

Locating Roots and Sign Changes

1 (2 marks)

$$f(1) = 1 + 1 - 3 = -1 \text{ and } f(2) = 8 + 2 - 3 = 7$$

M1

f is continuous and there is a change of sign, so there is a root between 1 and 2

A1

Answer: shown

2 marks

2 (2 marks)

$$\text{Let } f(x) = \ln x + x - 3. \quad f(2) = \ln 2 - 1 = -0.307 \text{ and } f(3) = \ln 3 = 1.099$$

M1

A change of sign and f is continuous on $[2, 3]$, so a root lies in the interval

A1

Answer: shown

2 marks

3 (2 marks)

$$f(2) = 8 - 6 - 5 = -3 \text{ and } f(3) = 27 - 9 - 5 = 13$$

M1

f is continuous and the sign changes, so there is a root between 2 and 3

A1

Answer: shown

2 marks

4 (2 marks)

$$\text{Let } f(x) = 2^x - 5. \quad f(2) = -1 \text{ and } f(3) = 3$$

M1

Sign change and f continuous, so a root lies in $(2, 3)$

A1

Answer: shown

2 marks

5 (2 marks)

$$f(2.6) = -0.240 < 0 \text{ and } f(2.7) = 0.290 > 0$$

M1

Sign change, so the root $\sqrt{7}$ of $f(x) = 0$ lies in $(2.6, 2.7)$

A1

Answer: shown

2 marks

6 (2 marks)

$$f(2) = 2 > 0 \text{ and } f(3) = -1 < 0$$

B1

A sign change for a continuous function: a root lies between $x = 2$ and $x = 3$

B1

Answer: between 2 and 3

2 marks

7 (5 marks)

$$(a) \quad f(1) = -2 \text{ and } f(2) = 1$$

M1

Change of sign and f continuous, so $1 < \alpha < 2$

A1

Answer: shown

2 marks

$$(b) \quad f(1.5) = -1.625 < 0, \text{ so } 1.5 < \alpha < 2$$

M1

$$f(1.75) = -0.641 < 0, \text{ so } 1.75 < \alpha < 2$$

M1

Interval $[1.75, 2]$

A1

Answer: $1.75 < \alpha < 2$

3 marks

8 (4 marks)

$$(a) \quad \text{At the intersection } e^\alpha = 4 - \alpha$$

M1

$$\text{So } e^\alpha + \alpha - 4 = 0$$

A1

Answer: shown

2 marks

$$(b) \quad \text{Let } g(x) = e^x + x - 4. \quad g(1) = e - 3 = -0.282 \text{ and } g(1.5) = 1.98$$

M1

Change of sign and g continuous, so $1 < \alpha < 1.5$

A1

Answer: shown

2 marks

9 (2 marks)

$$\text{Let } f(x) = \cos x - 2x. \quad f(0.4) = 0.121$$

M1

$f(0.5) = -0.122$; sign change and f continuous, so a root lies in $(0.4, 0.5)$

A1

Answer: shown

2 marks

10 (3 marks)

$f(x) = x^3 - x - 1$. Need the root in (1.315, 1.325)

M1

$f(1.315) = -0.0411$

M1

$f(1.325) = 0.00120$: sign change, so root rounds to 1.32

A1

Answer: shown

3 marks

11 (4 marks)

$f(1) = -6$, $f(2) = 3$: sign change, root in (1, 2)

B1

$f(1.5) = -2.625 < 0$, so root in (1.5, 2)

M1

$f(1.75) = -0.141 < 0$, so root in (1.75, 2)

M1

$f(1.875) = 1.34 > 0$, so root in (1.75, 1.875)

A1

Answer: $1.75 < \alpha < 1.875$

4 marks

12 (7 marks)

(a) $f(1) = \frac{1}{-1} = -1$ and $f(3) = \frac{1}{1} = 1$: opposite signs

B1

$\frac{1}{x-2} = 0$ would need $1 = 0$, which is impossible, so there is no root

M1

f has an asymptote at $x = 2$ and is not continuous on $[1, 3]$

E1

Answer: f is not continuous at $x = 2$

3 marks

(b) $f(2) = 0$, so $x = 2$ is a root

B1

$f(1) = 1 > 0$ and $f(3) = 1 > 0$: a repeated root touches the axis without changing sign

E1

Answer: same sign, but a repeated root at $x = 2$

2 marks

(c) $f(x)$ must be continuous on the interval $[a, b]$

E1

Then a change of sign between a and b guarantees at least one root in the interval

E1

Answer: f continuous on $[a, b]$, then a sign change implies a root

2 marks

13 (7 marks)

(a) Let $f(x) = \cos x - x$. $f(0.7) = 0.0648 > 0$ and $f(0.8) = -0.103 < 0$

M1

Change of sign and f continuous, so $0.7 < \alpha < 0.8$

A1

Answer: shown

2 marks

(b) Similar triangles: $\frac{\alpha - 0.7}{0.0648} = \frac{0.8 - \alpha}{0.103}$

M1

$\alpha = 0.7 + 0.1 \times \frac{0.0648}{0.0648 + 0.103}$

M1

$\alpha \approx 0.739$

A1

Answer: $\alpha \approx 0.739$

3 marks

(c) $f(0.735) = 0.00683 > 0$ and $f(0.745) = -0.00991 < 0$

M1

So $0.735 < \alpha < 0.745$, hence $\alpha = 0.74$ to 2 d.p.

A1

Answer: shown

2 marks

14 (6 marks)

(a) $f(1) = 1.56 > 0$, $f(2) = -2.19 < 0$

B1

$\tan x$ is not continuous at $x = \frac{\pi}{2} \approx 1.57$, which lies in (1, 2) (asymptote)

E1

The sign change comes from the asymptote, not a root; the conclusion is invalid

E1

Answer: $\tan x$ is discontinuous at $x = \pi/2$

3 marks

(b) $f(3) = -0.143 < 0$ and $f(4) = 1.16 > 0$

M1

The next asymptote is at $\frac{3\pi}{2} \approx 4.71 > 4$, so $\tan x$ is continuous on $[3, 4]$

E1

Sign change, so a root lies in (3, 4)

A1

Answer: shown

3 marks

15 (3 marks)

α is a root of $g(x) = 3 \sin x - x = 0$ (curves meet)

M1

$g(2) = 0.728 > 0$

M1

$g(3) = -2.58 < 0$; sign change and continuity, so $2 < \alpha < 3$

A1

Answer: shown

3 marks

16 (4 marks)

$$f(-2) = 1, f(-1) = -4: \text{sign change}$$

M1 A1

$$f(1) = -2, f(2) = 5: \text{sign change}$$

M1

f is a polynomial (continuous), so there is a root in each interval

A1

Answer: shown

4 marks

17 (6 marks)

$$(a) V = \pi r^2(r + 4)$$

M1

$$\pi r^3 + 4\pi r^2 = 100, \text{ so } \pi r^3 + 4\pi r^2 - 100 = 0$$

A1

Answer: shown

2 marks

$$(b) h(r) = \pi r^3 + 4\pi r^2 - 100: h(2) = -24.6$$

M1

$$h(3) = 97.9, \text{ sign change}$$

A1

Answer: shown

2 marks

$$(c) h(2.25) = -0.598 < 0$$

M1

$$h(2.3) = 4.70 > 0, \text{ so } 2.25 < r < 2.3, \text{ giving } r = 2.3 \text{ (1 d.p.)}$$

A1

Answer: shown

2 marks

18 (4 marks)

$$f(-2) = -1, f(-1) = 3, f(0) = 1, f(1) = -1, f(2) = 3$$

M1 A1

Sign changes in $(-2, -1)$, $(0, 1)$ and $(1, 2)$

M1

Polynomial, so continuous: one root in each interval, and a cubic has at most three roots

A1

Answer: roots in $(-2, -1)$, $(0, 1)$, $(1, 2)$

4 marks

19 (8 marks)

$$(a) y = e^x: \text{positive, increasing, passing through } (0, 1)$$

B1

$$y = 6 - x^2: \text{a maximum at } (0, 6), \text{ crossing the x-axis at } \pm\sqrt{6}$$

B1

The graphs meet once for $x < 0$ and once for $x > 0$, so there are two roots

B1

Answer: Two roots (one negative, one positive)

3 marks

$$(b) f(x) = e^x + x^2 - 6$$

M1

$$f(-2.5) = 0.332 > 0 \text{ and } f(-2.4) = -0.149 < 0$$

M1 A1

$$f(1.3) = -0.641 < 0 \text{ and } f(1.4) = 0.015 > 0$$

A1

f is continuous, so there is a sign change and hence a root in each interval

E1

Answer: Sign changes shown in both intervals

5 marks

20 (8 marks)

$$(a) f(-3) = -7 \text{ and } f(-2) = 6, \text{ a sign change}$$

M1 A1

$$f(0) = 2 \text{ and } f(1) = -3, \text{ a sign change}$$

A1

$$f(2) = -2 \text{ and } f(3) = 11, \text{ a sign change; } f \text{ continuous so a root lies in each interval}$$

A1

Answer: shown

4 marks

(b) A cubic equation has at most 3 real roots, and three have been found

E1

Answer: a cubic has at most three roots

1 mark

$$(c) f(0.335) = 0.0276 > 0$$

M1

$$f(0.345) = -0.0289 < 0$$

A1

So the root is in $(0.335, 0.345)$, so it is 0.34 to 2 d.p.

A1

Answer: 0.34

3 marks

21 (8 marks)

$$(a) f(1) = -4 - k$$

M1

$$f(2) = 3 - 2k$$

M1

A sign change needs $f(1)f(2) < 0: (-4 - k)(3 - 2k) < 0$, i.e. $(k + 4)(2k - 3) < 0$

M1

$$\text{Critical values } k = -4 \text{ and } k = \frac{3}{2}$$

A1

$$-4 < k < \frac{3}{2}$$

A1

Answer: $-4 < k < \frac{3}{2}$

5 marks

$$(b) f(x) = x^3 - x - 5: f(1.9) = -0.0410$$

M1

$f(1.95) = 0.465$, a change of sign, so $1.9 < \alpha < 1.95$	A1
$\alpha = 1.9$ to 1 d.p.	A1
Answer: 1.9	3 marks
22 (8 marks)	
(a) $f(0.6) = 0.0221 > 0$, $f(0.7) = -0.0862 < 0$	M1 A1
$f(1.5) = -0.0183 < 0$, $f(1.6) = 0.153 > 0$	M1 A1
Answer: shown	4 marks
(b) $f(0.615) = 0.00466 > 0$	M1
$f(0.625) = -0.00675 < 0$, so $0.615 < \alpha < 0.625$	A1
Answer: shown	2 marks
(c) $f(0) = 1 > 0$ and $f(2) = 1.39 > 0$: no sign change	B1
An even number of roots (here 2) gives no sign change over the interval	E1
Answer: same sign, two roots (an even number)	2 marks
23 (5 marks)	
(a) At a common point $\sin \alpha = 0.3\alpha$	B1
$d(x) = \sin x - 0.3x$: $d(2.3) = 0.0557$	M1
$d(2.4) = -0.0445$: sign change, $2.3 < \alpha < 2.4$	A1
Answer: shown	3 marks
(b) Let $d(x) = \sin x - 0.3x$. $d''(x) = -\sin x < 0$ for $0 < x < \pi$, so d is concave there. As $d(0) = 0$ and $d(\alpha) = 0$, a concave function is negative for $\alpha < x < \pi$	E1
For $\pi \leq x \leq 2\pi$, $\sin x \leq 0 < 0.3x$; for $x > 2\pi$, $0.3x > 1.8 > 1 \geq \sin x$. So $d(x) < 0$ for all $x > \alpha$, and there is no other root	E1
Answer: d is concave on $(0, \pi)$, so negative after α; and negative beyond π as $0.3x > \sin x$	2 marks
24 (7 marks)	
(a) $f(0.7) = -0.132 < 0$, $f(0.8) = 0.128 > 0$	M1
Sign change and continuity	A1
Answer: shown	2 marks
(b) Assume f is a straight line between the two points: $\frac{\alpha - 0.7}{0.8 - \alpha} = \frac{ f(0.7) }{f(0.8)}$	M1
$\alpha \approx 0.7 + 0.1 \times \frac{0.1319}{0.1319 + 0.1277} = 0.7508$	M1 A1
$\alpha \approx 0.751$	A1
Answer: 0.751	4 marks
(c) Error = $ 0.7549 - 0.7508 \approx 0.004$	B1
Answer: about 0.004	1 mark
25 (5 marks)	
(a) $g(1.1) = -0.235 < 0$	M1
$g(1.2) = 0.172 > 0$, and $\tan x$ is continuous on $[1.1, 1.2]$ as $\frac{\pi}{2} \approx 1.571 > 1.2$: sign change gives a root	M1 A1
Answer: shown	3 marks
(b) $g(1.5) > 0$ and $g(1.6) < 0$ but $\frac{\pi}{2} \approx 1.571$ lies between them	E1
g is not continuous there (asymptote), so no root can be concluded	E1
Answer: asymptote at $x = \pi/2$ between 1.5 and 1.6	2 marks
26 (8 marks)	
(a) $\frac{1}{x} = x^3 - 2$	M1
Multiply by x : $1 = x^4 - 2x$	M1
$x^4 - 2x - 1 = 0$	A1
Answer: shown	3 marks
(b) $g(x) = x^4 - 2x - 1$	M1
$g(1.3) = -0.744 < 0$ and $g(1.4) = 0.042 > 0$	M1 A1

$$g(-0.5) = 0.0625 > 0 \text{ and } g(-0.4) = -0.174 < 0$$

A1

g is continuous, so each sign change gives a root

E1

Answer: Sign changes shown in both intervals

5 marks

27 (10 marks)

(a) $g'(x) = e^x - 4 = 0$, so $x = \ln 4$

M1 A1

$$g(\ln 4) = 4 - 4 \ln 4$$

A1

Answer: $x = \ln 4$; $g = 4 - 4 \ln 4$

3 marks

(b) $g(0.3) = 0.150 > 0$ and $g(0.4) = -0.108 < 0$

M1 A1

$$g(2) = -0.611 < 0 \text{ and } g(2.5) = 2.18 > 0$$

M1

Continuous with sign changes, so the roots lie in the stated intervals

A1

Answer: shown

4 marks

(c) $4 - 4 \ln 4 = 4(1 - \ln 4) < 0$ as $\ln 4 > 1$, and $g(x) \rightarrow \infty$ as $x \rightarrow \infty$ (e^x dominates) and as $x \rightarrow -\infty$ ($-4x \rightarrow \infty$)

E1

So the graph is above the axis at both ends and below it at the minimum: at least two roots

E1

$g'(x) = 0$ only at $x = \ln 4$, so g decreases then increases: no more than two roots

E1

Answer: g has a single minimum, which is negative

3 marks

28 (9 marks)

(a) $f'(x) = 5x^4 - 5$

M1

$$f'(x) = 0 \text{ gives } x^4 = 1, \text{ so } x = \pm 1$$

A1

$$f(-1) = -1 + 5 + 1 = 5$$

A1

$$f(1) = 1 - 5 + 1 = -3. \text{ Stationary points } (-1, 5) \text{ and } (1, -3)$$

A1

Answer: $(-1, 5)$ and $(1, -3)$

4 marks

(b) $f(-2) = -21$ and $f(-1) = 5$, a sign change

M1

$$f(0) = 1 \text{ and } f(1) = -3, \text{ a sign change}$$

A1

$$f(1) = -3 \text{ and } f(2) = 23, \text{ a sign change}$$

A1

Answer: shown

3 marks

(c) The curve has only two turning points, a maximum at $(-1, 5)$ and a minimum at $(1, -3)$, so it changes direction twice

E1

A curve with two turning points can cross the x -axis at most three times, and three crossings have been found

E1

Answer: three roots exactly

2 marks

29 (10 marks)

(a) $f'(x) = e^x - k = 0$, so $x = \ln k$

M1 A1

$$f''(x) = e^x > 0, \text{ so a minimum}$$

M1 A1

Answer: minimum at $x = \ln k$

4 marks

(b) $f(\ln k) = e^{\ln k} - k \ln k$

M1

$$= k - k \ln k = k(1 - \ln k)$$

A1

Answer: shown

2 marks

(c) Roots exist iff the minimum value ≤ 0 : $1 - \ln k \leq 0$, so $k \geq e$

M1 A1

$k = e$: minimum = 0, exactly one (repeated) root

B1

$k = 1$: minimum = 1 > 0, no roots

B1

Answer: $k \geq e$; one root at $k = e$; none at $k = 1$

4 marks

30 (8 marks)

(a) $f(0.7) = -0.0648$, $f(0.8) = 0.103$

M1

Sign change and f continuous

A1

Answer: shown

2 marks

(b) $f'(x) = 1 + \sin x \geq 0$ for all x

M1 A1

$f'(x) = 0$ only at isolated points, so f is strictly increasing

E1

A strictly increasing function crosses zero at most once: α is the only root

E1

Answer: shown

4 marks

(c) $x = \cos x$ and $|\cos x| \leq 1$, so $|x| \leq 1$

M1

$$\alpha \in (0.7, 0.8) \subset [-1, 1]$$

A1

Answer: $|x| = |\cos x| \leq 1$ **2 marks**

Iteration

1 (3 marks)

$$x_1 = \sqrt{2 \times 2 + 3} = \sqrt{7} = 2.6458$$

B1

$$x_2 = \sqrt{2 \times 2.6458 + 3} = 2.8795$$

M1

$$x_3 = 2.9596$$

A1**Answer: $x_1 = 2.6458$, $x_2 = 2.8795$, $x_3 = 2.9596$** **3 marks**

2 (3 marks)

$$x^3 = x + 5$$

M1Take the cube root: $x = \sqrt[3]{x + 5}$ **A1**

$$x_1 = \sqrt[3]{1.5 + 5} = 1.8663$$

B1**Answer: $x_1 = 1.8663$** **3 marks**

3 (3 marks)

$$x_1 = 2 + \frac{1}{2} = 2.5$$

B1

$$x_2 = 2 + \frac{1}{2.5} = 2.4$$

B1

$$x_3 = 2 + \frac{1}{2.4} = 2.4167$$

B1**Answer: 2.5, 2.4, 2.4167****3 marks**

4 (3 marks)

Divide $x^2 = 3x + 1$ by x ($x \neq 0$): $x = 3 + \frac{1}{x}$ **M1 A1**

$$x_1 = 3 + \frac{1}{3} = \frac{10}{3} = 3.3333$$

B1**Answer: $x_1 = \frac{10}{3}$** **3 marks**

5 (2 marks)

$$x_1 = \ln 6 = 1.7918$$

B1

$$x_2 = \ln(1.7918 + 5) = 1.9157$$

B1**Answer: 1.7918, 1.9157****2 marks**

6 (3 marks)

$$x_1 = \cos 1 = 0.5403$$

B1

$$x_2 = 0.8576$$

B1

$$x_3 = 0.6543$$

B1**Answer: 0.5403, 0.8576, 0.6543****3 marks**

7 (5 marks)

(a) $x_1 = 2$, $x_2 = 1.5$, $x_3 = 1.6667$

M1 A1

$$x_4 = 1.6$$
, $x_5 = 1.625$, $x_6 = 1.6154$

A1**Answer: 2, 1.5, 1.6667, 1.6, 1.625, 1.6154****3 marks****(b)** The terms alternate either side of α , so $1.6154 < \alpha < 1.625$ **M1**Every number in this interval rounds to 1.62, so $\alpha = 1.62$ to 2 d.p.**E1****Answer: α lies between consecutive terms 1.6154 and 1.625****2 marks**

8 (5 marks)

$$e^x = 7 - x, \text{ so } x = \ln(7 - x)$$

M1 A1

$$x_1 = 1.6094, x_2 = 1.6846, x_3 = 1.6706$$

M1

$$x_4 = 1.6732, x_5 = 1.6727, x_6 = 1.6728$$

M1

Terms settle at 1.6728, so $\alpha = 1.673$ (3 d.p.)

A1

Answer: $\alpha = 1.673$

5 marks

9 (5 marks)

$$x^3 + 4x = 3, \text{ so } x(x^2 + 4) = 3$$

M1

$$x = \frac{3}{x^2 + 4}$$

A1

$$x_1 = \frac{3}{0.25 + 4} = 0.7059$$

B1

$$x_2 = 0.6669, x_3 = 0.6749$$

M1 A1

Answer: 0.7059, 0.6669, 0.6749

5 marks

10 (4 marks)

$$L = \frac{1}{2}L + 3$$

M1

$$L = 6$$

A1

$$x_1 = 3.5, x_2 = 4.75, x_3 = 5.375$$

M1

$$|6 - 5.375| = 0.625 < 1$$

A1

Answer: $L = 6$

4 marks

11 (7 marks)

(a) The staircase moves from the curve to the line $y = x$ and back, with each step closer to the point where the graphs meet

E1

The gradient of the curve at the intersection is less than 1, so the steps get smaller: the sequence converges

E1

Answer: staircase approaches the intersection

2 marks

(b) $x_1 = \sqrt{2 \times 0.5 + 3} = \sqrt{4} = 2$

B1

$$x_2 = \sqrt{7} = 2.65$$

B1

Answer: $x_1 = 2.00, x_2 = 2.65$

2 marks

(c) $L = \sqrt{2L + 3}$, so $L^2 = 2L + 3$

M1

$$L^2 - 2L - 3 = 0, \text{ so } (L - 3)(L + 1) = 0$$

A1

$L > 0$ as every term is positive, so $L = 3$

A1

Answer: $L = 3$

3 marks

12 (7 marks)

(a) $f(1.9) = e^{1.9} - 6.8 = -0.114 < 0$

M1

$$f(2) = e^2 - 7 = 0.389 > 0$$

A1

Answer: shown

2 marks

(b) $e^x = 2x + 3$

M1

Take natural logs: $x = \ln(2x + 3)$

A1

Answer: shown

2 marks

(c) $x_1 = 1.9169, x_2 = 1.9219$

M1

$$x_3 = 1.9233, x_4 = 1.9238, x_5 = 1.9239$$

M1

Terms settle at 1.924, so $\alpha = 1.924$ (3 d.p.)

A1

Answer: $\alpha = 1.924$

3 marks

13 (7 marks)

(a) $x^2 = 2 + \sin x$, so $x = \sqrt{2 + \sin x}$ ($x > 0$)

M1 A1

Answer: shown

2 marks

(b) $x_1 = 1.7313, x_2 = 1.7283$

M1

$$x_3 = 1.7285, x_4 = 1.7285$$

A1

$$\alpha = 1.728 \text{ (3 d.p.)}$$

Answer: $\alpha = 1.728$

A1

3 marks

(c) $f(x) = x^2 - 2 - \sin x$: $f(1.7280)$ and $f(1.7290)$ have opposite signs

M1 A1

Answer: shown

2 marks

14 (6 marks)

(a) $x_1 = 2.3069, x_2 = 2.1641, x_3 = 2.2280$

M1

$x_6 = 2.2061, x_7 = 2.2088, x_8 = 2.2076$

M1

$\alpha = 2.208 \text{ (3 d.p.)}$

A1

Answer: 2.208

3 marks

(b) $g'(x) = -\frac{1}{x}$

M1

$g'(\alpha) = -\frac{1}{2.208} = -0.453$

A1

$|g'(\alpha)| < 1$, so the iteration converges, with terms alternating about α

E1

Answer: $g'(\alpha) = -0.453, |g'| < 1$

3 marks

15 (5 marks)

(a) $x_1 = 3, x_2 = \frac{5}{3}, x_3 = \frac{11}{5}, x_4 = \frac{21}{11}$

M1 A1

Answer: $3, \frac{5}{3}, \frac{11}{5}, \frac{21}{11}$

2 marks

(b) The terms alternate above and below a value close to 2

E1

$L = 1 + \frac{2}{L}$, so $L^2 - L - 2 = 0$

M1

$(L - 2)(L + 1) = 0$ and terms are positive, so $L = 2$

A1

Answer: oscillating convergence to 2

3 marks

16 (6 marks)

(a) The path spirals inwards (a cobweb) around the intersection point

E1

Steps get smaller, so the iteration converges to the x-coordinate α of the intersection, oscillating either side of it

E1

Answer: cobweb spiral, convergent

2 marks

(b) $x_1 = \frac{4}{4} = 1, x_2 = \frac{4}{2} = 2, x_3 = \frac{4}{3} = 1.3333$

B1 B1

Answer: 1, 2, 1.3333

2 marks

(c) $\alpha = \frac{4}{\alpha + 1}$, so $\alpha^2 + \alpha - 4 = 0$

M1

$\alpha = \frac{-1 + \sqrt{17}}{2}$ (positive root)

A1

Answer: $\alpha = \frac{-1 + \sqrt{17}}{2}$

2 marks

17 (6 marks)

(a) $P_1 = 100 + 10(0.9) = 109.0$

B1

$P_2 = 118.7$

M1

$P_3 = 129.2$

A1

Answer: 109.0, 118.7, 129.2

3 marks

(b) $L = L + 0.1L(1 - \frac{L}{1000})$, so $0.1L(1 - \frac{L}{1000}) = 0$

M1

$L = 1000$ ($L \neq 0$)

A1

The maximum sustainable population (carrying capacity)

E1

Answer: $L = 1000$

3 marks

18 (10 marks)

(a) $f(x) = 6x - 2 \sin x - 3$

M1

$f(0.7) = -0.0884 < 0$ and $f(0.8) = 0.3653 > 0$

M1 A1

Sign change and f continuous, so $0.7 < \alpha < 0.8$

E1

Answer: shown

4 marks

(b) $6x = 3 + 2 \sin x$

M1

$$x = \frac{1}{2} + \frac{1}{3} \sin x$$

A1

Answer: shown

2 marks

(c) $x_1 = 0.5 + \frac{1}{3} \sin 0.75 = 0.7272$

M1 A1

$x_2 = 0.7216$

A1

$x_3 = 0.7202$

A1

Answer: $x_1 = 0.7272$, $x_2 = 0.7216$, $x_3 = 0.7202$

4 marks

19 (7 marks)

(a) $x^3 - 3x = 5$, so $x(x^2 - 3) = 5$

M1

$$x = \frac{5}{x^2 - 3}$$

A1

Answer: shown

2 marks

(b) $x_0^2 - 3 = 0$, so x_1 would need a division by zero

B1

Answer: Division by zero

1 mark

(c) $x_1 = \frac{5}{4 - 3} = 5$

M1 A1

$$x_2 = \frac{5}{25 - 3} = 0.2273, x_3 = -1.696 \text{ (3 d.p.)}$$

A1

The values jump about and do not settle, so the sequence does not converge to α

E1

Answer: 5, 0.2273, -1.696: not convergent

4 marks

20 (8 marks)

(a) $u_1 = 0.8 \times 100 + 50 = 130$

M1

$u_2 = 154, u_3 = 173.2$

A1

Answer: 130, 154, 173.2

2 marks

(b) $L = 0.8L + 50$

M1

$0.2L = 50$, so $L = 250$

A1

Answer: $L = 250$

2 marks

(c) $n = 0: 250 - 150 = 100 = u_0$, so true for $n = 0$

B1

Assume $u_k = 250 - 150(0.8)^k$

M1

$u_{k+1} = 0.8[250 - 150(0.8)^k] + 50 = 200 - 150(0.8)^{k+1} + 50$

M1

$= 250 - 150(0.8)^{k+1}$, so true for $n = k + 1$. True for $n = 0$ and $k \Rightarrow k + 1$, so true for all $n \geq 0$

A1

Answer: proved

4 marks

21 (8 marks)

(a) $L = \frac{1}{2}(L + \frac{7}{L})$

M1

$2L = L + \frac{7}{L}$, so $L = \frac{7}{L}$

A1

$L^2 = 7$; the terms are positive so $L = \sqrt{7}$

A1

Answer: $L = \sqrt{7}$

3 marks

(b) $x_1 = \frac{1}{2}(3 + \frac{7}{3}) = \frac{8}{3} = 2.66667$

M1

$$x_2 = \frac{1}{2}(2.66667 + \frac{7}{2.66667}) = 2.64583, x_3 = 2.64575$$

A1

Answer: 2.66667, 2.64583, 2.64575

2 marks

(c) $x_{n+1} - \sqrt{7} = \frac{x_n^2 + 7}{2x_n} - \sqrt{7} = \frac{x_n^2 - 2\sqrt{7}x_n + 7}{2x_n}$

M1

$$= \frac{(x_n - \sqrt{7})^2}{2x_n} \quad \text{A1}$$

The new error is roughly the square of the old error divided by $2x_n$, so the number of correct decimal places roughly doubles each step E1

Answer: shown 3 marks

22 (9 marks)

(a) (A): $x_1 = 2.3750$, $x_2 = 12.396$, $x_3 = 1904.0$ M1 A1

(B): $x_1 = 1.3572$, $x_2 = 1.3309$, $x_3 = 1.3259$ M1 A1

Answer: A: 2.375, 12.40, 1904; B: 1.3572, 1.3309, 1.3259 4 marks

(b) (A): $g'(x) = 3x^2$, $g'(\alpha) = 5.26 > 1$: diverges M1 A1

(B): $g'(x) = \frac{1}{3(x+1)^{2/3}}$, $g'(\alpha) = 0.190$, $|g'| < 1$: converges M1 A1

Answer: B converges ($|g'| = 0.190$); A diverges ($g' = 5.26$) 4 marks

(c) $g'(\alpha) > 0$, so successive terms approach α from one side: a staircase E1

Answer: $0 < g'(\alpha) < 1$ gives a staircase 1 mark

23 (7 marks)

(a) $e_0 = -0.0689$, $e_1 = 0.0185$ M1 A1

$e_2 = -0.00501$, $e_3 = 0.00136$ A1

Answer: -0.0689, 0.0185, -0.00501, 0.00136 3 marks

(b) Ratios: -0.268 , -0.272 , -0.271 M1 A1

$g'(x) = -\frac{2}{3(7-2x)^{2/3}}$, $g'(\alpha) = -0.271$ M1

The ratio is close to $g'(\alpha)$: $e_{n+1} \approx g'(\alpha)e_n$; negative, so terms alternate either side of α E1

Answer: ratio $\approx g'(\alpha) \approx -0.271$ 4 marks

24 (9 marks)

(a) $x_1 = 0.8$, $x_2 = 0.5120$, $x_3 = 0.7995$ M1 A1

$x_4 = 0.5129$, $x_5 = 0.7995$, $x_6 = 0.5130$ M1 A1

Answer: 0.8000, 0.5120, 0.7995, 0.5129, 0.7995, 0.5130 4 marks

(b) $L = 3.2L(1 - L)$, so $1 = 3.2(1 - L)$ ($L \neq 0$), $L = 1 - \frac{1}{3.2} = \frac{11}{16}$ M1 A1

$g'(x) = 3.2(1 - 2x)$, $g'(L) = 3.2 \times (-\frac{3}{8}) = -1.2$ M1

$|g'(L)| > 1$, so the fixed point repels; the terms settle into a cycle alternating about 0.5130 and 0.7995 E1 E1

Answer: $|g'(L)| = 1.2 > 1$ 5 marks

25 (9 marks)

(a) $x = 3 - \frac{1}{x}$, so $x^2 = 3x - 1$ M1

$x = \frac{3 \pm \sqrt{5}}{2}$, i.e. 2.618 and 0.3820 A1

Answer: $x = \frac{3 \pm \sqrt{5}}{2}$ 2 marks

(b) $g'(x) = \frac{1}{x^2}$ M1

Larger root 2.618: $g' = 0.146 < 1$: attracts M1 A1

Smaller root 0.382: $g' = 6.85 > 1$: repels A1

Answer: larger root attracts, smaller root repels 4 marks

(c) $x_1 = 3 - 2.5 = 0.5$, $x_2 = 3 - 2 = 1$ M1 A1

The terms move away from 0.382 and then converge towards 2.618 E1

Answer: moves away from 0.382 3 marks

26 (12 marks)

(a) $\frac{dy}{dx} = e^x - 10x$

M1 A1

$e^x = 10x$, so $x = \ln(10x)$

A1

Answer: shown

3 marks

(b) $k(x) = e^x - 10x$

M1

$k(3.5) = -1.885 < 0$ and $k(3.6) = 0.598 > 0$

A1

Sign change, so a root lies in (3.5, 3.6)

E1

Answer: shown

3 marks

(c) $x_1 = \ln 35 = 3.5553$

M1 A1

$x_2 = 3.5710$

A1

$x_3 = 3.5754$

A1

Answer: 3.5553, 3.5710, 3.5754

4 marks

(d) $\frac{d^2y}{dx^2} = e^x - 10$

M1

At $x \approx 3.58$, $e^x \approx 35.8$, so the second derivative is positive: a minimum

A1

Answer: Minimum

2 marks

27 (10 marks)

(a) $f(x) = x^3 - 4x - 2$; $f(2) = -2$ and $f(3) = 13$

M1

Sign change and f continuous

A1

Answer: shown

2 marks

(b) Divide $x^3 = 4x + 2$ by x ($x \neq 0$): $x^2 = 4 + \frac{2}{x}$

M1

$x = \sqrt{4 + \frac{2}{x}}$ (taking the positive root as $x > 0$)

A1

Answer: shown

2 marks

(c) $x_1 = 2.2361$, $x_2 = 2.2123$, $x_3 = 2.2145$

M1

$x_4 = 2.21430$, $x_5 = 2.21432$

M1

$\alpha = 2.2143$ (4 d.p.)

A1

Answer: $\alpha = 2.2143$

3 marks

(d) The terms move away from 2 and converge towards a different root, about -0.54

E1

For convergence to α we need $|g'(\alpha)| < 1$, where $g(x) = \frac{x^3 - 2}{4}$, $g'(x) = \frac{3x^2}{4}$

M1

At $\alpha \approx 2.21$, $g'(\alpha) \approx 3.7 > 1$, so (A) cannot converge to α

E1

Answer: $g'(\alpha) > 1$, so (A) moves away from α

3 marks

28 (10 marks)

(a) $u_2 = 1.8478$, $u_3 = 1.9616$

B1

Answer: 1.8478, 1.9616

1 mark

(b) $n = 1$: $u_1 = \sqrt{2} < 2$

B1

Assume $u_k < 2$

M1

Then $u_{k+1} = \sqrt{2 + u_k} < \sqrt{2 + 2} = 2$

M1

So true for $n = k + 1$; true for $n = 1$, hence for all $n \geq 1$ by induction

A1

Answer: proved

4 marks

(c) $u_{n+1}^2 - u_n^2 = 2 + u_n - u_n^2$

M1

$= (2 - u_n)(1 + u_n) > 0$ since $u_n < 2$ and $u_n > 0$

M1 A1

So the sequence is increasing and bounded above by 2, hence converges to a limit L

E1

$L = \sqrt{2 + L}$, so $L^2 - L - 2 = 0$, $(L - 2)(L + 1) = 0$, and $L = 2$

A1

Answer: increasing, bounded by 2, limit 2

5 marks

29 (11 marks)

(a) $u_2 = \sqrt{7} = 2.6458$

B1

$u_3 = 3.4550$

B1

Answer: 2.6458, 3.4550

2 marks

(b) $n = 1: u_1 = 1 < 4$

B1

Assume $u_k < 4$. Then $u_{k+1} = \sqrt{3u_k + 4} < \sqrt{16} = 4$

M1 A1

True for $n = 1$ and if true for k then $k + 1$, so true for all $n \geq 1$

E1

Answer: proved

4 marks

(c) $u_{n+1}^2 - u_n^2 = 3u_n + 4 - u_n^2 = (4 - u_n)(1 + u_n) > 0$

M1 A1

Increasing and bounded above by 4, so it converges to L

E1

$L^2 = 3L + 4, (L - 4)(L + 1) = 0, L = 4$

M1 A1

Answer: increasing; limit 4

5 marks

30 (10 marks)

(a) $x_1 = 0.60653, x_2 = 0.54524$

M1 A1

$x_3 = 0.57970, x_4 = 0.56006$

A1

Answer: 0.60653, 0.54524, 0.57970, 0.56006

3 marks

(b) $g'(x) = -e^{-x}$, and $e^{-\alpha} = \alpha$, so $g'(\alpha) = -\alpha = -0.567$

M1 A1

 $|g'(\alpha)| < 1$, so it converges (slowly, since 0.567 is not small), oscillating either side

E1

Answer: $g'(\alpha) = -\alpha \approx -0.567$

3 marks

(c) Numerator $(x_2 - x_1)^2 = 0.003757$

M1

Denominator $x_2 - 2x_1 + x_0 = -0.1678$

M1

Estimate = 0.56762, compared with $\alpha = 0.56714$: far more accurate than $x_4 = 0.56006$

A1 E1

Answer: Aitken estimate 0.56762, close to $\alpha = 0.56714$

4 marks

Newton-Raphson Method**1 (3 marks)**

$f(2.5) = 3.125$ and $f'(2.5) = 15.75$

B1

$x_1 = 2.5 - \frac{3.125}{15.75} = 2.3016$

M1 A1

Answer: 2.3016

3 marks

2 (3 marks)

$f'(x) = 3x^2 + 4$

B1

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

M1

$= x_n - \frac{x_n^3 + 4x_n - 9}{3x_n^2 + 4}$

A1

Answer: $x_{n+1} = x_n - \frac{x_n^3 + 4x_n - 9}{3x_n^2 + 4}$

3 marks

3 (3 marks)

$f'(x) = 2x$

B1

$x_1 = 3.3 - \frac{3.3^2 - 11}{2 \times 3.3} = 3.31667$

M1 A1

Answer: 3.3167

3 marks

4 (3 marks)

$f'(x) = e^x$

B1

$x_1 = 1.5 - \frac{e^{1.5} - 5}{e^{1.5}} = 1.61565$

M1 A1

Answer: 1.6157

3 marks

5 (2 marks)

$$x_1 = 2 - \frac{3}{12}$$

M1

$$x_1 = 1.75$$

A1

Answer: 1.75

2 marks

6 (3 marks)

x_1 is where the tangent at $(x_0, f(x_0))$ meets the x-axis

B1

The tangent has gradient $f'(x_0)$, so $0 - f(x_0) = f'(x_0)(x_1 - x_0)$, giving $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

M1

Near the root the tangent approximates the curve, so its x-intercept is closer to the root

E1

Answer: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

3 marks

7 (5 marks)

$$f'(x) = 3x^2 - 2$$

M1 A1

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 2 - \frac{-1}{10} = 2.10000$$

M1 A1

$$x_2 = 2.09457$$

A1

Answer: $x_1 = 2.1000$, $x_2 = 2.0946$

5 marks

8 (5 marks)

$$f'(x) = 1 + \frac{2}{x}$$

M1 A1

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 3 - \frac{0.1972}{1.6667} = 2.88167$$

M1 A1

$$x_2 = 2.88261$$

A1

Answer: $x_1 = 2.8817$, $x_2 = 2.8826$

5 marks

9 (5 marks)

$$f'(x) = -\sin x - 2$$

M1 A1

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 0.5 - \frac{-0.1224}{-2.4794} = 0.45063$$

M1 A1

$$x_2 = 0.45018$$

A1

Answer: $x_1 = 0.4506$, $x_2 = 0.4502$

5 marks

10 (5 marks)

$$f'(x) = 2x$$

B1

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n} = \frac{2x_n^2 - x_n^2 + 10}{2x_n}$$

M1

$$= \frac{x_n^2 + 10}{2x_n} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right)$$

A1

$$x_1 = \frac{1}{2}\left(3 + \frac{10}{3}\right) = 3.16667, x_2 = 3.16228$$

M1 A1

Answer: 3.16228

5 marks

11 (5 marks)

$$f'(x) = e^x - 3$$

M1 A1

$$x_1 = 0.610060, x_2 = 0.618997$$

M1

$$x_3 = 0.619061, x_4 = 0.619061$$

M1

$$\alpha = 0.6191 \text{ (4 d.p.)}$$

A1

Answer: 0.6191

5 marks

12 (5 marks)

$$f(x) = x^3 + 4x - 7, f'(x) = 3x^2 + 4$$

B1

$$x_1 = 1.2 - \frac{-0.472}{8.32} = 1.256731$$

M1 A1

$$x_2 = 1.255384, x_3 = 1.255383$$

A1

Iterates agree to 6 d.p., so the root is 1.255383

A1

Answer: 1.255383

5 marks

13 (8 marks)

$$(a) f'(x) = -\frac{1}{x^2}$$

B1

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = x_n + x_n^2 \left(\frac{1}{x_n} - a \right)$$

M1 A1

$$= x_n + x_n - a x_n^2 = x_n(2 - a x_n)$$

A1

Answer: shown

4 marks

$$(b) x_1 = 0.1(2 - 0.7) = 0.1300$$

M1 A1

$$x_2 = 0.141700, x_3 = 0.14284777$$

A1

$$\frac{1}{7} = 0.14285714: \text{ agrees to 6 d.p. after three steps}$$

E1

Answer: 0.13, 0.141700, 0.142848

4 marks

14 (8 marks)

$$(a) f'(x) = 3x^2 - 2$$

B1

$$x_1 = 0 - \frac{2}{-2} = 1$$

M1 A1

$$x_2 = 1 - \frac{1}{1} = 0$$

A1

Answer: $x_1 = 1, x_2 = 0$

4 marks

(b) The iterates alternate 0, 1, 0, 1, ...: a cycle that never converges

E1

The tangents at $x = 0$ and $x = 1$ lead back to each other, away from the root

E1

Answer: cycles between 0 and 1

2 marks

$$(c) x_1 = -2 - \frac{-2}{10} = -1.8$$

M1

$$x_2 = -1.8 - \frac{f(-1.8)}{f'(-1.8)} = -1.7699$$

A1

Answer: -1.8, -1.7699

2 marks

15 (5 marks)

$$(a) f'(x) = 3x^2 - 3, f'(1) = 0$$

M1

The formula would divide by zero: the tangent at $x = 1$ is horizontal and never meets the x-axis

E1

Answer: $f'(1) = 0$

2 marks

$$(b) x_1 = 2 - \frac{3}{9} = 1.66667$$

M1 A1

$$x_2 = 1.54861$$

A1

Answer: 1.6667, 1.5486

3 marks

16 (8 marks)**(a)** Box dimensions $(10 - 2x)$, $(8 - 2x)$, x

M1

$$x(10 - 2x)(8 - 2x) = 50 \text{ and expanding: } 4x^3 - 36x^2 + 80x = 50$$

M1 A1

Answer: shown

3 marks

(b) $f'(x) = 12x^2 - 72x + 80$

B1

$$x_1 = 1 - \frac{-2}{20} = 1.1000$$

M1 A1

$$x_2 = 1.115405, x_3 = 1.115774$$

M1

$$x = 1.116 \text{ (3 d.p.)}$$

A1

Answer: 1.116

5 marks

17 (8 marks)**(a)** $f(2) = 2 \ln 2 - 2 = -0.614 < 0$

M1

 $f(3) = 3 \ln 3 - 2 = 1.296 > 0$; sign change, f continuous

A1

Answer: shown

2 marks

(b) $f'(x) = \ln x + 1$

M1 A1

$$x_1 = 2.362464, x_2 = 2.345783, x_3 = 2.345751$$

M1 A1

$$f(2.34570) = -0.000093 < 0, f(2.34580) = 0.000093 > 0, \text{ so } \alpha = 2.3458 \text{ (4 d.p.)}$$

M1 A1

Answer: 2.3458

6 marks

18 (5 marks)

$$f'(x) = 2^x \ln 2$$

M1 A1

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 3 - \frac{-2}{5.5452} = 3.36067$$

M1 A1

$$x_2 = 3.32244$$

A1

Answer: $x_1 = 3.3607$, $x_2 = 3.3224$

5 marks

19 (8 marks)**(a)** $f(x) = x^2 - e^{-x}$

M1

$$f(0.7) = -0.0066 < 0 \text{ and } f(0.71) = 0.0125 > 0$$

M1 A1

Sign change, so $0.7 < \alpha < 0.71$

E1

Answer: shown

4 marks

(b) $f'(x) = 2x + e^{-x}$

B1

$$x_1 = 0.7 - \frac{f(0.7)}{f'(0.7)} = 0.703472$$

M1 A1

$$x_2 = 0.703467$$

A1

Answer: $x_1 = 0.703472$, $x_2 = 0.703467$

4 marks

20 (7 marks)**(a)** Tangent: $y - f(a) = f'(a)(x - a)$

M1

At the x -axis $y = 0$: $-f(a) = f'(a)(b - a)$

M1

$$b = a - \frac{f(a)}{f'(a)}$$

A1

Answer: shown

3 marks

$$\text{(b) } f'(a) = 2a, \text{ so } b = a - \frac{a^2 - k}{2a} = \frac{a^2 + k}{2a} = \frac{1}{2}\left(a + \frac{k}{a}\right)$$

M1 A1

$$b - \sqrt{k} = \frac{(a - \sqrt{k})^2}{2a} \geq 0$$

M1 A1

Answer: shown

4 marks

21 (8 marks)

(a) $f(x) = x \sin x - 1$

B1

$f'(x) = \sin x + x \cos x$ (product rule)

M1 A1

Substitute into $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

A1

Answer: shown

4 marks

(b) $x_1 = 1.114729$

M1

$x_2 = 1.114157, x_3 = 1.114157$

M1 A1

$\alpha = 1.1142$ (4 d.p.)

A1

Answer: 1.1142

4 marks

22 (7 marks)

(a) $f'(x) = \frac{1}{3}x^{-2/3}$

B1

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n$$

M1 A1

$= -2x_n$

A1

Answer: shown

4 marks

(b) $x_1 = -0.2, x_2 = 0.4, x_3 = -0.8$

M1 A1

The iterates double in size and alternate in sign: they diverge from the root. The tangent is nearly vertical at the root, so the method is unsuitable here

E1

Answer: diverges

3 marks

23 (9 marks)

(a) $f'(x) = 2x - 5$

B1

$$x_{n+1} = x_n - \frac{x_n^2 - 5x_n + 6}{2x_n - 5}$$

M1

$$= \frac{2x_n^2 - 5x_n - x_n^2 + 5x_n - 6}{2x_n - 5} = \frac{x_n^2 - 6}{2x_n - 5}$$

A1

Answer: shown

3 marks

(b) $x_0 = 4: x_1 = \frac{10}{3} = 3.3333, x_2 = 3.0667$

M1 A1

$$x_0 = 1: x_1 = \frac{-5}{-3} = 1.6667, x_2 = 1.9333$$

M1 A1

Answer: from 4 towards 3; from 1 towards 2

4 marks

(c) $x_{n+1} - 3 = \frac{x_n^2 - 6 - 3(2x_n - 5)}{2x_n - 5}$

M1

$$= \frac{x_n^2 - 6x_n + 9}{2x_n - 5} = \frac{(x_n - 3)^2}{2x_n - 5}$$

A1

Answer: shown

2 marks

24 (7 marks)

(a) $f(0.8) = -0.2196 < 0$

M1

$f(0.9) = 0.2136 > 0$; sign change and continuity

A1

Answer: shown

2 marks

(b) $f'(x) = e^x + xe^x = (x + 1)e^x$

M1 A1

$x_1 = 0.867879$

M1

$x_2 = 0.852783, x_3 = 0.8526055$

M1

root = 0.85261 (5 d.p.)

A1

Answer: 0.85261

5 marks

25 (11 marks)

(a) $f'(x) = 1 + e^{-x}$ B1

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{x_n + x_n e^{-x_n} - x_n + e^{-x_n}}{1 + e^{-x_n}}$$
 M1

$$= \frac{e^{-x_n}(x_n + 1)}{1 + e^{-x_n}}; \text{ multiply top and bottom by } e^{x_n}: \frac{x_n + 1}{e^{x_n} + 1}$$
 M1 A1

Answer: shown 4 marks

(b) Newton: 0.566311, 0.567143, 0.567143 M1 A1

Simple: 0.606531, 0.545239, 0.579703 M1 A1

Answer: Newton: 0.566311, 0.567143, 0.567143 4 marks

(c) Newton's errors: 8e-04, 1e-07, 3e-15 M1

Simple iteration errors: 4e-02, 2e-02, 1e-02 M1

Newton-Raphson converges far faster (quadratic against linear convergence) E1

Answer: Newton converges much faster 3 marks

26 (10 marks)

(a) $\frac{dy}{dx} = e^x + xe^x - 3$ M1 A1

Set equal to 0: $(x + 1)e^x - 3 = 0$ A1

Answer: shown 3 marks

(b) $f(x) = (x + 1)e^x - 3$, $f'(x) = (x + 2)e^x$ M1 A1

$$x_{n+1} = x_n - \frac{(x_n + 1)e^{x_n} - 3}{(x_n + 2)e^{x_n}}$$
 A1

Answer: $x_{n+1} = x_n - \frac{(x_n + 1)e^{x_n} - 3}{(x_n + 2)e^{x_n}}$ 3 marks

(c) $x_1 = 0.7 - \frac{f(0.7)}{f'(0.7)} = 0.62213$ M1 A1

$x_2 = 0.61766$, $x_3 = 0.61764$ A1

$\alpha = 0.6176$ A1

Answer: $x_1 = 0.62213$, $x_2 = 0.61766$, $x_3 = 0.61764$, $\alpha = 0.6176$ 4 marks

27 (9 marks)

(a) $f(x) = x^n - a$, $f'(x) = nx^{n-1}$ B1

$$x_{k+1} = x_k - \frac{x_k^n - a}{nx_k^{n-1}} = x_k - \frac{x_k}{n} + \frac{a}{nx_k^{n-1}}$$
 M1 A1

$$= \frac{1}{n}((n-1)x_k + \frac{a}{x_k^{n-1}})$$
 A1

Answer: shown 4 marks

(b) $x_{k+1} = \frac{1}{4}(3x_k + \frac{100}{x_k^3})$ M1

$x_1 = 3.175926$, $x_2 = 3.162365$ M1 A1

$x_3 = 3.1622777$, so $\sqrt[4]{100} = 3.16228$ (5 d.p.) M1 A1

Answer: 3.16228 5 marks

28 (9 marks)

(a) $x_{n+1} - \sqrt{2} = \frac{x_n^2 + 2}{2x_n} - \sqrt{2}$ M1

$$= \frac{x_n^2 - 2\sqrt{2}x_n + 2}{2x_n} = \frac{(x_n - \sqrt{2})^2}{2x_n}$$
 M1 A1

Answer: shown 3 marks

(b) $e_0 = 0.0858$, $e_1 = 0.00245$ M1 A1

$e_2 = 0.00000212$, $e_3 = 0.00000000000159$ A1

Answer: 0.0858, 0.00245, 0.00000212, 0.00000000000159 3 marks

(c) $e_{n+1} = \frac{e_n^2}{2x_n} \approx \frac{e_n^2}{2.83}$, so each error is about the square of the previous error

M1

Order of magnitude: 9×10^{-2} , 2×10^{-3} , 2×10^{-6} , 2×10^{-12} : the exponent roughly doubles each step

E1 A1

Answer: error is roughly squared each step

3 marks

29 (9 marks)

(a) $f'(x) = 2(x-1)(x+2) + (x-1)^2 = (x-1)(3x+3) = 3(x-1)(x+1)$

M1 A1

$$x_{n+1} - 1 = (x_n - 1) - \frac{(x_n - 1)(x_n + 2)}{3(x_n + 1)}$$

M1

$$= (x_n - 1) \frac{3(x_n + 1) - (x_n + 2)}{3(x_n + 1)} = \frac{(x_n - 1)(2x_n + 1)}{3(x_n + 1)}$$

A1

Answer: shown

4 marks

(b) x_1 to x_5 : 1.555556, 1.297907, 1.155390, 1.079562, 1.040288

M1 A1

Ratios of errors: 0.556, 0.536, 0.522, 0.512

M1

Answer: 1.555556, 1.297907, 1.155390, 1.079562, 1.040288; ratios 0.556, 0.536, 0.522, 0.512

3 marks

(c) As $x_n \rightarrow 1$, $\frac{2x_n + 1}{3(x_n + 1)} \rightarrow \frac{1}{2}$: the error roughly halves each step (linear convergence)

E1

At a repeated root $f'(\alpha) = 0$, so the tangents are nearly horizontal and the quadratic convergence is lost

E1

Answer: error only halves each step since $f'(\alpha) = 0$

2 marks

30 (9 marks)

(a) $V = 1000$, so $f(h) = 0$

B1

$f(6) = -95.22 < 0$

M1

$f(7) = 180.19 > 0$; sign change

A1

$f'(h) = \frac{\pi}{3}(60h - 3h^2) = \pi h(20 - h) > 0$ for $0 < h < 10$, so f is increasing: one root

E1

Answer: shown

4 marks

(b) $f'(h) = \pi h(20 - h)$

B1

$h_1 = 6.36971$

M1 A1

$h_2 = 6.35502$, $h_3 = 6.35501$

M1

$h = 6.36$ cm (2 d.p.)

A1

Answer: 6.36 cm

5 marks

Trapezium Rule

1 (3 marks)

$h = 1$

B1

$T = \frac{1}{2}[3 + 2 + 2(5 + 8 + 6)]$

M1

$= \frac{1}{2} \times 43 = 21.5$

A1

Answer: 21.5

3 marks

2 (3 marks)

$h = 2$

B1

$T = \frac{2}{2}[1.2 + 0.6 + 2(3.4 + 2.8)]$

M1

$= 14.2$

A1

Answer: 14.2

3 marks

3 (3 marks)

$$y = 1, 1.4142, 1.7321, 2, 2.2361$$

B1

$$h = 1; T = \frac{1}{2}[1 + 2.2361 + 2(1.4142 + 1.7321 + 2)]$$

M1

$$= 6.7643 = 6.76$$

A1

Answer: 6.76

3 marks

4 (2 marks)

$$h = \frac{2-1}{5} = 0.2$$

B1

$$x = 1, 1.2, 1.4, 1.6, 1.8, 2$$

B1

Answer: h = 0.2; x = 1, 1.2, 1.4, 1.6, 1.8, 2

2 marks

5 (3 marks)

$$h = 1; \text{ordinates } 1, 2, 4, 8$$

B1

$$T = \frac{1}{2}[1 + 8 + 2(2 + 4)]$$

M1

$$= 10.5$$

A1

Answer: 10.5

3 marks

6 (5 marks)

$$(a) h = 1, \text{ordinates } 0, 1, 4$$

B1

$$T = \frac{1}{2}[0 + 4 + 2(1)] = 3$$

M1 A1

Answer: 3

3 marks

$$(b) \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

B1

$$3 > \frac{8}{3}: \text{overestimate}$$

B1

Answer: $\frac{8}{3}$; overestimate

2 marks

7 (6 marks)

$$(a) h = 0.5; \text{ordinates } 1, 0.6667, 0.5, 0.4, 0.3333$$

B1

$$T = \frac{0.5}{2}[1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)]$$

M1

$$= 1.1167$$

A1

Answer: 1.1167

3 marks

$$(b) I = [\ln x]_1^3 = \ln 3 = 1.0986$$

B1

$$\text{Error} = \frac{1.1167 - 1.0986}{1.0986} \times 100 = 1.6\%$$

M1 A1

Answer: $\ln 3$; 1.6%

3 marks

8 (5 marks)

$$h = \frac{1-0}{4} = 0.25$$

B1

$$\text{Ordinates: } y_0 = 1.0000, y_1 = 1.0645, y_2 = 1.2840, y_3 = 1.7551, y_4 = 2.7183$$

M1 A1

$$T = \frac{0.25}{2}[1.0000 + 2.7183 + 2(1.0645 + 1.2840 + 1.7551)]$$

M1

$$= 1.4907 = 1.49 \text{ (3 s.f.)}$$

A1

Answer: 1.49

5 marks

9 (6 marks)

$$(a) h = \frac{\pi}{8}; \text{ordinates } 0, 0.3827, 0.7071, 0.9239, 1$$

B1 M1

$$T = \frac{\pi}{16}[0 + 1 + 2(0.3827 + 0.7071 + 0.9239)] = 0.9871$$

M1 A1

Answer: 0.9871

4 marks

$$(b) \text{Error} = \frac{1 - 0.9871}{1} \times 100 = 1.29\%$$

M1 A1

Answer: 1.29%**2 marks****10 (5 marks)**

$$h = \frac{6-2}{4} = 1$$

B1

$$\text{Ordinates: } y_0 = 0.6931, y_1 = 1.0986, y_2 = 1.3863, y_3 = 1.6094, y_4 = 1.7918$$

M1 A1

$$T = \frac{1}{2}[0.6931 + 1.7918 + 2(1.0986 + 1.3863 + 1.6094)]$$

M1

$$= 5.3368 = 5.34 \text{ (3 s.f.)}$$

A1**Answer: 5.34****5 marks****11 (5 marks)****(a)** $h = 1$; ordinates 0, 1, 1.4142, 1.7321, 2**B1**

$$T = \frac{1}{2}[0 + 2 + 2(1 + 1.4142 + 1.7321)] = 5.146$$

M1 A1**Answer: 5.146****3 marks****(b)** The curve is concave (bends downwards), so each chord lies below the curve**E1**

So each trapezium has less area than the region under the curve: underestimate

E1**Answer: curve is concave: underestimate****2 marks****12 (3 marks)**

$$h = 1; T = \frac{1}{2}[2 + 9 + 2(k + 7)]$$

M1

$$\frac{1}{2}(25 + 2k) = 17, \text{ so } 25 + 2k = 34$$

M1

$$k = 4.5$$

A1**Answer: k = 4.5****3 marks****13 (5 marks)**

$$h = 2$$

B1

$$A = \frac{2}{2}[0 + 0 + 2(1.2 + 2.1 + 1.6)]$$

M1

$$= 9.8 \text{ m}^2$$

A1

$$\text{Volume per second} = \text{area} \times \text{speed} = 9.8 \times 0.6 = 5.88 \text{ m}^3$$

M1

$$\text{In 30 s: } 176.4 \text{ m}^3$$

A1**Answer: area 9.8 m²; volume 176.4 m³****5 marks****14 (5 marks)**

$$h = 10; T = \frac{10}{2}[0 + 12 + 2(4.2 + 7.5 + 9.8 + 11 + 11.6)]$$

M1 A1

$$= 5 \times 100.2 = 501 \text{ m}$$

A1

$$\text{Mean speed} = \frac{501}{60} = 8.35 \text{ m/s}$$

M1 A1**Answer: 501 m; 8.35 m/s****5 marks****15 (9 marks)****(a)** $h = 0.5$; ordinates 4, 3.2, 2**B1**

$$T = \frac{0.5}{2}[4 + 2 + 2(3.2)] = 3.10$$

M1 A1**Answer: 3.10****3 marks****(b)** $h = 0.25$; ordinates 4, 3.7647, 3.2, 2.56, 2**B1**

$$T = \frac{0.25}{2}[4 + 2 + 2(3.7647 + 3.2 + 2.56)] = 3.1312$$

M1 A1**Answer: 3.1312****3 marks****(c)** $|T_4 - \pi| = 0.0104 < |T_2 - \pi| = 0.0416$ **M1**

$$\text{Error} = \frac{0.0104}{\pi} \times 100 = 0.33\%$$

M1 A1**Answer: 0.33%****3 marks**

16 (5 marks)

(a) $h = 0.5$

B1

$$T = \frac{0.5}{2}[1 + 2 + 2(3 + 2.5 + 4)] = 5.50$$

M1 A1

Answer: 5.50

3 marks

(b) Estimate $= 3T + 2 \times (3 - 1)$

M1

$$= 3 \times 5.50 + 4 = 20.50$$

A1

Answer: 20.50

2 marks

17 (4 marks)

Strips of widths 1, 2, 3

B1

$$\frac{1}{2}(1)(2 + 3.5) + \frac{1}{2}(2)(3.5 + 4) + \frac{1}{2}(3)(4 + 1)$$

M1 A1

$$= 2.75 + 7.5 + 7.5 = 17.75$$

A1

Answer: 17.75

4 marks

18 (8 marks)

(a) $h = 0.5$; ordinates 0, 0.3033, 0.3679, 0.3347, 0.2707

M1 A1

$$I \approx \frac{0.5}{2}[0 + 0.2707 + 2(0.3033 + 0.3679 + 0.3347)]$$

M1

$$= 0.571$$

A1

Answer: 0.571

4 marks

(b) $\int 1 \, dx = 2$ over the interval

B1

$$2 + 4(0.5706) = 4.28 \text{ (3 s.f.)}$$

A1

Answer: 4.28

2 marks

(c) The substitution $u = x - 3$ gives $\int_0^2 u e^{-u} \, du$

M1

Estimate is 0.571

A1

Answer: 0.571

2 marks

19 (7 marks)

(a) $I = [e^x]_0^1 = e - 1 = 1.7183$

M1 A1

Answer: $e - 1$

2 marks

(b) $h = 0.5$; $T = \frac{0.5}{2}[1 + e + 2e^{0.5}] = 1.7539$

M1 A1

$$T > I \text{ (1.7539 > 1.7183)}$$

B1

Answer: 1.7539

3 marks

(c) The second derivative $e^x > 0$, so the curve is convex

E1

Chords lie above a convex curve, so the trapezia have more area than the region under it

E1

Answer: convex curve: chords above

2 marks

20 (7 marks)

(a) $h = 2$; ordinates 5, 3.352, 2.247, 1.506, 1.010, 0.6767

B1 M1

$$V = \frac{2}{2}[5 + 0.6767 + 2(3.352 + 2.247 + 1.506 + 1.010)]$$

M1

$$= 21.9 \text{ litres}$$

A1

Answer: 21.9 litres

4 marks

(b) $V = \int_0^{10} 5e^{-0.2t} \, dt = [-25e^{-0.2t}]_0^{10} = 25(1 - e^{-2}) = 21.62$

M1 A1

The curve $y = 5e^{-0.2t}$ is convex (second derivative > 0), so the trapezium rule overestimates

E1

Answer: 21.62 litres; convex curve

3 marks

21 (7 marks)

(a) $h = 0.5$; ordinates 0.3679, 0.7788, 1, 0.7788, 0.3679

B1 M1

$$T = \frac{0.5}{2}[0.3679 + 0.3679 + 2(0.7788 + 1 + 0.7788)] = 1.4627$$

M1 A1

Answer: 1.4627

4 marks

(b) e^{-x^2} is an even function, so $I = 2 \int_0^1 e^{-x^2} \, dx$

E1

$$2 \times \frac{0.5}{2} [1 + 0.3679 + 2(0.7788)] = 1.4627, \text{ the same}$$

M1 A1

Answer: even function: I = 2 × integral on [0, 1]

3 marks

22 (8 marks)

(a) $y = \frac{1}{x}$: second derivative $\frac{2}{x^3} > 0$, convex, chord above the curve: overestimate

M1 A1

$y = \ln x$: second derivative $-\frac{1}{x^2} < 0$, concave, chord below the curve: underestimate

M1 A1

Answer: 1/x: overestimate; ln x: underestimate

4 marks

(b) $\frac{1}{x}: \frac{2}{2}(1 + \frac{1}{3}) = \frac{4}{3} = 1.3333$

M1

$\ln x: \frac{2}{2}(0 + \ln 3) = 1.0986$

A1

Error: $\frac{1.3333 - 1.0986}{1.0986} \times 100 = 21.4\%$

M1 A1

Answer: 1.3333; 1.0986; 21.4%

4 marks

23 (8 marks)

(a) $u = x, \frac{d v}{d x} = e^x: I = [x e^x]_0^1 - \int_0^1 e^x d x$

M1 A1

$= e - (e - 1) = 1$

A1

Answer: I = 1

3 marks

(b) $h = 0.25$; ordinates 0, 0.3210, 0.8244, 1.5888, 2.7183

B1 M1

$T = \frac{0.25}{2} [0 + 2.7183 + 2(0.3210 + 0.8244 + 1.5888)] = 1.0231$

M1 A1

Error = 2.3%

A1

Answer: 1.0231; 2.3%

5 marks

24 (7 marks)

(a) $h = 2$

B1

$V = \frac{2}{2} [0 + 0 + 2(1.8 + 3.1 + 3.4 + 2.2)]$

M1

$= 21.0 \text{ m}^3$

A1

Answer: 21.0 m³

3 marks

(b) Mass = $21.0 \times 400 = 8400 \text{ kg}$

M1 A1

Mean area = $\frac{21.0}{10} = 2.10 \text{ m}^2$

A1

Answer: 8400 kg; 2.10 m²

3 marks

(c) Narrower strips fit the curve more closely, so the chords deviate less from the true cross-section profile and the error in the area falls

E1

Answer: narrower strips: smaller error

1 mark

25 (9 marks)

(a) $h = 0.5$; ordinates $a, \frac{2a}{3}, \frac{a}{2}, \frac{2a}{5}, \frac{a}{3}$

B1

$\frac{0.5}{2} [a + \frac{a}{3} + 2(\frac{2a}{3} + \frac{a}{2} + \frac{2a}{5})] = 3.35$

M1

$\frac{67a}{60} = 3.35$

A1

$a = 3$

A1

Answer: a = 3

4 marks

(b) $\int \frac{3}{1+x} dx = 3 \ln 3$

M1 A1

$3 \ln 3 = 3.296 < 3.35$

B1

The curve is convex (concave up), so the chords lie above it

E1

Answer: 3 ln 3 = 3.296; the curve is convex

4 marks

(c) Use more strips of smaller width

B1

Answer: More strips

1 mark

26 (11 marks)

(a) $h = \frac{1}{n}$, ordinates $y_r = (\frac{r}{n})^2$, $y_0 = 0$, $y_n = 1$

B1

$$T_n = \frac{h}{2} [0 + 1 + 2 \sum_{r=1}^{n-1} (\frac{r}{n})^2]$$

M1

$$= \frac{1}{n} [\frac{1}{2} + \sum_{r=1}^{n-1} (\frac{r}{n})^2]$$

A1

Answer: shown

3 marks

(b) $\sum_{r=1}^{n-1} r^2 = \frac{(n-1)n(2n-1)}{6}$

M1

$$T_n = \frac{1}{n} [\frac{1}{2} + \frac{(n-1)(2n-1)}{6n}]$$

M1

$$= \frac{1}{n} \times \frac{3n + 2n^2 - 3n + 1}{6n} = \frac{2n^2 + 1}{6n^2}$$

M1 A1

$$= \frac{1}{3} + \frac{1}{6n^2}$$

A1

Answer: shown

5 marks

(c) Error = $\frac{1}{6n^2} < 0.001$

M1

$$n^2 > \frac{1000}{6} = 166.7, \text{ so } n > 12.9$$

M1

$$n = 13$$

A1

Answer: n = 13

3 marks

27 (12 marks)

(a) Ordinates $\ln 1, \ln 2, \dots, \ln n$ with $h = 1$

B1

$$T = \frac{1}{2} [\ln 1 + \ln n + 2(\ln 2 + \dots + \ln(n-1))]$$

M1

$$= \ln 2 + \dots + \ln(n-1) + \frac{1}{2} \ln n = \ln(n!) - \ln n + \frac{1}{2} \ln n = \ln(n!) - \frac{1}{2} \ln n$$

M1 A1

Answer: shown

4 marks

(b) $\int \ln x \, dx = x \ln x - \int 1 \, dx = x \ln x - x$

M1 A1

$$\text{Limits: } (n \ln n - n) - (0 - 1) = n \ln n - n + 1$$

A1

Answer: shown

3 marks

(c) Concave: chords below the curve, so $T < \text{integral}$

E1

$$\ln(n!) - \frac{1}{2} \ln n < n \ln n - n + 1, \text{ giving the result}$$

M1 A1

$$n = 6: \ln 720 = 6.5793 < 6.5 \ln 6 - 5 = 6.6464$$

M1 A1

Answer: shown

5 marks

28 (14 marks)

(a) The end ordinates are $\sin 0 = \sin \pi = 0$

B1

$$T_n = \frac{h}{2} \times 2 \sum_{r=1}^{n-1} \sin(rh) = h \sum_{r=1}^{n-1} \sin(rh)$$

M1 A1

$$T_4 = \frac{\pi}{4} (0.7071 + 1 + 0.7071) = 1.8961$$

M1 A1

Answer: 1.8961

5 marks

(b) $2 \sin(\frac{h}{2}) \sin(rh) = \cos((r - \frac{1}{2})h) - \cos((r + \frac{1}{2})h)$

M1 A1

$$\text{The sum telescopes: } \cos(\frac{h}{2}) - \cos((n - \frac{1}{2})h)$$

M1

$$(n - \frac{1}{2})h = \pi - \frac{h}{2}, \text{ so } \cos((n - \frac{1}{2})h) = -\cos(\frac{h}{2}); \text{ sum} = \frac{2\cos(\frac{h}{2})}{2\sin(\frac{h}{2})} = \cot(\frac{h}{2})$$

M1 A1

Answer: shown

5 marks

$$(c) T_4 = \frac{\pi}{4} \cot\left(\frac{\pi}{8}\right) = 1.8961, \text{ matching}$$

M1 A1

$$\text{For small } h, \cot\left(\frac{h}{2}\right) \approx \frac{2}{h}, \text{ so } T_n \approx h \times \frac{2}{h} = 2 = \text{exact value of } I$$

M1 A1

Answer: $T_n \rightarrow 2$

4 marks

29 (9 marks)

$$(a) T_2 = \frac{0.5}{2}[4 + 2 + 2(3.2)] = 3.1000$$

M1 A1

$$T_4 = 3.13118$$

A1

Answer: 3.1000, 3.13118

3 marks

$$(b) \text{ Subtract: } T_4 - T_2 = \frac{c}{16} - \frac{c}{4} = -\frac{3c}{16}, \text{ so } c = \frac{16(T_2 - T_4)}{3}$$

M1 A1

$$I \approx T_4 - \frac{c}{16} = T_4 - \frac{T_2 - T_4}{3} = \frac{4T_4 - T_2}{3}$$

M1 A1

Answer: shown

4 marks

$$(c) \frac{4(3.13118) - 3.1000}{3} = 3.14157$$

M1

$$\pi = 3.14159; \text{ the error is now about } 0.00002, \text{ much smaller than for } T_4 \text{ (about } 0.01)$$

A1

Answer: 3.14157

2 marks

30 (13 marks)

$$(a) h = \frac{1}{n}; \text{ ordinates } 1, q, q^2, \dots, q^n = 3$$

B1

$$T_n = \frac{1}{2n}[1 + 3 + 2(q + q^2 + \dots + q^{n-1})]$$

M1

$$q + \dots + q^{n-1} = \frac{q(q^{n-1} - 1)}{q - 1} = \frac{3 - q}{q - 1}$$

M1 A1

$$T_n = \frac{1}{2n} \times \frac{4(q - 1) + 2(3 - q)}{q - 1} = \frac{1}{n} \times \frac{q + 1}{q - 1}$$

M1 A1

Answer: shown

6 marks

$$(b) q = \sqrt{3}, T_2 = \frac{1}{2} \times \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

M1

$$\text{Rationalise the denominator: } = \frac{1}{2} \times \frac{(\sqrt{3} + 1)^2}{2}$$

M1

$$T_2 = 1 + \frac{\sqrt{3}}{2}$$

A1

$$\text{Answer: } T_2 = 1 + \frac{\sqrt{3}}{2}$$

3 marks

$$(c) \frac{2}{\ln 3} = 1.8205$$

B1

$$T_2 = 1.8660$$

B1

$$\text{Error} = \frac{1.8660 - 1.8205}{1.8205} \times 100 = 2.5\%$$

M1 A1

Answer: 2.5%

4 marks

Convergence and Error**1 (3 marks)**

$$\text{Absolute error} = \left| \frac{22}{7} - \pi \right| = 0.00126 = 0.0013 \text{ (2 s.f.)}$$

M1 A1

$$\text{Percentage error} = \frac{0.00126}{\pi} \times 100 = 0.040\%$$

A1

Answer: 0.0013; 0.040%

3 marks

2 (3 marks)

$$2.65 \leq \alpha < 2.75$$

B1 B1

Maximum error 0.05

B1

Answer: $2.65 \leq \alpha < 2.75$; 0.05

3 marks

3 (3 marks)

(a) Terms alternate either side of about 0.667 and get closer: converges

B1

(b) Increases without bound: diverges

B1

(c) Alternates between two values: oscillates without a limit

B1

Answer: (a) converges, (b) diverges, (c) oscillates

3 marks

4 (4 marks)

 $|0.3| < 1$: converges

B1

 $|-0.8| < 1$: converges (oscillating)

B1

 $|1.5| > 1$: diverges

B1

 $|-2| > 1$: diverges (oscillating)

B1

Answer: (a) converges, (b) converges, (c) diverges, (d) diverges

4 marks

5 (2 marks)

 $f(1.75)$ and $f(1.85)$ have opposite signs, so (f continuous) $1.75 < \alpha < 1.85$

M1

Every number in this interval rounds to 1.8 to 1 d.p.

E1

Answer: root in (1.75, 1.85)

2 marks

6 (2 marks)

Error = 0.4

M1

$$\text{Percentage error} = \frac{0.4}{25} \times 100 = 1.6\%$$

A1

Answer: 1.6%

2 marks

7 (3 marks)

Need the root in (2.195, 2.205)

M1

$$f(2.195) = -0.0344$$

M1

 $f(2.205) = 0.1308$: sign change, so root = 2.20 (2 d.p.)

A1

Answer: shown

3 marks

8 (6 marks)

(a) $x^3 = x^2 + 2$, divide by x^2 : $x = 1 + \frac{2}{x^2}$

M1

$$x_1 = 1.8889, x_2 = 1.5606, x_3 = 1.8212$$

M1 A1

Answer: 1.8889, 1.5606, 1.8212

3 marks

(b) $g'(x) = -4x^{-3}$, $g'(\alpha) = -0.820$

M1 A1

 $-1 < g'(\alpha) < 0$: converges, alternating; but $|g'|$ is close to 1, so the error shrinks slowly

E1

Answer: $g'(\alpha) = -0.820$

3 marks

9 (6 marks)

(a) (A): staircase; terms increase steadily towards the root and converge

B1 B1

(B): cobweb (spiral); terms alternate either side of the root and converge

B1 B1

Answer: (A) staircase, (B) cobweb; both converge

4 marks

(b) (A): $g'(\alpha) > 0$ (staircase) and $0 < g'(\alpha) < 1$

B1

(B): $g'(\alpha) < 0$ (cobweb) and $-1 < g'(\alpha) < 0$

B1

Answer: (A) $0 < g' < 1$; (B) $-1 < g' < 0$

2 marks

10 (4 marks)

Differences: 0.08740, -0.02347, 0.00637

M1 A1

Ratios: -0.269, -0.271

M1

Constant ratio ≈ -0.27 ($|\text{ratio}| < 1$, negative): linear convergence with alternating terms

E1

Answer: ratio ≈ -0.27 : linear convergence

4 marks

11 (3 marks)

After n bisections the width is 2^{-n}

M1

$2^{-n} < 0.001$, so $2^n > 1000$

M1

$2^{10} = 1024$, so $n = 10$

A1

Answer: 10

3 marks

12 (5 marks)

$T_2 = \frac{0.5}{2}[0 + 1 + 2(0.125)] = 0.3125$, error 25.0%

M1 A1

$T_4 = 0.265625$, error 6.25%

M1 A1

The error falls to about a quarter

E1

Answer: 25%, 6.25%: error falls by a factor of 4

5 marks

13 (5 marks)

$a \in [4.365, 4.375]$, $b \in [1.515, 1.525]$

B1

$a - b$: least $4.365 - 1.525 = 2.84$, greatest $4.375 - 1.515 = 2.86$

M1 A1

$\frac{a}{b}$: least $\frac{4.365}{1.525} = 2.862$, greatest $\frac{4.375}{1.515} = 2.888$

M1 A1

Answer: (a) 2.84 to 2.86; (b) 2.862 to 2.888

5 marks

14 (4 marks)

$x_1 = 1.60944$, $x_2 = 1.68465$, $x_3 = 1.67060$

M1

$x_4 = 1.67324$, $x_5 = 1.67274$

M1

Differences $|x_5 - x_4| = 0.00050 < 0.0005$

M1

5 iterations; $x_5 = 1.6727$

A1

Answer: 5 iterations; 1.6727

4 marks

15 (8 marks)

(a) A: 1.500000, 1.400000, 1.416667

M1 A1

B: 1.500000, 1.416667, 1.414216

M1 A1

Answer: A: 1.500000, 1.400000, 1.416667; B: 1.500000, 1.416667, 1.414216

4 marks

(b) A: ratios -0.166 , -0.173 , -0.171

M1 A1

B: $e_n = 0.086$, 0.0025 , 0.0000021 , so $e_{n+1} \approx 0.35e_n^2$

M1 A1

Answer: A linear (ratio ≈ -0.17), B quadratic

4 marks

16 (4 marks)

$x_1 = 0.5$, $x_2 = 2$, $x_3 = 0.5$, $x_4 = 2$

M1 A1

$g'(x) = -\frac{1}{x^2}$, so $g'(1) = -1$

M1

$|g'(\alpha)| = 1$ is the borderline case: the terms neither approach nor leave 1; they cycle between 2 and 0.5

E1

Answer: cycles 2, 0.5, 2, 0.5; $g'(1) = -1$

4 marks

17 (7 marks)

(a) $x_1 = 1.680$, $x_2 = 1.848$

M1 A1

$x_3 = 1.613$, $x_4 = 1.942$

A1

Answer: 1.680, 1.848, 1.613, 1.942

3 marks

(b) $\alpha = 4.2 - 1.4\alpha$, so $\alpha = \frac{4.2}{2.4} = 1.75$

M1 A1

$g' = -1.4$, $|g'| > 1$: errors multiply by -1.4 each step

M1

The terms alternate either side and move outwards: a diverging cobweb

E1

Answer: $\alpha = 1.75$; diverging cobweb

4 marks

18 (6 marks)

(a) $A = \pi r^2$: least $\pi(4.45)^2 = 62.2$

M1 A1

greatest $\pi(4.55)^2 = 65.0$

A1

Answer: 62.2 to 65.0 cm²

3 marks

$$(b) A = \pi(4.5)^2 = 63.62$$

B1

$$\text{Max error} = 1.42, \text{ percentage} = \frac{1.42}{63.62} \times 100 = 2.2\%$$

M1 A1

Answer: 2.2%

3 marks

19 (4 marks)

$$f(x) = x^3 - x^2 - 1$$

B1

$$f(1.46555) = -0.000075$$

M1

$$f(1.46565) = 0.000277$$

M1

Sign change, so $1.46555 < \text{root} < 1.46565$ and the root is 1.4656 (4 d.p.)

A1

Answer: shown

4 marks

20 (10 marks)

$$(a) f(1) = -3 < 0 \text{ and } f(2) = 3 > 0$$

B1

$$f(1.5) = -1.125 < 0, \text{ so } \alpha \in [1.5, 2]$$

M1 A1

$$f(1.75) = 0.609 > 0, \text{ so } \alpha \in [1.5, 1.75]$$

A1

Answer: $\alpha \in [1.5, 1.75]$

4 marks

$$(b) \text{ After } n \text{ bisections the interval has width } \frac{1}{2^n}$$

M1

$$\text{Midpoint error} \leq \text{half the width} = \frac{1}{2^7}$$

M1

$$= \frac{1}{128} = 0.0078 \text{ (2 s.f.)}$$

A1

Answer: $\frac{1}{128} = 0.0078$

3 marks

$$(c) \frac{1}{2^{n+1}} < 0.0005, \text{ so } 2^{n+1} > 2000$$

M1

$$2^{11} = 2048 > 2000 \text{ but } 2^{10} = 1024 < 2000$$

M1

$$n + 1 = 11, \text{ so } n = 10$$

A1

Answer: 10 bisections

3 marks

21 (6 marks)

$$(a) N'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}$$

M1 A1

$$= \frac{[f'(x)]^2 - [f'(x)]^2 + f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

A1

Answer: shown

3 marks

$$(b) f(\alpha) = 0 \text{ and } f'(\alpha) \neq 0, \text{ so } N'(\alpha) = 0$$

M1 A1

The error $e_{n+1} \approx N'(\alpha)e_n$ would be zero to first order, so the convergence is faster than linear (quadratic)

E1

Answer: $N'(\alpha) = 0$: faster than linear

3 marks

22 (6 marks)

$$(a) \text{ Width after } n \text{ steps} = 2^{-n}$$

M1

$$2^{-10} = \frac{1}{1024} < 0.001, \text{ while } 2^{-9} = \frac{1}{512} > 0.001$$

M1 A1

Answer: 10 steps

3 marks

$$(b) \text{ Errors: } 0.023, 0.00048, 0.00000021$$

M1 A1

After 3 Newton iterations the root is correct to about 6 d.p., while 10 bisections give only about 3 d.p.: Newton is far more efficient

E1

Answer: Newton: far fewer iterations

3 marks

23 (9 marks)

$$(a) \text{ Divide } x^3 = 6x + 4 \text{ by } x^2: x = \frac{6}{x} + \frac{4}{x^2}$$

M1 A1

Answer: shown

2 marks

$$(b) (A): g' = \frac{x^2}{2}, g'(\alpha) = 3.73 > 1$$

M1 A1

$$(B): g' = \frac{2}{(6x+4)^{2/3}}, g'(\alpha) = 0.268 < 1$$

M1 A1

$$(C): g' = -\frac{6}{x^2} - \frac{8}{x^3}, |g'(\alpha)| = 1.20 > 1$$

M1 A1

Only (B) converges

A1

Answer: only (B)

7 marks

24 (8 marks)

$$(a) f' = 2x, f'' = 2: \frac{2}{2(2\alpha)} = \frac{1}{2\alpha}$$

M1 A1

$$\alpha = \sqrt{3}, \text{ so } \frac{1}{2\sqrt{3}} = 0.2887$$

A1

Answer: shown

3 marks

$$(b) e_0 = 2 - \sqrt{3} = 0.26795$$

B1

$$\text{Predicted } e_1 \approx 0.2887 \times (0.26795)^2 = 0.02073$$

M1 A1

$$\text{Actual } x_1 = 1.75000, e_1 = 0.01795: \text{ close to the prediction}$$

M1 A1

Answer: predicted 0.02073; actual 0.01795

5 marks

25 (10 marks)

$$(a) f''(x) = e^x, \text{ so } M = e \text{ on } [0, 1]$$

M1 A1

$$|\text{error}| \leq \frac{1 \times 0.25^2}{12} \times e = 0.01416$$

M1 A1

Answer: |error| ≤ 0.0142

4 marks

$$(b) T = \frac{0.25}{2} [1 + e + 2(e^{0.25} + e^{0.5} + e^{0.75})] = 1.72722$$

M1 A1

$$\text{Actual error} = 1.72722 - 1.71828 = 0.00894 < 0.01416$$

M1 A1

Answer: actual error 0.00894 ≤ 0.01416

4 marks

$$(c) \frac{e}{12n^2} < 0.0001, \text{ so } n^2 > 2265.2$$

M1

$$n > 47.59, \text{ so } n = 48$$

A1

Answer: 48 strips

2 marks

26 (9 marks)

$$(a) \text{ Ratios } \frac{0.05}{0.1} = \frac{0.025}{0.05} = \frac{0.0125}{0.025} = 0.5$$

M1 A1

Error proportional to the previous error: first order (linear) convergence

E1

Answer: ratio 0.5: linear

3 marks

$$(b) \frac{0.01}{0.1^2} = \frac{0.0001}{0.01^2} = \frac{10^{-8}}{0.0001^2} = 1$$

M1 A1

Error proportional to the square of the previous error: second order (quadratic) convergence

E1

Answer: ratio 1: quadratic

3 marks

$$(c) P: 0.0125 \times 0.5 = 0.00625$$

B1

$$Q: (10^{-8})^2 = 10^{-16}$$

B1

Method Q: its error falls far faster

E1

Answer: P: 0.00625; Q: 10⁻¹⁶; choose Q

3 marks

27 (10 marks)

$$(a) f(\alpha + e) = f(\alpha) + f'(\alpha)e + \frac{1}{2}f''(\alpha)e^2 + \dots, \text{ with } f(\alpha) = 0$$

M1 A1

$$f'(\alpha + e) = f'(\alpha) + f''(\alpha)e + \dots$$

A1

Answer: shown

3 marks

$$(b) e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} \approx e_n - \frac{f'e + \frac{1}{2}f''e^2}{f' + f''e}$$

M1

$$= \frac{e(f' + f''e) - f'e - \frac{1}{2}f''e^2}{f' + f''e} = \frac{\frac{1}{2}f''e^2}{f' + f''e}$$

M1 A1

$$\approx \frac{f''(\alpha)}{2f'(\alpha)} e^2 \text{ for small } e$$

A1

Answer: shown

4 marks

(c) $f' = 3x^2$, $f'' = 6x$: $\frac{6\alpha}{6\alpha^2} = \frac{1}{\alpha} = 0.7937$

M1 A1

$$e_1 \approx 0.7937 \times 10^{-4} = 7.94e-05$$

A1

Answer: constant $1/\alpha = 0.7937$; $e_1 \approx 7.9 \times 10^{-5}$

3 marks

28 (9 marks)

(a) $\cos$ is decreasing on $[0, 1]$

M1

$\cos 1 = 0.5403 > 0$, so $\cos x \in [0.5403, 1] \subset [0, 1]$

A1

Answer: shown

2 marks

(b) $g'(x) = -\sin x$

B1

On $[0, 1]$, $0 \leq \sin x \leq \sin 1 = 0.8415 < 1$

M1 A1

Answer: shown

3 marks

(c) Since $|g'| \leq k < 1$ the error is multiplied by at most k each step: convergence to a fixed point

E1

If there were two fixed points, the mean value theorem would give $|\alpha - \beta| = |g'(c)||\alpha - \beta| < |\alpha - \beta|$, a contradiction: unique

E1

Iterating from $x_0 = 1$: terms settle at 0.7391 (4 d.p.)

M1 A1

Answer: 0.7391

4 marks

29 (10 marks)

(a) $g'(x) = 1 - \frac{2x}{m}$

M1 A1

At $\alpha = \sqrt{2}$: $1 - \frac{2\sqrt{2}}{m}$

A1

Answer: shown

3 marks

(b) Need $|1 - \frac{2\sqrt{2}}{m}| < 1$

M1

$-1 < 1 - \frac{2\sqrt{2}}{m} < 1$: the right-hand inequality holds for $m > 0$; the left gives $\frac{2\sqrt{2}}{m} < 2$, so $m > \sqrt{2}$

M1 A1

Answer: $m > \sqrt{2}$

3 marks

(c) $x_1 = 1.33333$, $x_2 = 1.40741$, $x_3 = 1.41381$

M1 A1

$g'(\alpha) = 1 - \frac{2.828}{3} = 0.057$ for $m = 3$, but $1 - \frac{2.828}{6} = 0.529$ for $m = 6$

M1

The smaller $|g'(\alpha)|$ (closer to 0) means a faster reduction in error

E1

Answer: $m = 3$: $|g'| = 0.057$; $m = 6$: $|g'| = 0.529$

4 marks

30 (10 marks)

(a) $x_{n+1} - \alpha = g(x_n) - g(\alpha) = g'(c)(x_n - \alpha)$ for some c , so $|x_{n+1} - \alpha| \leq k|x_n - \alpha|$

M1 A1

$|x_n - \alpha| \leq |x_n - x_{n+1}| + |x_{n+1} - \alpha| \leq |x_{n+1} - x_n| + k|x_n - \alpha|$

M1

Hence $(1 - k)|x_n - \alpha| \leq |x_{n+1} - x_n|$

A1

Answer: shown

4 marks

(b) Apply the second result of (a) with $n - 1$ in place of n : $(1 - k)|x_{n-1} - \alpha| \leq |x_n - x_{n-1}|$

M1

So $|x_n - \alpha| \leq k|x_{n-1} - \alpha| \leq \frac{k}{1 - k}|x_n - x_{n-1}|$

A1

Answer: shown

2 marks

(c) $|x_4 - x_3| = 0.001724$

M1

Bound $= \frac{0.3}{0.7} \times 0.001724 = 0.000739$

M1 A1

Actual error $|x_4 - \alpha| = 0.000368 < \text{bound}$

A1

Answer: bound 0.000739; actual 0.000368

4 marks