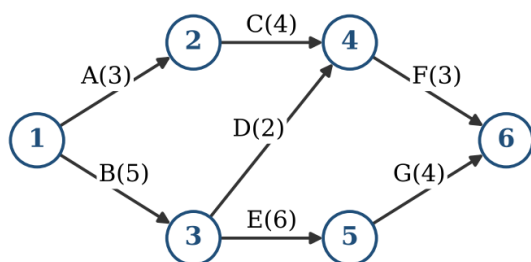
- 30** The total float of an activity X is $LS - ES$. Its free float is the amount by which X can be delayed without affecting the earliest start of any following activity.

Activity	Duration	Immediately preceding
A	3	-
B	5	-
C	4	A
D	6	A, B
E	2	B
F	5	C, D
G	4	D, E
H	3	F
I	6	F, G
J	2	H, I

- (a) For the project in the table, find the free float of each of A, C, E in terms of the earliest start times of their successors. [4]
- (b) Explain why the free float of an activity can never exceed its total float. [2]

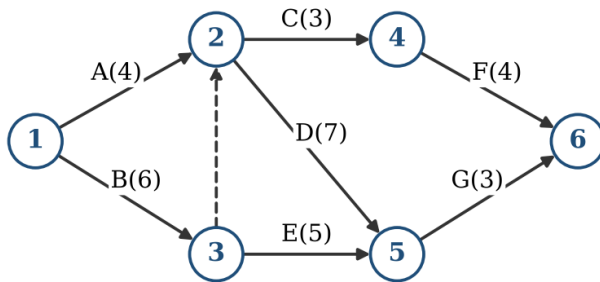
Activity Networks

- 1** In an activity network drawn with activities on arcs, state the purpose of a dummy activity. [2]
- 2** In an activity network the early time of event 4 is 11 and the late time is 14. An activity of duration 5 leaves event 4 and goes to event 7, with early time 16 and late time 19. Find the total float of that activity. [2]
- 3** Activity P (duration 5) and activity Q (duration 7) both start at event 1. P ends at event 2 and Q ends at event 3. Explain how to show that activity R, which starts at event 4, can begin only after both P and Q are finished. [2]
- 4** In the activity network shown, read off the activities that must be completed before F can start.



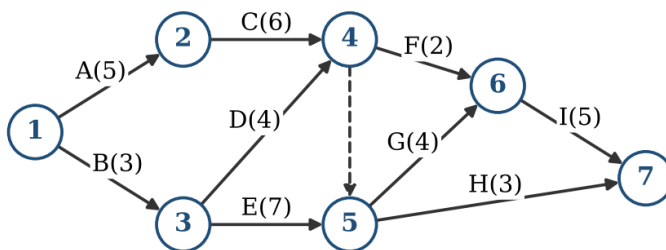
- 5** A cascade (Gantt) chart shows an activity as a bar from its earliest start to its earliest finish. State what is shown by a hatched or thin section after the bar. [2]
- 6** A project has total duration of all activities 52 hours and the minimum project time is 13 hours. Find the minimum number of workers needed for completion in the minimum time, if each activity needs one worker. [2]

- 7 Carry out a forward pass and a backward pass on the activity network shown, writing down the early and late times for each event. State the minimum project time.



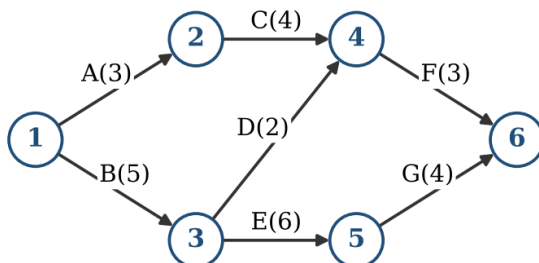
[5]

- 8 For the activity network shown, find the early and late times of every event and list the critical activities.



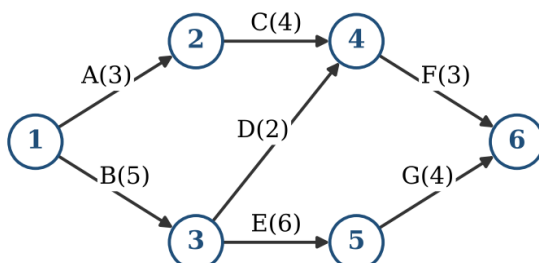
[5]

- 9 Find the minimum completion time for the project shown and the total float of activity C.



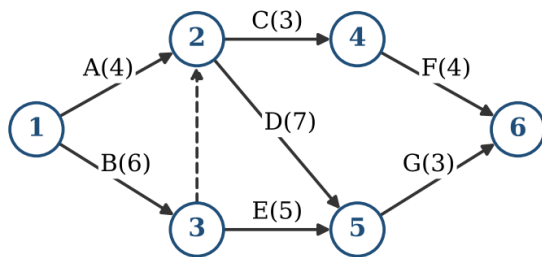
[5]

- 10 The activity network shown for a project is drawn with activities on arcs.



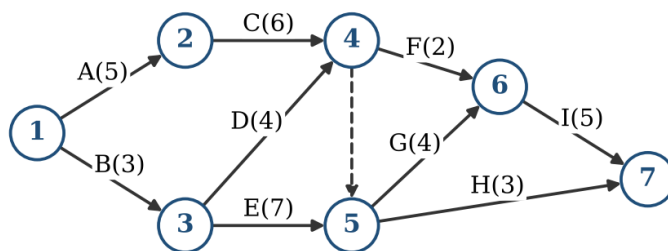
- (a) Complete a precedence table for the activities. [2]
 (b) State the earliest time at which the project can finish. [1]

- 11** Draw a cascade (Gantt) chart for the project of the network shown (assume each activity starts at its earliest time). State, from the chart, the number of activities in progress at time 5.



[3]

- 12** In the activity network shown, the late event time at event 3 is 4 and the early event time is 3. Explain what this tells you about activities B and D, and find the total float of D.



[3]

- 13** Draw (or describe) an activity-on-arc network for these activities, using a dummy if needed: A and B start the project; C follows A; D follows A and B.

[3]

- 14** The precedence table gives the activities of a project.

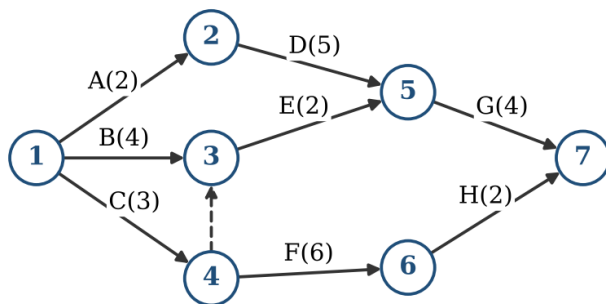
Activity	Duration	Immediately preceding
A	5	-
B	3	-
C	6	A
D	4	B
E	7	B
F	2	C, D
G	4	C, D, E
H	3	C, D, E
I	5	F, G

- (a) Explain why a dummy activity is needed when this project is drawn as an activity network with activities on arcs.
- (b) Draw the network, carry out forward and backward passes and find the minimum time.

[2]

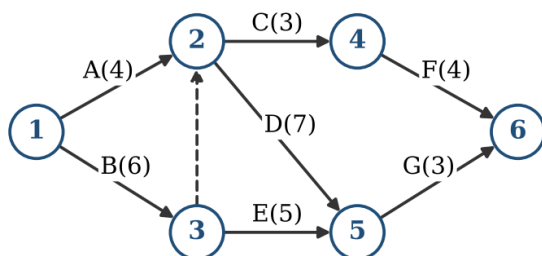
[5]

- 15** The activity network shown for a project has a dummy activity.



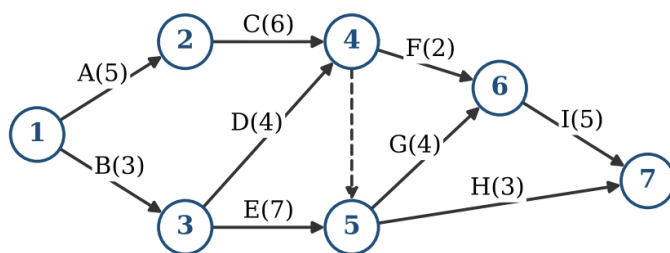
- (a) Write down the precedence table for the activities (the activities that must be completed immediately before each starts). [3]
 (b) Find the early and late event times and the minimum project time. [4]

- 16** Each activity of the project shown needs one worker. All activities start at their earliest times.



- (a) Draw a cascade chart for the project. [2]
 (b) Draw a resource histogram showing the number of workers needed at each time and state the greatest number needed. [3]

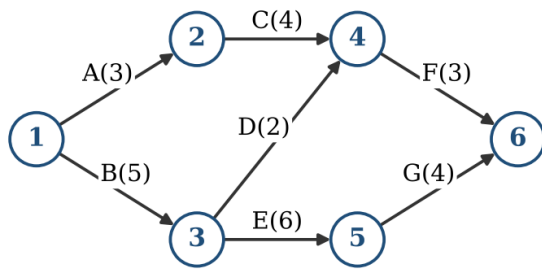
- 17** The project shown uses one worker for each activity.



- (a) Find the early start time of each activity. [2]
 (b) Find the number of workers needed at time 8, assuming earliest starts, and the least number of workers which could complete the project in the minimum time (a lower bound). [4]

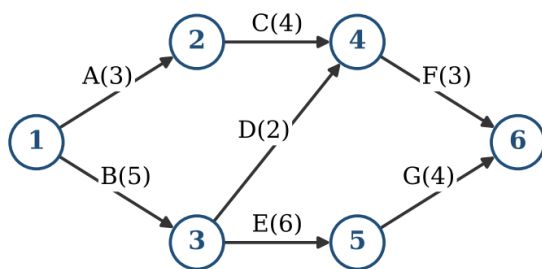
- 18** Explain, with reference to the cascade chart, how the float of a non-critical activity can be used to move the resource histogram to reduce the greatest number of workers. [2]

- 19** A student draws the activity network N for a project and obtains these early event times: 1: 0, 2: 3, 3: 5, 4: 7, 5: 9, 6: 13. The activities are as in the network shown.



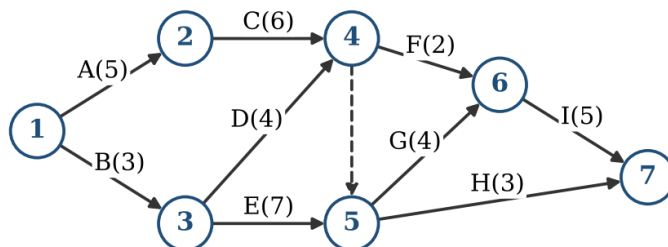
- (a) Find the correct early and late event times. [4]
 (b) Identify the student's error(s) and state the correct minimum time. [3]

- 20** The activity network shown has activity durations in days. Find the minimum project time, the critical activities and the total float of each other activity.



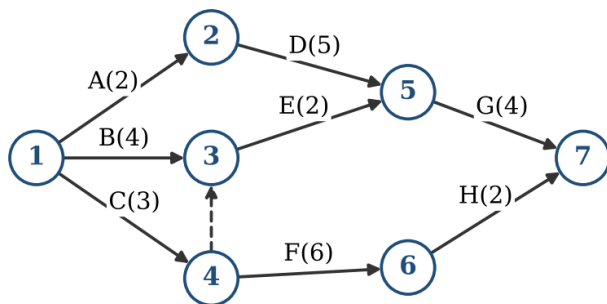
[6]

- 21** The project in the network shown is to be completed by a team of workers; each activity needs exactly one worker and cannot be interrupted.



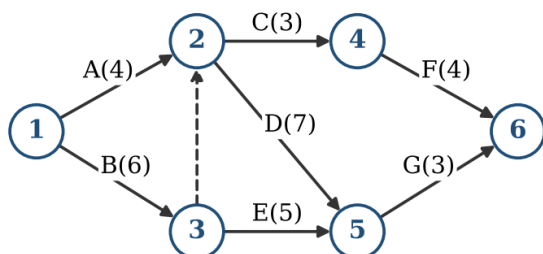
- (a) Find the minimum project time and the total of all the activity durations. [2]
 (b) Show that at least 2 workers are needed to complete the project in the minimum time. [2]
 (c) Show that two workers cannot complete the project in the minimum time. [4]

- 22** The project in the table can be modelled by the activity network shown. Each activity needs one worker.



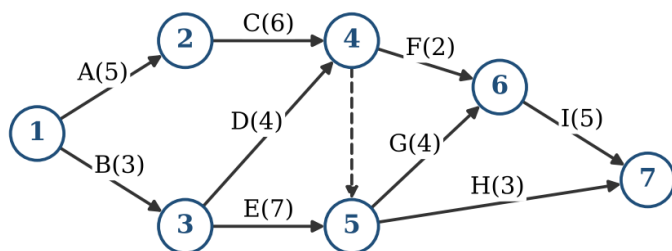
- (a) Find the minimum project time and the critical activities. [5]
 (b) Draw the resource histogram for earliest starts and state the greatest number of workers needed. [3]
 (c) Show that the project cannot be completed in the minimum time with only two workers. [3]

- 23** The project shown has the activity network in the diagram. A delay occurs: activity C is delayed so that it starts 6 days later than planned.



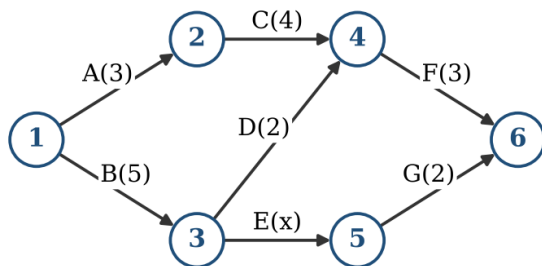
- (a) Find the total float of C and state the delay to the project. [2]
 (b) Find the new minimum project time. [1]

- 24** For the activity network shown, an engineer says that the late time at event 3 is 3, because the early time is 3.



- (a) Show by a backward pass that the late time at event 3 is 4. [2]
 (b) Explain why the early and late times at an event need not be equal. [1]
 (c) State whether B is critical and why. [2]

- 25 In the activity network shown the duration of activity E is x days, $x > 0$.



- (a) Find the minimum project time when $x = 2$, and the total float of E. [2]
 (b) Find the range of values of x for which E is critical. [2]
 (c) Find the project time when $x = 8$. [1]

- 26 For the project in the table (each activity needs one worker), find the shortest time in which the project can be completed if only two workers are available.

Activity	Duration	Immediately preceding
A	4	-
B	3	-
C	5	A
D	6	A, B
E	2	B
F	4	C, D
G	3	D, E
H	5	F, G

- (a) Show that the minimum time with two workers is at least 19. [2]
 (b) Give a schedule for two workers that finishes in 19 (or the best you can find) and state its finish time. [3]

- 27 A project has the activities in the table, each needing one worker.

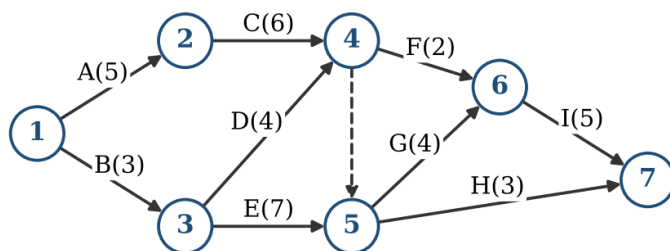
Activity	Duration	Immediately preceding
A	4	-
B	3	-
C	5	A
D	6	A, B
E	2	B
F	4	C, D
G	3	D, E
H	5	F, G

- (a) Find the minimum project time and the critical path. [3]
 (b) Find the lower bound for the number of workers needed to finish in the minimum time. [2]
 (c) Show by giving a schedule (start time and worker for each activity) that 2 workers can complete the project in the minimum time. [4]

- 28** In a project network each activity needs one worker. Let T be the minimum project time (with unlimited workers) and W the total of the activity durations.

- (a) Prove that at least $\lceil W \div T \rceil$ workers are needed to complete the project in time T . [3]
 (b) Explain why $\lceil W \div T \rceil$ workers may still not be enough. [2]

- 29** The project in the network shown uses one worker per activity.



- (a) Draw resource histograms for (i) every activity at its earliest start and (ii) every activity at its latest start. Give the greatest number of workers in each. [5]
 (b) Explain why moving a non-critical activity within its float can change the greatest number of workers needed, and state whether the greatest number can be reduced to 2 without delaying the project. [4]

- 30** A project has the activities shown in the table.

Activity	Duration	Immediately preceding
A	3	-
B	4	-
C	2	A
D	5	A, B
E	3	B
F	4	C, D
G	2	D, E
H	3	F, G

- (a) Explain why dummy activities are needed to draw the activity network with activities on arcs. [2]
 (b) Find the minimum project time and the critical path. [3]
 (c) Find the total float of every non-critical activity. [2]

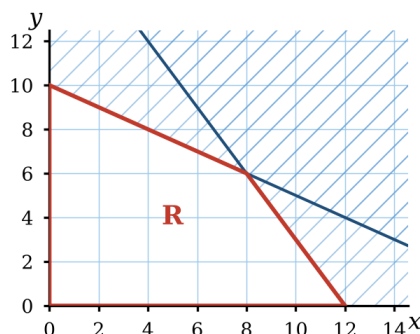
Linear Programming

- 1** Write down the coordinates of the points where the line $3x + 2y = 18$ meets the axes. [2]
2 The inequality $x + 2y \leq 14$ is to be shown on a graph by shading the unwanted region. State whether the point $(6, 5)$ lies in the wanted region, with a reason. [2]
3 A baker makes x loaves and y cakes. Each loaf takes 3 minutes of oven time and each cake takes 5 minutes. At most 120 minutes of oven time are available. Write down an inequality for this. [2]
4 A shop sells x pens at 40p profit each and y notebooks at 90p profit each. Write down the objective function for the total profit P in pence that is to be maximised. [2]

5 Find the point where the lines $x + y = 10$ and $3x + y = 18$ meet. [2]

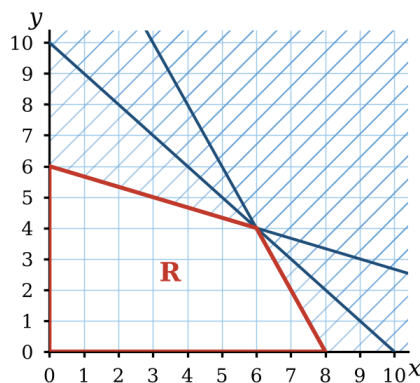
6 Explain why, in a linear programming problem, the maximum of the objective function over a feasible region with straight edges is always found at a vertex of the region. [2]

7 The feasible region R is defined by $x \geq 0$, $y \geq 0$, $3x + 2y \leq 36$ and $x + 2y \leq 20$, as shown. Find the coordinates of the vertices of R and the maximum value of $P = 5x + 4y$.



[4]

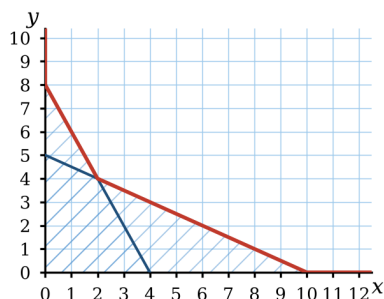
8 R is the region $x \geq 0$, $y \geq 0$, $x + y \leq 10$, $x + 3y \leq 18$ and $2x + y \leq 16$. Find the greatest value of $3x + 4y$ in R and the point where it occurs.



[4]

9 The feasible region R is defined by $x \geq 0$, $y \geq 0$, $2x + y \geq 8$ and $x + 2y \geq 10$, as shown, and $C = 3x + 2y$ is to be minimised. The region is unbounded, so check the vertices on the axes and at the intersection of the lines. Find the minimum value of C.

R



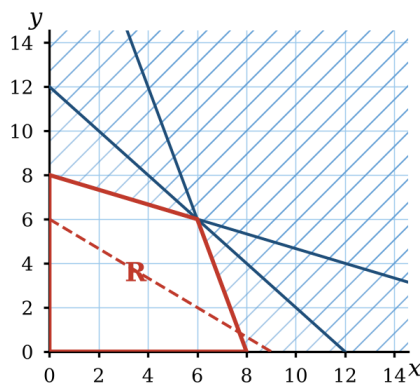
[4]

10 A farmer grows wheat on x hectares and barley on y hectares. At most 40 hectares are available. The wheat needs at least 10 hectares. Barley requires at most twice the area of wheat. The profit is £300 per hectare for wheat and £200 per hectare for barley.

(a) Write down the constraints and the objective function. [3]

(b) Find the maximum profit by testing the vertices of the feasible region. [3]

- 11** Use the objective line method to solve this problem: maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 3y \leq 24$, $3x + y \leq 24$, $x \geq 0$, $y \geq 0$. State the coordinates of the optimal vertex and the value of P , and give the gradient of the objective line.



[4]

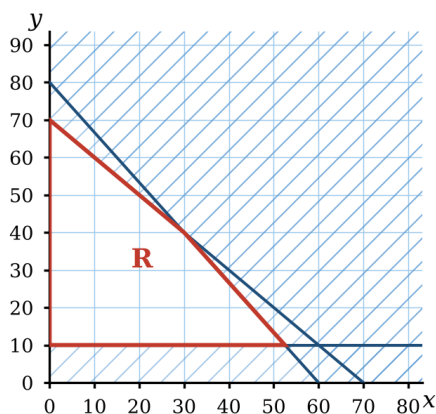
- 12** Find the minimum value of $C = 4x + y$ subject to $x + 2y \leq 16$, $3x + y \leq 21$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$.

[4]

- 13** A company makes two products, X and Y, using x and y units a day respectively. Product X needs 2 hours of machine time and Y needs 3 hours; 60 hours are available. Product X needs 1 hour of labour and Y needs 1 hour; 25 hours of labour are available. Write down the constraints and find the greatest total number of units made per day.

[5]

- 14** A workshop makes tables (x per week) and chairs (y per week). Each table needs 4 hours of carpentry and each chair needs 3 hours; 240 hours are available. At most 70 items can be made. At least 10 chairs must be made. Profit is £30 per table and £25 per chair.

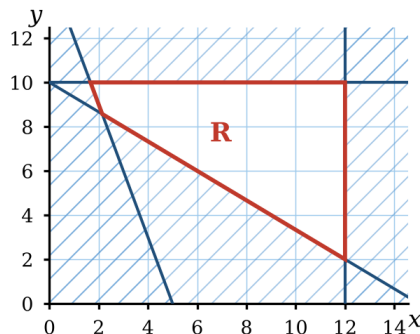


- (a) Write down the constraints and the objective function, and say what the region R shown represents.
- (b) Find the number of tables and chairs that maximises weekly profit, and the maximum profit.

[3]

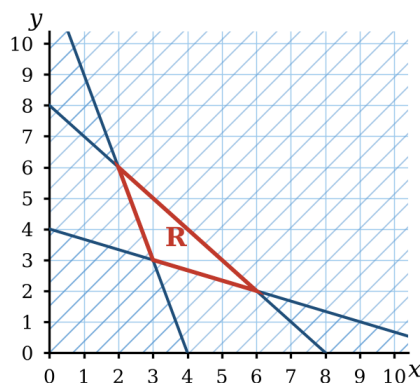
[4]

- 15** A diet is made from x kg of food A and y kg of food B. Each kg of A has 2 units of protein and each kg of B has 3 units; at least 30 units are needed. Each kg of A has 3 units of iron and each kg of B has 1 unit; at least 15 units are needed. No more than 12 kg of A and 10 kg of B can be used. Food A costs £5 per kg and food B £4 per kg. Find the cheapest diet and its cost.



[6]

- 16** The region R is defined by $x + y \leq 8$, $x + 3y \geq 12$, $3x + y \geq 12$, $x \geq 0$ and $y \geq 0$.



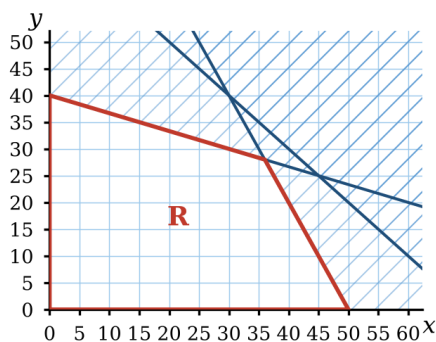
(a) Find the coordinates of the vertices of R.

[3]

(b) Find the maximum and the minimum value of $2x + y$ over R.

[3]

- 17** A company makes standard and deluxe cases. In a day it makes x standard and y deluxe cases. Cutting needs 2 minutes per standard case and 1 minute per deluxe case; 100 minutes are available. Assembly needs 1 minute per case; 70 minutes are available. Finishing needs 1 minute for a standard case and 3 minutes for a deluxe case; 120 minutes are available. Profits are £4 per standard case and £3 per deluxe case.



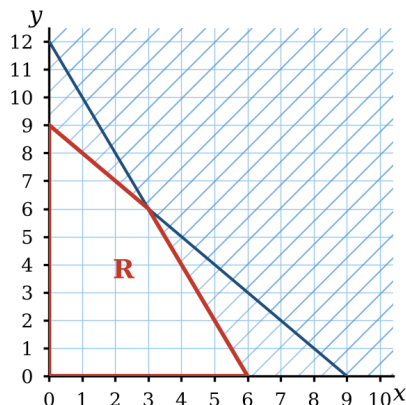
(a) Show that the constraints are $2x + y \leq 100$, $x + y \leq 70$, $x + 3y \leq 120$.

[3]

(b) Find the numbers of each type that give the maximum profit, and the profit.

[4]

- 18** In a linear programming problem $x + y \leq 9$, $2x + y \leq 12$ and $x, y \geq 0$. The objective is to maximise $P = ax + y$.

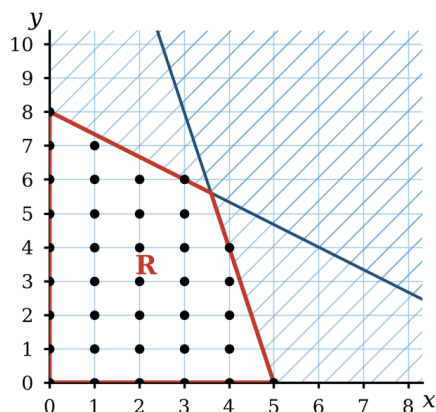


- (a) Find the maximum of P when $a = 1$. [4]
 (b) Find the maximum of P when $a = 3$, and the vertex where it occurs. [2]

- 19** A linear programming problem has the feasible region with vertices $(0, 0)$, $(8, 0)$, $(6, 4)$ and $(0, 6)$. The objective function is $P = 3x + 2y$. Show that P is largest at $(6, 4)$, and find the range of values of k for which the objective function $P = kx + 2y$ is still largest at $(6, 4)$. [5]

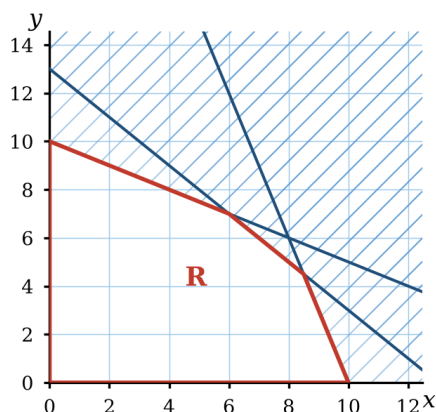
- 20** A feasible region is defined by $y \geq x - 2$, $y \leq x + 3$, $x + y \leq 12$ and $x, y \geq 0$.
 (a) Find the coordinates of its vertices. [3]
 (b) Find the maximum of $2x + 3y$. [2]

- 21** A company produces x small and y large units, both whole numbers. The constraints are $2x + 3y \leq 24$ and $4x + y \leq 20$, with $x, y \geq 0$, and the profit $P = 5x + 4y$ is to be maximised.



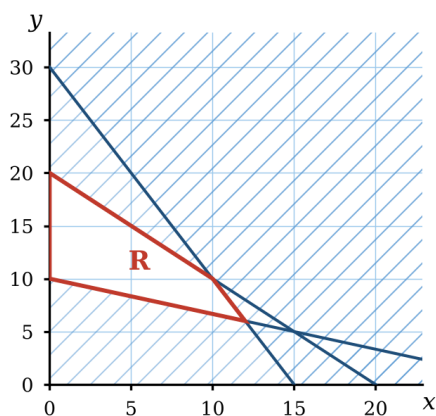
- (a) Ignoring the whole number condition, find the optimal vertex and the maximum profit. [4]
 (b) Hence find the optimal integer solution, checking nearby integer points. [4]

- 22** The feasible region R of a linear programming problem is defined by $x + y \leq 13$, $x + 2y \leq 20$, $3x + y \leq 30$, $x \geq 0$, $y \geq 0$.



- (a) Find the vertices of R . [3]
 (b) Find the maximum of $P = 5x + 6y$ and the point where it occurs. [2]

- 23** A company makes x units of A and y units of B. The constraints are $x + y \leq 20$, $2x + y \leq 30$, $x + 3y \geq 30$, $x \geq 0$, $y \geq 0$. The cost is $C = 3x + 2y$.

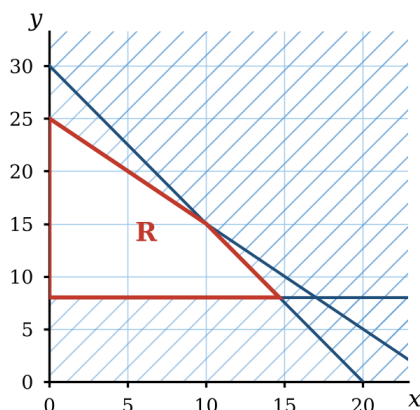


- (a) Find the vertices of the feasible region. [3]
 (b) Find the minimum cost and the point where it occurs. [2]

- 24** A student solves a linear programming problem and finds the optimal point to be at the intersection of $x + y = 8$ and $2x + y = 12$, namely $(4, 4)$, for the objective $P = 5x + 2y$ with constraints $x + y \leq 8$, $2x + y \leq 12$, $x, y \geq 0$.

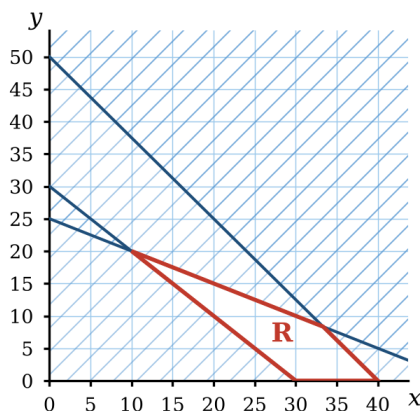
- (a) Show that $(4, 4)$ is a vertex of the feasible region. [2]
 (b) Show that the student's optimal point is wrong, by finding the true optimum. [2]

- 25** A company packs x boxes of type A and y boxes of type B. Machine time: $3x + 2y \leq 60$. Packing staff: $x + y \leq 25$. At least 8 boxes of type B are required. Profit per box is £7 for A and £5 for B.



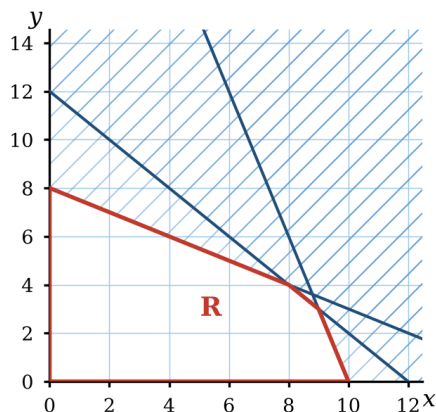
- (a) Find the optimum numbers and the maximum profit. [4]
 (b) The profit on type A falls to £4 per box. Find the new optimum. [2]

- 26** A recycling plant processes x tonnes of glass and y tonnes of paper per day. Machine hours: $5x + 4y \leq 200$. Sorting staff: $x + 2y \leq 50$. At least 30 tonnes in total must be processed. The cost per day is $C = 3x + 2y$, in hundreds of pounds, and is to be minimised.



- (a) Find the vertices of the feasible region. [3]
 (b) Find the minimum cost, giving the amounts of glass and paper processed. [2]

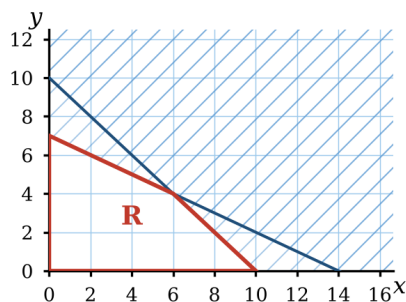
- 27** The feasible region of a linear programming problem is given by $x + y \leq 12$, $x + 2y \leq 16$, $3x + y \leq 30$, $x \geq 0$, $y \geq 0$. The objective function is $P = kx + y$, where $k > 0$.



- (a) Find the vertices of the feasible region. [3]
 (b) Find the maximum value of P when $k = 2$ and the point where it occurs. [3]
 (c) Find the range of values of k for which the maximum of P occurs at $(9, 3)$. [5]

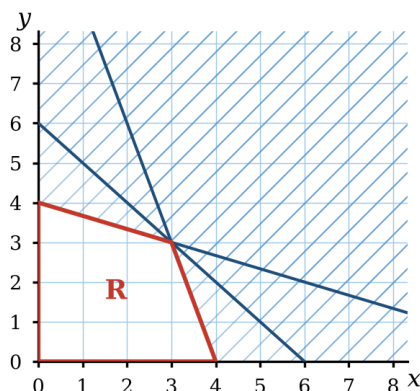
- 28** A manufacturer makes x chairs and y tables. Each chair needs 2 hours of cutting and 1 hour of finishing, each table needs 3 hours of cutting and 2 hours of finishing. Cutting is limited to 120 hours and finishing to 70 hours. Profit is £ a per chair and £ b per table.
- Find the vertices of the feasible region. [3]
 - When $a = 5$ and $b = 8$, find the maximum profit. [2]
 - Find the range of the ratio $a : b$ (that is, of $a \div b$) for which the optimal vertex is (30, 20). [4]

- 29** A student claims: "In the problem to maximise $P = 2x + 5y$ subject to $x + y \leq 10$, $x + 2y \leq 14$, $x, y \geq 0$, the maximum is at the vertex (10, 0) because it is furthest from the origin."



- Show that the claim is wrong, and find the true maximum. [4]
- Explain why the distance from the origin is not the correct test. [1]
- Find the range of c for which $P = cx + 5y$ has its maximum at (6, 4). [4]

- 30** The lines $x + y = 6$, $x + 3y = 12$ and $3x + y = 12$ bound a feasible region R ($x, y \geq 0$, each constraint $\leq$).



- Show that the three lines are concurrent, and find the vertices of R . [4]
- Find the maximum of $P = x + y$ and explain what is unusual about the solution. [3]
- State the integer solutions of the problem, i.e. whole number points, with $x + y = 6$. [2]

Optimisation with Constraints

- The constraint $3x + 2y \leq 18$ is to be used in the simplex algorithm. Write it as an equation using a slack variable r , and state what r represents. [2]
- The objective function $P = 5x + 4y$ is to be maximised using the simplex algorithm. Write down the objective row equation in the form $P - ax - by = 0$. [1]
- In a simplex tableau the objective row is $P: 1, -3, -7, 0, 0 \mid 0$ for columns $P, x, y, r, s \mid$ value. State which column is the pivot column and explain why. [2]
- The pivot column of a tableau has entries 2, 5 in the constraint rows with values 12 and 20. Find the pivot row using the ratio test. [2]
- Explain why the ratio test uses the smallest non-negative ratio. [2]

- 6** A simplex problem has 3 variables x, y, z and 4 constraints of the form $\leq$. State the number of slack variables and the number of columns (excluding the value column and including the P column) in the initial tableau. [2]
- 7** Write down the initial simplex tableau for: maximise $P = 5x + 4y$ subject to $3x + 2y \leq 36, x + 2y \leq 20, x, y \geq 0$, using slack variables r and s . [3]
- 8** The initial tableau for a simplex problem (columns P, x, y, r, s , value; rows r, s, P) is
- $$\begin{pmatrix} 0 & 3 & 2 & 1 & 0 & 36 \\ 0 & 1 & 2 & 0 & 1 & 20 \\ 1 & -5 & -4 & 0 & 0 & 0 \end{pmatrix}$$
- Carry out one iteration, stating the pivot, and write down the new tableau. [4]
- 9** A simplex problem is to maximise $P = 5x + 4y$ subject to $3x + 2y \leq 36, x + 2y \leq 20, x, y \geq 0$. After the first iteration the tableau (columns P, x, y, r, s , value; rows x, s, P) is
- $$\begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 12 \\ 0 & 0 & \frac{4}{3} & \frac{-1}{3} & 1 & 8 \\ 1 & 0 & \frac{-2}{3} & \frac{5}{3} & 0 & 60 \end{pmatrix}$$
- Carry out the second iteration and state the optimal values of x, y and P . [3]
- 10** A simplex problem is to maximise $P = 3x + 4y$ subject to $x + y \leq 10, x + 3y \leq 18, 2x + y \leq 16, x, y \geq 0$. The initial tableau (columns P, x, y, r, s, t , value; rows r, s, t, P) is
- $$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 10 \\ 0 & 1 & 3 & 0 & 1 & 0 & 18 \\ 0 & 2 & 1 & 0 & 0 & 1 & 16 \\ 1 & -3 & -4 & 0 & 0 & 0 & 0 \end{pmatrix}$$
- Carry out one iteration, writing down the new tableau, and state the values of x, y and P that it represents. [4]
- 11** The final simplex tableau (columns P, x, y, r, s , value; basic variables x, y) of a problem is
- $$\begin{pmatrix} 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & 8 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{3}{4} & 6 \\ 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 64 \end{pmatrix}$$
- Read off the optimal solution and state which constraints are binding (have zero slack). [3]
- 12** After one iteration of a simplex algorithm the objective row is $P: 1, 0, -2, \frac{3}{2}, 0 \mid 40$. Is the solution optimal? If not, state the next pivot column. [2]
- 13** Write down the initial tableau for: maximise $P = 2x + 3y$ subject to $x + y \leq 12, x + 3y \leq 24, 3x + y \leq 24$ (slack variables r, s, t), and state the first pivot. [3]
- 14** Use the simplex algorithm to maximise $P = 6x + 5y$ subject to $x + y \leq 12, x + 2y \leq 16, 3x + y \leq 30, x, y \geq 0$. Show each tableau. [8]
- 15** A workshop makes x tables and y chairs per week, with profits £30 and £25 respectively. Carpentry: $4x + 3y \leq 240$ hours. Total items: $x + y \leq 70$. Chairs: $y \leq 60$. Use the simplex algorithm to find the production plan that maximises profit. [8]
- 16** Maximise $P = 7x + 5y$ subject to $3x + 2y \leq 60, x + y \leq 25, x, y \geq 0$.
- (a) Use the simplex algorithm to find the optimal solution. [8]
- (b) State the value of each slack variable at the optimum and interpret it. [2]

- 17** A company makes three products with quantities x , y , z per day. The constraints are $x + y + z \leq 20$, $2x + y \leq 30$ and $y + 3z \leq 36$, and the profit is $P = 5x + 4y + 2z$.
- (a) Write down the initial tableau (slack variables r , s , t). [3]
- (b) State the pivot column and pivot row for the first iteration. [2]
-
- 18** Explain why the simplex algorithm, starting from the origin, moves from vertex to vertex along the edges of the feasible region, and why it stops when there are no negative entries in the objective row. [3]
-
- 19** A company makes x , y and z units of three products. The problem is to maximise $P = 2x + 5y + 3z$ subject to $x + 2y + z \leq 30$, $2x + z \leq 40$, $x + y + 3z \leq 45$, $x, y, z \geq 0$. Slack variables r , s , t are used for the three constraints, and the final tableau (columns P , x , y , z , r , s , t , value; rows y , s , z , P) is
- $$\begin{array}{c|ccccccc} 0 & \frac{2}{5} & 1 & 0 & \frac{3}{5} & 0 & -\frac{1}{5} & 9 \\ 0 & \frac{9}{5} & 0 & 0 & \frac{1}{5} & 1 & -\frac{2}{5} & 28 \\ 0 & \frac{1}{5} & 0 & 1 & -\frac{1}{5} & 0 & \frac{2}{5} & 12 \\ 1 & \frac{3}{5} & 0 & 0 & \frac{12}{5} & 0 & \frac{1}{5} & 81 \end{array}$$
- (a) Read off the optimal values of x , y , z and P . [2]
- (b) State which resources are fully used and the amount of the other resource that is unused. [2]
- (c) Use the objective row to find how much P increases per extra unit of the first resource, and say how much P would increase if the first resource limit rose by 5 units (assuming the same basis). [3]
-
- 20** Use the simplex algorithm to maximise $P = 4x + 5y$ subject to $x + 2y \leq 14$, $3x + y \leq 18$, $x, y \geq 0$. [8]
-
- 21** Use the simplex algorithm to maximise $P = 5x + 4y + 2z$ subject to $x + y + z \leq 20$, $2x + y \leq 30$, $y + 3z \leq 36$, $x, y, z \geq 0$. [10]
-
- 22** Maximise $P = 2x + 5y + 3z$ subject to $x + 2y + z \leq 30$, $2x + z \leq 40$, $x + y + 3z \leq 45$, $x, y, z \geq 0$, using the simplex algorithm. [8]
-
- 23** Maximise $P = 5x + 4y$ subject to $2x + 3y \leq 24$, $4x + y \leq 20$, $x, y \geq 0$, where x and y must be integers.
- (a) Solve the problem ignoring the integer condition, using the simplex algorithm. [8]
- (b) Explain why rounding the answer to (a) does not necessarily give the optimal integer solution, and find the optimal integer solution. [4]
-
- 24** Maximise $P = 5x + 4y$ subject to $3x + 2y \leq b$, $x + 2y \leq 20$, $x, y \geq 0$. When $b = 36$ the optimum is $P = 64$ at $(8, 6)$.
- (a) Find the maximum value of P when $b = 37$ (the first constraint is relaxed by one unit). [3]
- (b) Hence state the shadow price of the first constraint, that is the increase in P per extra unit of that resource, and say for what range of b it applies. [2]
-
- 25** Maximise $P = 2x + y$ subject to $x - y \leq 3$, $x \geq 0$, $y \geq 0$, using the simplex algorithm.
- (a) Carry out one iteration and show that the problem is unbounded. [4]
- (b) Explain graphically why the problem is unbounded. [1]
-
- 26** A company wishes to minimise $C = x - 3y$ subject to $x + 2y \leq 12$, $3x + y \leq 15$, $x, y \geq 0$. To use the simplex algorithm the problem is rewritten as maximising $P = -C = 3y - x$.
- (a) Write down the objective row equation and the initial tableau, using slack variables r and s . [3]
- (b) Carry out the simplex algorithm and state the optimal values of x , y and P . [4]
- (c) State the minimum value of C and the values of x and y at which it occurs. [1]

- 27** Use the simplex algorithm to maximise $P = 5x + 4y + 3z$ subject to $2x + 3y + z \leq 5$, $4x + y + 2z \leq 11$, $3x + 4y + 2z \leq 8$, $x, y, z \geq 0$. Show each tableau and interpret the final slack variables. [9]
-
- 28** A company makes three products X, Y, Z in quantities x, y, z . Constraints: $x + y + 2z \leq 4$, $2x + 3z \leq 5$, $2x + y + 3z \leq 7$. Profit $P = 3x + 2y + 4z$. Use the simplex algorithm to find the maximum profit and the quantities produced. [10]
-
- 29** In a simplex problem with two variables the optimal vertex is the intersection of the constraints $3x + 2y \leq 36$ and $x + 2y \leq 20$, with objective $P = px + 4y$.
(a) Find the range of values of p for which the optimal vertex is $(8, 6)$. [5]
(b) Explain what happens at $p = 6$ and at $p = 2$. [2]
-
- 30** Prove that, in the simplex algorithm, if the ratio test chooses a row with the smallest positive ratio then all the basic variables remain non-negative after the iteration.
(a) Consider a constraint row with entries a (in the pivot column) and value $v \geq 0$, and the pivot row with entry $p > 0$ and value u . Show that after the iteration the value in this row is $v - \frac{a u}{p}$. [2]
(b) Show this is non-negative for every row, using the ratio test. [4]
-