

Locating Roots and Sign Changes

- 1 $f(x) = x^3 + x - 3$.
Show that the equation $f(x) = 0$ has a root between 1 and 2. [2]

- 2 Show that the equation $\ln x + x - 3 = 0$ has a root α in the interval $2 < \alpha < 3$. [2]

- 3 $f(x) = x^3 - 3x - 5$. Show that $f(x) = 0$ has a root between 2 and 3. [2]

- 4 Show that the equation $2^x = 5$ has a root between 2 and 3. [2]

- 5 By considering $f(x) = x^2 - 7$, show that $\sqrt{7}$ lies between 2.6 and 2.7. [2]

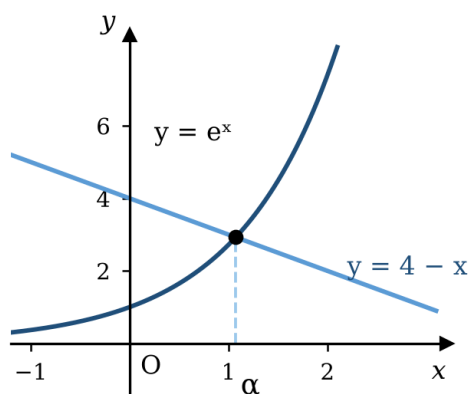
- 6 The table gives values of a continuous function f . Write down the interval of length 1 in which $f(x) = 0$ must have a root, and give a reason.

$x = 1$	$x = 2$	$x = 3$	$x = 4$
5	2	-1	-3

[2]

- 7 $f(x) = x^3 - 4x + 1$.
(a) Show that $f(x) = 0$ has a root α between 1 and 2. [2]
(b) Use interval bisection twice, starting with the interval $[1, 2]$, to find an interval of width 0.25 that contains α . [3]

- 8 The diagram shows the curves $y = e^x$ and $y = 4 - x$. They intersect at one point, with x -coordinate α .



- (a) Show that α is a root of the equation $e^x + x - 4 = 0$. [2]
(b) Show that $1 < \alpha < 1.5$. [2]

- 9 Show that the equation $\cos x = 2x$, where x is in radians, has a root between 0.4 and 0.5. [2]

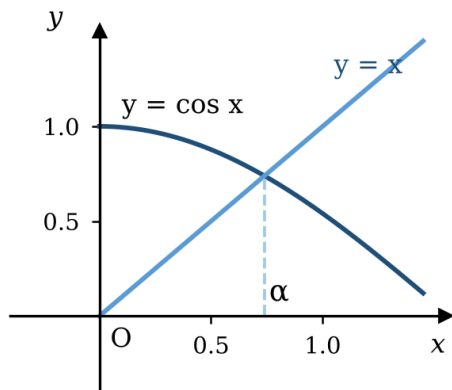
- 10 Show that the root of $x^3 - x - 1 = 0$ is 1.32 correct to 2 decimal places. [3]

- 11 $f(x) = x^3 + 2x - 9$. Show that $f(x) = 0$ has a root in $(1, 2)$. Use interval bisection three times to find an interval of width 0.125 containing the root. [4]

- 12** A student states: 'If $f(a)$ and $f(b)$ have opposite signs then $f(x) = 0$ has a root between a and b .'

- (a) Show that for $f(x) = \frac{1}{x-2}$, $f(1)$ and $f(3)$ have opposite signs, but $f(x) = 0$ has no root between 1 and 3. [3]
- (b) Show that $f(x) = (x-2)^2$ has a root at $x = 2$ even though $f(1)$ and $f(3)$ have the same sign. [2]
- (c) Complete the student's statement so that it is always true. [2]

- 13** The diagram shows the graphs of $y = \cos x$ and $y = x$, where x is in radians. They meet at the point with x -coordinate α .

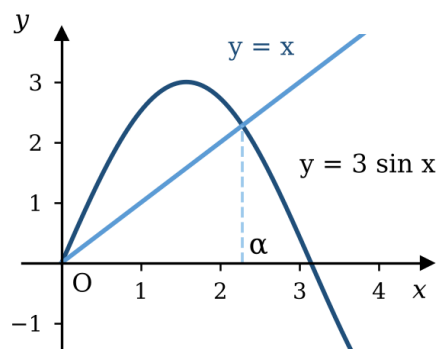


- (a) Show that $0.7 < \alpha < 0.8$. [2]
- (b) Use linear interpolation on $f(x) = \cos x - x$ between $x = 0.7$ and $x = 0.8$ to estimate α , giving your answer to 3 decimal places. [3]
- (c) By evaluating $f(0.735)$ and $f(0.745)$, show that $\alpha = 0.74$ to 2 decimal places. [2]

- 14** A student argues: ' $f(x) = \tan x$ has $f(1) > 0$ and $f(2) < 0$, so $\tan x = 0$ has a root between 1 and 2.'

- (a) Calculate $f(1)$ and $f(2)$ (radians) and explain why the student's conclusion is wrong. [3]
- (b) Show that $\tan x = 0$ has a root between 3 and 4. [3]

- 15** The diagram shows the curves $y = 3 \sin x$ and $y = x$, for x in radians, which meet at the origin and at the point with x -coordinate α . Show that $2 < \alpha < 3$.



[3]

- 16** $f(x) = x^4 - 3x^2 + x - 1$. Show that $f(x) = 0$ has a root in each of the intervals $(-2, -1)$ and $(1, 2)$. [4]

- 17** A solid cylinder has radius r cm and height $(r + 4)$ cm. Its volume is 100 cm^3 .

- (a) Show that r satisfies $\pi r^3 + 4\pi r^2 - 100 = 0$. [2]
- (b) Show that r lies between 2 and 3. [2]
- (c) Show that $r = 2.3$ to 1 decimal place. [2]

- 18** Show that the equation $x^3 - 3x + 1 = 0$ has three real roots, and write down an interval of length 1 containing each. [4]

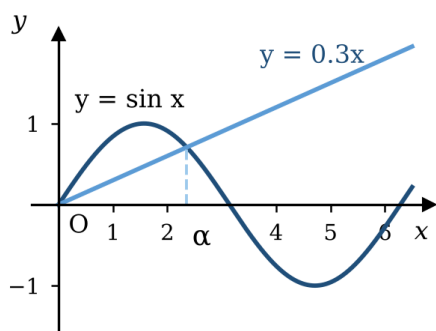
- 19** The equation $e^x = 6 - x^2$ has roots α and β , where $\alpha < \beta$.
- (a) By sketching the graphs of $y = e^x$ and $y = 6 - x^2$ on the same axes, show that the equation has exactly two real roots. [3]
- (b) Show that α lies in the interval $(-2.5, -2.4)$ and that β lies in the interval $(1.3, 1.4)$. [5]

- 20** $f(x) = x^3 - 6x + 2$.
- (a) Show that the equation $f(x) = 0$ has a root in each of the intervals $(-3, -2)$, $(0, 1)$ and $(2, 3)$. [4]
- (b) Explain why these are all of the real roots of the equation. [1]
- (c) Find the root in $(0, 1)$ correct to 2 decimal places, justifying your answer. [3]

- 21** $f(x) = x^3 - kx - 5$, where k is a constant.
- (a) Given that $f(x) = 0$ has a root between $x = 1$ and $x = 2$ because of a change of sign, find the range of possible values of k . [5]
- (b) For $k = 1$, show that the root of $f(x) = 0$ lies in $(1.9, 1.95)$ and hence give it to 1 decimal place. [3]

- 22** $f(x) = e^x - 3x$. The equation $f(x) = 0$ has two roots, α and β , with $\alpha < \beta$.
- (a) Show that $0.6 < \alpha < 0.7$ and $1.5 < \beta < 1.6$. [4]
- (b) Show that $\alpha = 0.62$ correct to 2 decimal places. [2]
- (c) Explain why $f(0)$ and $f(2)$ alone would not show that f has two roots. [2]

- 23** The diagram shows part of the graphs of $y = \sin x$ (x in radians) and $y = 0.3x$. They meet at the origin and at the point where $x = \alpha$, with $\alpha > 0$.

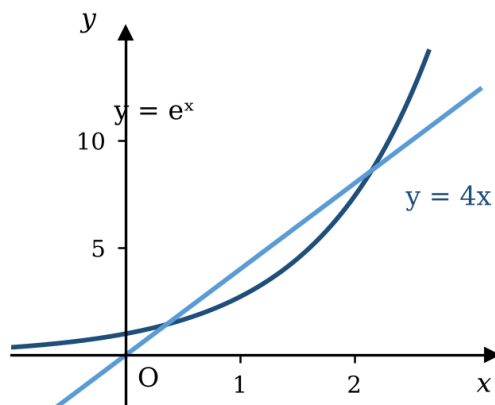


- (a) Show that α is a root of $\sin x - 0.3x = 0$ and that $2.3 < \alpha < 2.4$. [3]
- (b) Explain, using the diagram, why there is no other root with $x > 0$. [2]
- 24** $f(x) = x^5 + x - 1$. The equation $f(x) = 0$ has a root α .
- (a) Show that $0.7 < \alpha < 0.8$. [2]
- (b) Use linear interpolation on the interval $[0.7, 0.8]$ to estimate α , to 3 decimal places. [4]
- (c) The true root is 0.7549 to 4 decimal places. Find the error in your estimate. [1]

- 25** $g(x) = \tan x - 2x$, where x is in radians.
- (a) Show that $g(x) = 0$ has a root between 1.1 and 1.2. [3]
- (b) A student calculates $g(1.5) = 11.1$ and $g(1.6) = -37.4$, and concludes that there is another root between 1.5 and 1.6. Explain the error. [2]

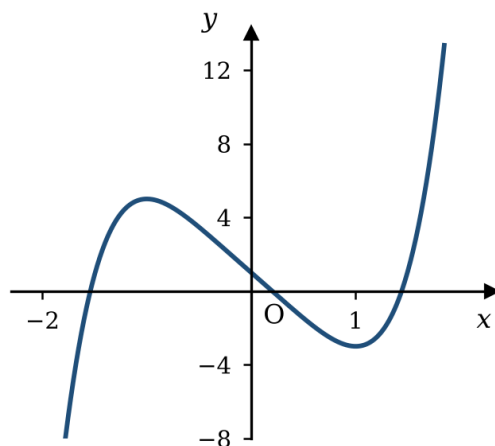
- 26** The curves $y = \frac{1}{x}$ and $y = x^3 - 2$ are sketched for $x \neq 0$.
- (a) Show that the x -coordinates of the points of intersection satisfy $x^4 - 2x - 1 = 0$. [3]
- (b) Show that this equation has a positive root in $(1.3, 1.4)$ and a negative root in $(-0.5, -0.4)$. [5]

- 27** The curves $y = e^x$ and $y = 4x$ are shown. The equation $g(x) = 0$, where $g(x) = e^x - 4x$, has two roots α and β with $\alpha < \beta$.



- Find the exact x -coordinate of the stationary point of $y = g(x)$, and show that the value of g there is $4 - 4 \ln 4$. [3]
- Show that $0.3 < \alpha < 0.4$ and $2 < \beta < 2.5$. [4]
- Explain why $g(\ln 4) < 0$ guarantees that there are exactly two roots, and no more. [3]

- 28** $f(x) = x^5 - 5x + 1$. The graph of $y = f(x)$ is shown.



- Find the coordinates of the stationary points of the curve. [4]
- Use sign changes to show that $f(x) = 0$ has a root in each of the intervals $(-2, -1)$, $(0, 1)$ and $(1, 2)$. [3]
- Hence explain why $f(x) = 0$ has exactly three real roots. [2]

- 29** Let $f(x) = e^x - kx$, where $k > 0$ is a constant.

- Show that f has a single stationary point, at $x = \ln k$, and that it is a minimum. [4]
- Show that the minimum value is $k(1 - \ln k)$. [2]
- Hence find the set of values of k for which $f(x) = 0$ has real roots, and state the number of roots when $k = e$ and when $k = 1$. [4]

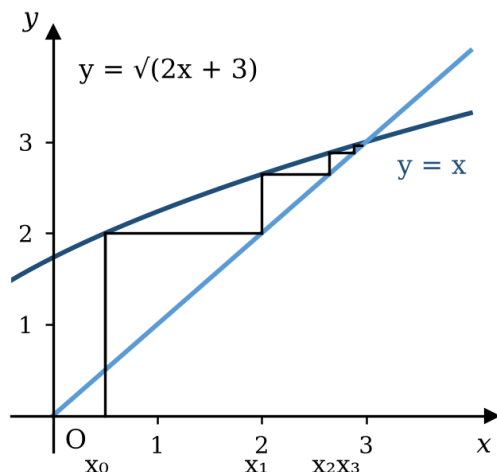
- 30** Let $f(x) = x - \cos x$, where x is in radians.

- Show that $f(x) = 0$ has a root α in the interval $(0.7, 0.8)$. [2]
- Show that f is an increasing function on $\mathbb{R}$, and deduce that α is the only real root. [4]
- Explain why any root of $f(x) = 0$ must lie in $[-1, 1]$, and hence why the argument in (b) is consistent with the sign change in (a). [2]

Iteration

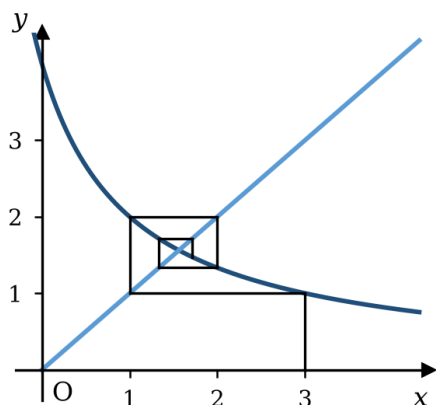
- 1 A sequence is defined by $x_{n+1} = \sqrt{2x_n + 3}$, with $x_0 = 2$.
Find x_1 , x_2 and x_3 , giving each to 4 decimal places. [3]
- 2 (a) Show that the equation $x^3 - x - 5 = 0$ can be rearranged as $x = \sqrt[3]{x + 5}$.
(b) Use $x_{n+1} = \sqrt[3]{x_n + 5}$ with $x_0 = 1.5$ to find x_1 to 4 decimal places. [3]
- 3 A sequence is defined by $x_{n+1} = 2 + \frac{1}{x_n}$ with $x_0 = 2$. Find x_1 , x_2 and x_3 , giving x_3 as a decimal to 4 decimal places. [3]
- 4 Show that $x^2 - 3x - 1 = 0$ can be rearranged into $x = 3 + \frac{1}{x}$, and find x_1 when $x_0 = 3$ in $x_{n+1} = 3 + \frac{1}{x_n}$. [3]
- 5 $x_{n+1} = \ln(x_n + 5)$ with $x_0 = 1$. Find x_1 and x_2 to 4 decimal places. [2]
- 6 $x_{n+1} = \cos x_n$, with x in radians and $x_0 = 1$. Find x_1 , x_2 and x_3 to 4 decimal places. [3]
- 7 The sequence defined by $x_{n+1} = 1 + \frac{1}{x_n}$ with $x_0 = 1$ converges to a limit α .
(a) Find x_1 , x_2 , x_3 , x_4 , x_5 and x_6 , giving each to 4 decimal places. [3]
(b) Explain why the terms suggest that $\alpha = 1.62$ to 2 decimal places. [2]
- 8 The equation $e^x + x - 7 = 0$ has a root α between 1 and 2.
(a) Show that it can be written as $x = \ln(7 - x)$.
(b) Use $x_{n+1} = \ln(7 - x_n)$ with $x_0 = 2$ to find α to 3 decimal places. [5]
- 9 (a) Show that $x^3 + 4x - 3 = 0$ can be rearranged as $x = \frac{3}{x^2 + 4}$.
(b) Starting with $x_0 = 0.5$, find x_1 , x_2 and x_3 in $x_{n+1} = \frac{3}{x_n^2 + 4}$, to 4 decimal places. [5]
- 10 A sequence is defined by $x_{n+1} = \frac{1}{2}x_n + 3$ with $x_0 = 1$. Given that it converges to a limit L , find L , and show that x_3 is within 1 of L . [4]

- 11** The diagram shows the graphs of $y = x$ and $y = \sqrt{2x + 3}$, and the staircase formed by the iteration $x_{n+1} = \sqrt{2x_n + 3}$ starting with $x_0 = 0.5$.



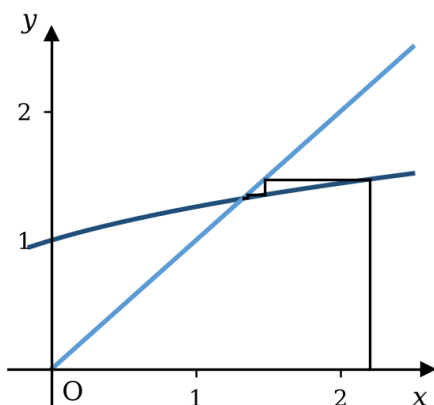
- Explain how the diagram shows that the iteration converges. [2]
 - Find x_1 and x_2 to 2 decimal places. [2]
 - Given that the sequence converges to a limit L , find the exact value of L . [3]
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- 12** $f(x) = e^x - 2x - 3$. The equation $f(x) = 0$ has a root α in the interval $(1.9, 2)$.
- Show that $f(1.9) < 0$ and $f(2) > 0$. [2]
 - Show that $f(x) = 0$ can be rearranged as $x = \ln(2x + 3)$. [2]
 - Use $x_{n+1} = \ln(2x_n + 3)$ with $x_0 = 1.9$ to find α correct to 3 decimal places. [3]
-
- 13** The equation $x^2 = 2 + \sin x$, where x is in radians, has a positive root α .
- Show that α is a root of the iteration $x_{n+1} = \sqrt{2 + \sin x_n}$. [2]
 - Using $x_0 = 1.5$, find x_1 , x_2 , x_3 and x_4 to 4 decimal places, and state α to 3 decimal places. [3]
 - Verify your value by a change of sign. [2]
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- 14** The equation $\ln x + x - 3 = 0$ has a root α near 2.
- Use $x_{n+1} = 3 - \ln x_n$ with $x_0 = 2$ to find α to 3 decimal places. [3]
 - Show that the iteration converges, by evaluating $g'(\alpha)$ for $g(x) = 3 - \ln x$. [3]
-
- 15** A sequence is defined by $x_{n+1} = 1 + \frac{2}{x_n}$, with $x_0 = 1$.
- Find x_1 , x_2 , x_3 and x_4 . [2]
 - Describe the behaviour of the sequence and find its limit L exactly. [3]

- 16** The diagram shows the graphs of $y = x$ and $y = \frac{4}{x+1}$, with the iteration $x_{n+1} = \frac{4}{x_n + 1}$ starting at $x_0 = 3$.



- (a) Describe the pattern of the iteration shown in the diagram and what it tells you about convergence. [2]
- (b) Find x_1 , x_2 and x_3 . [2]
- (c) Find the exact value of the limit α . [2]
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- 17** A population is modelled by $P_{n+1} = P_n + 0.1P_n(1 - \frac{P_n}{1000})$, with $P_0 = 100$, where P_n is the population after n years.
- (a) Find P_1 , P_2 and P_3 , to 1 decimal place. [3]
- (b) Given that $P_n \rightarrow L$, find the non-zero value of L and say what it represents. [3]
-
- 18** The equation $6x - 2 \sin x - 3 = 0$, where x is in radians, has a root α .
- (a) Show that α lies between 0.7 and 0.8. [4]
- (b) Show that the equation can be written as $x = \frac{1}{2} + \frac{1}{3} \sin x$. [2]
- (c) Use $x_{n+1} = \frac{1}{2} + \frac{1}{3} \sin x_n$ with $x_0 = 0.75$ to find x_1 , x_2 and x_3 , giving each to 4 decimal places. [4]
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- 19** The equation $x^3 - 3x - 5 = 0$ has a root α between 2 and 3. A student rearranges it as $x_{n+1} = \frac{5}{x_n^2 - 3}$.
- (a) Show that this rearrangement is correct. [2]
- (b) Explain why the student cannot use $x_0 = \sqrt{3}$. [1]
- (c) Use $x_0 = 2$ to find x_1 , x_2 and x_3 , and state what happens to the sequence. [4]
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- 20** A sequence is defined by $u_{n+1} = 0.8u_n + 50$, with $u_0 = 100$.
- (a) Find u_1 , u_2 and u_3 . [2]
- (b) Given that the sequence converges to a limit L , show that $L = 250$. [2]
- (c) Prove by induction that $u_n = 250 - 150(0.8)^n$ for all integers $n \geq 0$. [4]
-
- 21** A sequence is defined by $x_{n+1} = \frac{1}{2}(x_n + \frac{7}{x_n})$, with $x_0 = 3$.
- (a) Given that $x_n \rightarrow L$ as $n \rightarrow \infty$, show that $L = \sqrt{7}$. [3]
- (b) Find x_1 , x_2 and x_3 to 5 decimal places. [2]
- (c) Show that $x_{n+1} - \sqrt{7} = \frac{(x_n - \sqrt{7})^2}{2x_n}$, and explain why this shows that the sequence converges very quickly. [3]

- 22** The equation $x^3 - x - 1 = 0$ has a root $\alpha \approx 1.3247$. Two rearrangements are (A) $x = x^3 - 1$ and (B) $x = \sqrt[3]{x+1}$.



- (a) Starting with $x_0 = 1.5$, find x_1 , x_2 and x_3 for each iteration, to 4 decimal places where sensible. [4]
 (b) Show, using the gradient of the iteration function at α , which iteration converges. [4]
 (c) The diagram shows the staircase for (B) starting at $x_0 = 2.2$. Explain its shape. [1]

- 23** The iteration $x_{n+1} = \sqrt[3]{7 - 2x_n}$, with $x_0 = 1.5$, converges to the root α of $x^3 + 2x - 7 = 0$. The first terms are shown (to 5 s.f.). Take $\alpha = 1.56895$.

x_0	x_1	x_2	x_3	x_4
1.5000	1.5874	1.5639	1.5703	1.5686

- (a) Calculate the errors $e_n = x_n - \alpha$ for $n = 0$ to 3, to 3 significant figures. [3]
 (b) Show that $\frac{e_{n+1}}{e_n}$ is approximately constant, and compare it with $g'(\alpha)$ for $g(x) = \sqrt[3]{7 - 2x}$. [4]

- 24** A sequence is defined by $x_{n+1} = 3.2x_n(1 - x_n)$, $x_0 = 0.5$.

- (a) Find x_1 to x_6 , to 4 decimal places. [4]
 (b) Show that $L = \frac{11}{16}$ would be a fixed point, but that the sequence does not converge to it. [5]

- 25** Consider the iteration $x_{n+1} = 3 - \frac{1}{x_n}$.

- (a) Show that the fixed points satisfy $x^2 - 3x + 1 = 0$ and solve it exactly. [2]
 (b) Find $g'(x)$ for $g(x) = 3 - \frac{1}{x}$ and decide which fixed point attracts the iteration. [4]
 (c) Starting from $x_0 = 0.4$, find x_1 and x_2 and say what happens. [3]

- 26** The curve C has equation $y = e^x - 5x^2$.

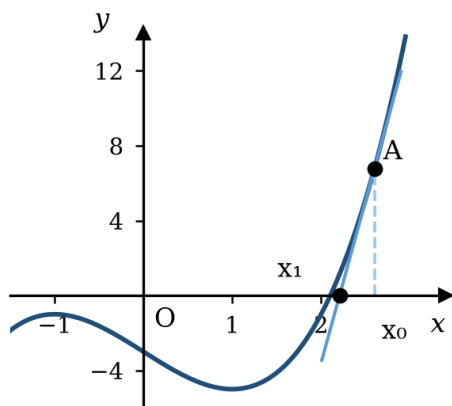
- (a) Show that the x-coordinate of a stationary point of C satisfies $x = \ln(10x)$. [3]
 (b) Show that the stationary point with $x > 3$ has its x-coordinate in the interval (3.5, 3.6). [3]
 (c) Use $x_{n+1} = \ln(10x_n)$ with $x_0 = 3.5$ to find x_1 , x_2 and x_3 to 4 decimal places. [4]
 (d) Determine whether this stationary point is a maximum or a minimum. [2]

- 27** The equation $x^3 - 4x - 2 = 0$ has a root α between 2 and 3. Three rearrangements are
 (A) $x = \frac{x^3 - 2}{4}$, (B) $x = \sqrt[3]{4x + 2}$, (C) $x = \sqrt{4 + \frac{2}{x}}$.
- (a) Show that α lies between 2 and 3. [2]
 (b) Show that (C) is a correct rearrangement of the equation. [2]
 (c) Use iteration (C) with $x_0 = 2$ to find α to 4 decimal places, showing the terms you calculate. [3]
 (d) A student uses (A) with $x_0 = 2$ and obtains $x_1 = 1.500$, $x_2 = 0.3438$, $x_3 = -0.4898$, $x_4 = -0.5294$. Explain what is happening and why (A) does not find α . [3]
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- 28** A sequence is defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2 + u_n}$.
- (a) Find u_2 and u_3 to 4 decimal places. [1]
 (b) Prove by induction that $u_n < 2$ for all $n \geq 1$. [4]
 (c) Show that $u_{n+1} > u_n$ for all n , and deduce that the sequence converges to 2. [5]
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- 29** A sequence is defined by $u_1 = 1$ and $u_{n+1} = \sqrt{3u_n + 4}$.
- (a) Find u_2 and u_3 , to 4 decimal places. [2]
 (b) Prove by induction that $u_n < 4$ for all $n \geq 1$. [4]
 (c) Show that $u_{n+1} > u_n$ for all n , and hence find the limit of the sequence. [5]
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- 30** The root α of the equation $x = e^{-x}$ can be found by iteration $x_{n+1} = e^{-x_n}$, with $x_0 = 0.5$.
- (a) Calculate x_1 to x_4 , to 5 decimal places. [3]
 (b) Show that the iteration converges to α by finding $g'(\alpha)$, where $g(x) = e^{-x}$. Take $\alpha = 0.56714$. [3]
 (c) Aitken's formula estimates the limit as $x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0}$. Use it with x_0, x_1, x_2 and compare the result with α . [4]

Newton-Raphson Method

- 1** $f(x) = x^3 - 3x - 5$ and $f'(x) = 3x^2 - 3$.
 Use the Newton-Raphson formula with $x_0 = 2.5$ to find x_1 . [3]
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- 2** Differentiate $f(x) = x^3 + 4x - 9$ and write down the Newton-Raphson iteration formula for solving $f(x) = 0$. [3]
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- 3** Use one step of the Newton-Raphson method on $f(x) = x^2 - 11$ with $x_0 = 3.3$ to find a better approximation to $\sqrt{11}$, to 4 decimal places. [3]
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- 4** $f(x) = e^x - 5$.
 Use the Newton-Raphson method with $x_0 = 1.5$ to find x_1 , to 4 decimal places. [3]
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- 5** A function f has $f(2) = 3$ and $f'(2) = 12$. Starting with $x_0 = 2$, find x_1 using the Newton-Raphson method. [2]

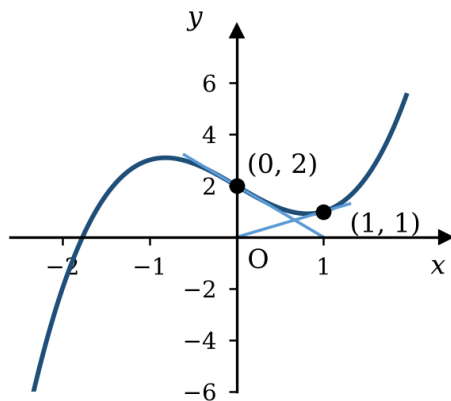
- 6** The diagram shows the graph of $y = f(x)$ and the tangent at the point A, where $x = x_0$. The tangent meets the x -axis at $x = x_1$.
Explain how x_1 is obtained from x_0 in the Newton-Raphson method and why x_1 is usually a better approximation to the root.



[3]

- 7** $f(x) = x^3 - 2x - 5$.
Use the Newton-Raphson method with $x_0 = 2$ to find x_1 and x_2 , giving each answer to 4 decimal places. [5]
- 8** $f(x) = 2 \ln x + x - 5$.
Use the Newton-Raphson method with $x_0 = 3$ to find x_1 and x_2 , giving each answer to 4 decimal places. [5]
- 9** $f(x) = \cos x - 2x$.
Use the Newton-Raphson method with $x_0 = 0.5$ to find x_1 and x_2 , giving each answer to 4 decimal places. [5]
- 10** (a) Show that applying the Newton-Raphson method to $f(x) = x^2 - 10$ gives $x_{n+1} = \frac{1}{2}(x_n + \frac{10}{x_n})$.
(b) Use $x_0 = 3$ to find x_2 , to 5 decimal places, as an approximation to $\sqrt{10}$. [5]
- 11** $f(x) = e^x - 3x$ has a root α near 0.6. Use the Newton-Raphson method with $x_0 = 0.5$ to find α correct to 4 decimal places. [5]
- 12** Use the Newton-Raphson method with $x_0 = 1.2$ to find a root of $x^3 + 4x = 7$, giving your answer correct to 6 decimal places. Show your iterates. [5]
- 13** The reciprocal of a number a can be found without division by solving $f(x) = \frac{1}{x} - a = 0$.
(a) Show that the Newton-Raphson iteration is $x_{n+1} = x_n(2 - ax_n)$. [4]
(b) Use this with $a = 7$ and $x_0 = 0.1$ to find x_1 , x_2 and x_3 , and compare x_3 with $\frac{1}{7}$. [4]

- 14** $f(x) = x^3 - 2x + 2$. The graph of $y = f(x)$ is shown, and meets the x -axis once.

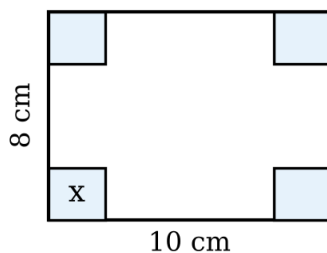


- (a) Show that the Newton-Raphson method with $x_0 = 0$ gives $x_1 = 1$ and $x_2 = 0$. [4]
 (b) Describe what happens if the method is continued and explain why $x_0 = 0$ is a poor choice, given that the root is near -1.77 . [2]
 (c) Use $x_0 = -2$ to find x_1 and x_2 , to 4 decimal places. [2]

- 15** $f(x) = x^3 - 3x + 1$.

- (a) Explain why the Newton-Raphson method cannot be used with $x_0 = 1$. [2]
 (b) Use $x_0 = 2$ to find x_1 and x_2 , to 4 decimal places. [3]

- 16** A rectangular sheet of card measures 10 cm by 8 cm. A square of side x cm is cut from each corner and the sides are folded up to make an open box of volume 50 cm^3 .



- (a) Show that x satisfies $4x^3 - 36x^2 + 80x - 50 = 0$. [3]
 (b) Use the Newton-Raphson method with $x_0 = 1$ to find the smaller solution for x , correct to 3 decimal places. [5]
- 17** $f(x) = x \ln x - 2$, for $x > 0$.
 (a) Show that $f(x) = 0$ has a root α in the interval $(2, 3)$. [2]
 (b) Use the Newton-Raphson method with $x_0 = 2$ to find α to 4 decimal places, and verify your answer by a change of sign. [6]

- 18** Solve $2^x = 10$ numerically: let $f(x) = 2^x - 10$.
 Use the Newton-Raphson method with $x_0 = 3$ to find x_1 and x_2 , giving each answer to 4 decimal places. [5]

- 19** The graphs of $y = e^{-x}$ and $y = x^2$ meet at a single point with x -coordinate $\alpha > 0$.
 (a) Show that α lies between 0.7 and 0.71. [4]
 (b) Use the Newton-Raphson method with $x_0 = 0.7$ to find x_1 and x_2 to 6 decimal places. [4]

- 20** The tangent to the curve $y = f(x)$ at the point where $x = a$ meets the x -axis at $x = b$.
 (a) Show that $b = a - \frac{f(a)}{f'(a)}$, where $f'(a) \neq 0$. [3]
 (b) Hence show that for $f(x) = x^2 - k$, $k > 0$, with $a > 0$, $b = \frac{1}{2}(a + \frac{k}{a})$ and $b \geq \sqrt{k}$. [4]

- 21** The equation $x \sin x = 1$ (x in radians) has a root α near 1.1.
- (a) Show that the Newton-Raphson iteration for this equation is $x_{n+1} = x_n - \frac{x_n \sin x_n - 1}{\sin x_n + x_n \cos x_n}$. [4]
- (b) Use $x_0 = 1$ to find α correct to 4 decimal places. [4]
-
- 22** Consider applying the Newton-Raphson method to $f(x) = \sqrt[3]{x}$, which has a root at $x = 0$.
- (a) Show that $f'(x) = \frac{1}{3}x^{-2/3}$, and that the iteration simplifies to $x_{n+1} = -2x_n$. [4]
- (b) Starting at $x_0 = 0.1$, find x_1 , x_2 and x_3 and explain what happens. [3]
-
- 23** $f(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$.
- (a) Show that the Newton-Raphson iteration is $x_{n+1} = \frac{x_n^2 - 6}{2x_n - 5}$. [3]
- (b) Find x_1 and x_2 when $x_0 = 4$, and when $x_0 = 1$. [4]
- (c) Show that $x_{n+1} - 3 = \frac{(x_n - 3)^2}{2x_n - 5}$. [2]
-
- 24** $f(x) = x e^x - 2$.
- (a) Show that $f(x) = 0$ has a root between 0.8 and 0.9. [2]
- (b) Show that $f'(x) = (x + 1)e^x$, and use the Newton-Raphson method with $x_0 = 1$ to find the root to 5 decimal places. [5]
-
- 25** The equation $x = e^{-x}$ has a root α . Let $f(x) = x - e^{-x}$.
- (a) Show that the Newton-Raphson iteration for $f(x) = 0$ is $x_{n+1} = \frac{x_n + 1}{e^{x_n} + 1}$. [4]
- (b) Take $x_0 = 0.5$. Find x_1 , x_2 and x_3 by this method, and by the simpler iteration $x_{n+1} = e^{-x_n}$, to 6 decimal places. [4]
- (c) Given $\alpha = 0.567143$, comment on the relative speed of the two methods. [3]
-
- 26** The curve C has equation $y = xe^x - 3x$ and has a stationary point at $x = \alpha$.
- (a) Show that α is a root of the equation $(x + 1)e^x - 3 = 0$. [3]
- (b) Write down the Newton-Raphson formula for this equation, with $f'(x)$ simplified. [3]
- (c) Starting with $x_0 = 0.7$, find x_1 , x_2 and x_3 to 5 decimal places, and state α to 4 decimal places. [4]
-
- 27** The Newton-Raphson method is used to find the positive n -th root of a , by solving $x^n - a = 0$.
- (a) Show that $x_{k+1} = \frac{1}{n}((n - 1)x_k + \frac{a}{x_k^{n-1}})$. [4]
- (b) Use this with $n = 4$, $a = 100$ and $x_0 = 3$ to find $\sqrt[4]{100}$ to 5 decimal places. [5]
-
- 28** The Newton-Raphson method is applied to $f(x) = x^2 - 2$ with $x_0 = 1.5$, giving $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. Let $e_n = x_n - \sqrt{2}$.
- (a) Show that $e_{n+1} = \frac{e_n^2}{2x_n}$. [3]
- (b) Calculate e_0 , e_1 , e_2 , e_3 to 3 significant figures. [3]
- (c) Explain how the pattern of these errors shows that the number of correct decimal places roughly doubles each iteration (quadratic convergence). [3]

- 29** $f(x) = (x - 1)^2(x + 2)$ has a repeated root at $x = 1$. The Newton-Raphson method is used with $x_0 = 2$.
- (a) Show that $f'(x) = 3(x - 1)(x + 1)$, and hence that $x_{n+1} - 1 = \frac{(x_n - 1)(2x_n + 1)}{3(x_n + 1)}$. [4]
- (b) Calculate x_1 to x_5 to 6 decimal places and the ratios $\frac{e_{n+1}}{e_n}$ where $e_n = x_n - 1$. [3]
- (c) Explain why the convergence is much slower than for a simple root. [2]
-
- 30** A hemispherical bowl of radius 10 cm is held with its rim horizontal and is filled with water to a depth of h cm. The volume of water is $V = \frac{\pi}{3}(30h^2 - h^3)$ cm³. The bowl is filled with 1000 cm³ of water.
- (a) Show that h satisfies $f(h) = 0$, where $f(h) = \frac{\pi}{3}(30h^2 - h^3) - 1000$, and show that it has exactly one solution for $0 < h < 10$, lying in the interval (6, 7). [4]
- (b) Use the Newton-Raphson method with $h_0 = 7$ to find h to 2 decimal places, showing your iterations. [5]

Trapezium Rule

- 1** The table shows values of $y = f(x)$. Use the trapezium rule with all the values to estimate the area under the curve between $x = 0$ and $x = 4$.
- | $x = 0$ | $x = 1$ | $x = 2$ | $x = 3$ | $x = 4$ |
|---------|---------|---------|---------|---------|
| 3 | 5 | 8 | 6 | 2 |
- [3]
-
- 2** Use the trapezium rule with the values in the table to estimate $\int_0^6 y \, dx$.
- | $x = 0$ | $x = 2$ | $x = 4$ | $x = 6$ |
|---------|---------|---------|---------|
| 1.2 | 3.4 | 2.8 | 0.6 |
- [3]
-
- 3** (a) Complete a table of values of $y = \sqrt{x}$ for $x = 1, 2, 3, 4$ and 5 , giving each value to 4 decimal places.
- (b) Use the trapezium rule with 4 strips to estimate $\int_1^5 \sqrt{x} \, dx$ to 3 significant figures. [3]
-
- 4** The trapezium rule is used to estimate $\int_1^2 f(x) \, dx$ with 5 strips of equal width. Write down the width of each strip and the x -values of the ordinates. [2]
-
- 5** Use the trapezium rule with 3 strips to estimate $\int_0^3 2^x \, dx$. [3]
-
- 6** Use the trapezium rule with 2 strips to estimate $\int_0^2 x^2 \, dx$.
- (a) Find the estimate. [3]
- (b) Find the exact value of the integral and state whether the estimate is an over- or under-estimate. [2]

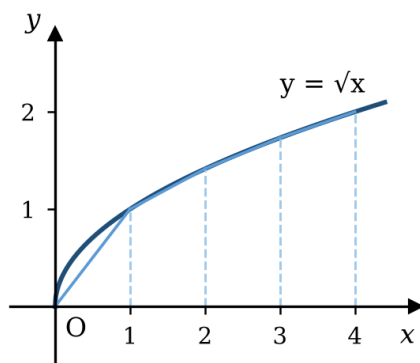
- 7 The integral $I = \int_1^3 \frac{1}{x} \, dx$.
- (a) Use the trapezium rule with 4 strips to estimate I , to 4 decimal places. [3]
- (b) Find the exact value of I and the percentage error in the estimate, to 2 significant figures. [3]

- 8 Use the trapezium rule with 4 strips of equal width to estimate $\int_0^1 e^{x^2} \, dx$, giving your answer to 3 significant figures. [5]

- 9 The integral $I = \int_0^{\pi/2} \sin x \, dx$, with x in radians.
- (a) Use the trapezium rule with 4 strips to estimate I , to 4 decimal places. [4]
- (b) Find the percentage error, given that the exact value is 1. [2]

- 10 Use the trapezium rule with 4 strips of equal width to estimate $\int_2^6 \ln x \, dx$, giving your answer to 3 significant figures. [5]

- 11 The diagram shows the curve $y = \sqrt{x}$ with the four trapeziums used to estimate $\int_0^4 \sqrt{x} \, dx$.



- (a) Calculate the trapezium rule estimate, to 3 decimal places. [3]
- (b) Explain, using the diagram, why the estimate is smaller than the exact value $\frac{16}{3}$. [2]

- 12 The trapezium rule with 3 strips, using the values in the table, gives an estimate of 17 for $\int_1^4 y \, dx$. Find the value of k .

$x = 1$	$x = 2$	$x = 3$	$x = 4$
2	k	7	9

[3]

- 13 A river is 8 m wide. The depth d metres is measured at 2 m intervals across the river (first row: distance from the bank in metres; second row: depth in metres). Estimate the area of the cross-section of the river using the trapezium rule, and the volume of water passing a point in 30 seconds if the river flows at a constant speed of 0.6 m/s.

0	2	4	6	8
0	1.2	2.1	1.6	0

[5]

- 14** The speed v m/s of a train at times t seconds after leaving a station is recorded every 10 seconds (first row t , second row v).
Use the trapezium rule to estimate the distance travelled in the first 60 seconds, and hence the mean speed over this period.

0	10	20	30	40	50	60
0.0	4.2	7.5	9.8	11.0	11.6	12.0

[5]

- 15** The integral $I = \int_0^1 \frac{4}{1+x^2} dx$ has exact value π .

- (a) Estimate I using the trapezium rule with 2 strips. [3]
 (b) Estimate I using 4 strips, to 4 decimal places. [3]
 (c) Show that the estimate with 4 strips is closer to π , and find its percentage error to 2 s.f. [3]

- 16** The trapezium rule with the values in the table (equal strips) gives an estimate of a certain integral $\int_1^3 f(x) dx$.

$x = 1$	$x = 1.5$	$x = 2$	$x = 2.5$	$x = 3$
1	3	2.5	4	2

- (a) Find the estimate T . [3]
 (b) Hence, using the same ordinates, estimate $\int_1^3 (3f(x) + 2) dx$. [2]

- 17** The values of y at $x = 0, 1, 3$ and 6 are shown (first row x , second row y). Use trapezia of the appropriate widths to estimate $\int_0^6 y dx$.

0	1	3	6
2.0	3.5	4.0	1.0

[4]

- 18** Use the trapezium rule with 4 strips of equal width to estimate $I = \int_0^2 xe^{-x} dx$.

- (a) Complete a table of values of x and xe^{-x} (4 decimal places) and find the estimate for I , giving your answer to 3 significant figures. [4]
 (b) Hence write down an estimate for $\int_0^2 (1 + 4xe^{-x}) dx$. [2]
 (c) Write down an estimate for $\int_3^5 (x - 3)e^{3-x} dx$, explaining your answer. [2]

- 19** Let $I = \int_0^1 e^x dx$.

- (a) Find I exactly. [2]
 (b) Use the trapezium rule with 2 strips to estimate I and show that it is an overestimate. [3]
 (c) Explain, without calculation, why the trapezium rule overestimates every integral of e^x . [2]

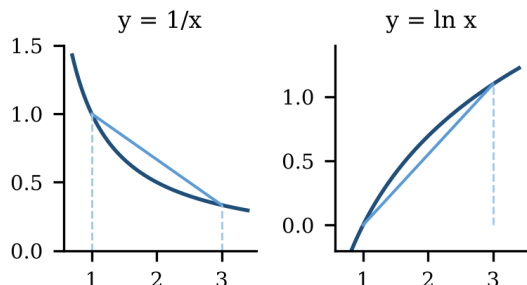
- 20** Water flows into a tank at a rate of r litres per second, where $r = 5e^{-0.2t}$ and t is the time in seconds.

- (a) Use the trapezium rule with 5 strips to estimate the volume of water entering the tank in the first 10 seconds, to 3 significant figures. [4]
 (b) Find the exact volume, and explain why your estimate is greater. [3]

21 $I = \int_{-1}^1 e^{-x^2} dx.$

- (a) Use the trapezium rule with 4 strips to estimate I, to 4 decimal places. [4]
 (b) Explain how symmetry allows the same answer from two strips on $[0, 1]$, and verify it. [3]

22 The diagrams show the curves $y = \frac{1}{x}$ and $y = \ln x$ for $1 \leq x \leq 3$, each with a single trapezium (a chord) joining the points at $x = 1$ and $x = 3$.



- (a) State, with a reason, whether the trapezium rule overestimates or underestimates $\int_1^3 \frac{1}{x} dx$ and the same for $\int_1^3 \ln x dx$. [4]
 (b) Using one strip for each, find the two trapezium estimates, and the percentage by which the first exceeds the exact value $\ln 3$. [4]

23 Let $I = \int_0^1 x e^x dx.$

- (a) Use integration by parts to show that $I = 1$. [3]
 (b) Use the trapezium rule with 4 strips to estimate I, and find the percentage error. [5]

24 The cross-sectional area $A \text{ m}^2$ of a hull at distances $x \text{ m}$ from the bow of a small boat is given in the table (first row x , second row A).

0	2	4	6	8	10
0.0	1.8	3.1	3.4	2.2	0.0

- (a) Use the trapezium rule to estimate the volume of the hull, $V = \int_0^{10} A dx$. [3]
 (b) The boat is made of material of density 400 kg per m^3 filling the whole hull. Find its mass, and the mean cross-sectional area. [3]
 (c) Explain why using more measurements would give a more reliable estimate. [1]

25 The trapezium rule with 4 strips of equal width is used to estimate $\int_0^2 \frac{a}{1+x} dx$, where a is a positive constant. The estimate is 3.35.

- (a) Find the value of a . [4]
 (b) Find the exact value of the integral and explain, without further calculation, why the trapezium rule gives an overestimate. [4]
 (c) State how a more accurate estimate could be obtained. [1]

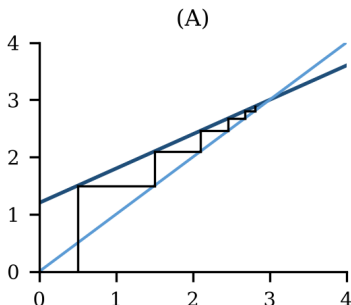
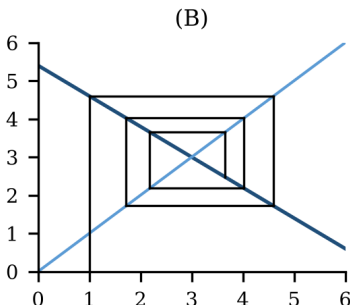
26 The trapezium rule with n strips of equal width is used to estimate $\int_0^1 x^2 dx$.

- (a) Show that the estimate is $T_n = \frac{1}{n} \left[\frac{1}{2} + \sum_{r=1}^{n-1} \left(\frac{r}{n} \right)^2 \right]$. [3]
 (b) Using $\sum_{r=1}^N r^2 = \frac{N(N+1)(2N+1)}{6}$, prove that $T_n = \frac{1}{3} + \frac{1}{6n^2}$. [5]
 (c) Find the least number of strips for which the estimate is within 0.001 of the exact value. [3]

- 27** The trapezium rule with strips of width 1 is applied to $\int_1^n \ln x \, dx$, where n is an integer greater than 1.
- (a) Show that the estimate is $\ln(n!) - \frac{1}{2} \ln n$. [4]
- (b) Use integration by parts to show that $\int_1^n \ln x \, dx = n \ln n - n + 1$. [3]
- (c) Given that $\ln x$ is concave, deduce that $\ln(n!) < (n + \frac{1}{2}) \ln n - n + 1$, and verify this for $n = 6$. [5]
-
- 28** The trapezium rule with n strips is used to estimate $I = \int_0^\pi \sin x \, dx$. Let $h = \frac{\pi}{n}$.
- (a) Show that the estimate is $T_n = h \sum_{r=1}^{n-1} \sin(rh)$, and find T_4 to 4 decimal places. [5]
- (b) By multiplying each term by $2 \sin(\frac{h}{2})$ and using $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$, show that $\sum_{r=1}^{n-1} \sin(rh) = \cot(\frac{h}{2})$. [5]
- (c) Hence show that $T_n = h \cot(\frac{h}{2})$, check it for $n = 4$, and find the limit of T_n as $n \rightarrow \infty$. [4]
-
- 29** For smooth functions the error in the trapezium rule with n strips is approximately $\frac{c}{n^2}$, for a constant c . Let T_2 and T_4 be the estimates of $I = \int_0^1 \frac{4}{1+x^2} \, dx$ using 2 and 4 strips.
- (a) Calculate T_2 and T_4 . [3]
- (b) Using $I \approx T_2 - \frac{c}{4}$ and $I \approx T_4 - \frac{c}{16}$, eliminate c to show that $I \approx \frac{4T_4 - T_2}{3}$. [4]
- (c) Calculate this improved estimate and compare with π . [2]
-
- 30** Let T_n be the trapezium rule estimate for $\int_0^1 3^x \, dx$ using n strips of equal width. Let $q = 3^{1/n}$.
- (a) Show that $T_n = \frac{1}{n} \times \frac{q+1}{q-1}$. [6]
- (b) Find T_2 in the form $a + b\sqrt{3}$. [3]
- (c) The exact value of the integral is $\frac{2}{\ln 3}$. Find the percentage error in T_2 , to 2 significant figures. [4]

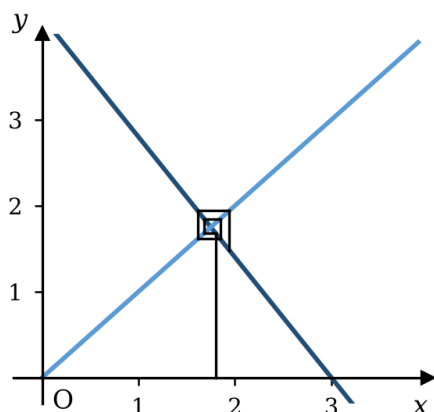
Convergence and Error

- 1** $22/7$ is used as an approximation to $\pi = 3.14159\dots$. Find the absolute error and the percentage error, to 2 significant figures. [3]
-
- 2** A root α of an equation is 2.7 correct to 1 decimal place. Write down the interval in which α lies, and the maximum possible error if 2.7 is used as an approximation. [3]
-
- 3** State whether each sequence converges, diverges, or oscillates without a limit.
 (a) 1, 0.5, 0.75, 0.625, 0.6875, ... (b) 1, 2, 4, 8, 16, ... (c) 1, -1, 1, -1, ... [3]
-
- 4** An iteration $x_{n+1} = g(x_n)$ has a fixed point α . State whether the iteration will converge to α for each value of $g'(\alpha)$.
 (a) 0.3 (b) -0.8 (c) 1.5 (d) -2 [4]
-
- 5** A root α of $f(x) = 0$ is stated as 1.8 correct to 1 decimal place. A student shows that $f(1.75) < 0$ and $f(1.85) > 0$. Explain why this confirms the statement. [2]

- 6** A trapezium rule estimate of an integral is 25.4. The exact value is 25. Find the percentage error in the estimate. [2]
- 7** $f(x) = x^3 + 2x - 15$. Show that the root of $f(x) = 0$ is 2.20 correct to 2 decimal places. [3]
- 8** The equation $x^3 - x^2 - 2 = 0$ has a root $\alpha \approx 1.6956$. The iteration $x_{n+1} = 1 + \frac{2}{x_n^2}$ is used with $x_0 = 1.5$.
 (a) Show that the iteration comes from the equation, and find x_1 , x_2 and x_3 , to 4 decimal places. [3]
 (b) Find $g'(\alpha)$ for $g(x) = 1 + 2x^{-2}$ and explain why the iteration converges slowly, with terms alternating about α . [3]
- 9** The diagrams show two iterations $x_{n+1} = g(x_n)$ with different functions $g(x)$, each starting from a different x_0 .
 (A)  (B) 
 (a) Describe the type of diagram in (A) and in (B), and what each shows about convergence. [4]
 (b) State whether $g'(\alpha)$ is positive or negative in each case, and what its size must be. [2]
- 10** The terms of an iteration are $x_0 = 1.5$, $x_1 = 1.58740$, $x_2 = 1.56393$, $x_3 = 1.57030$, $x_4 = 1.56858$. Show that the ratio of successive differences $\frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}$ is approximately constant and explain what this says about the convergence. [4]
- 11** Interval bisection starts with an interval of width 1 containing a root. How many bisections are needed to be sure the root lies in an interval of width less than 0.001? [3]
- 12** The exact value of $\int_0^1 x^3 dx$ is $\frac{1}{4}$. Find the percentage error of the trapezium rule estimates using 2 strips and 4 strips, and describe how the error changes when the number of strips is doubled. [5]
- 13** $a = 4.37$ and $b = 1.52$, each correct to 2 decimal places. Find the greatest and least possible values of (a) $a - b$, (b) $\frac{a}{b}$, to 3 decimal places. [5]
- 14** The iteration $x_{n+1} = \ln(7 - x_n)$ with $x_0 = 2$ is stopped when $|x_{n+1} - x_n| < 0.0005$. Find how many iterations are needed, and state the final approximation to 4 decimal places. [4]
- 15** Two methods approximate $\sqrt{2}$. Method A: $x_{n+1} = 1 + \frac{1}{1 + x_n}$. Method B (Newton-Raphson): $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. Both start with $x_0 = 1$, and $e_n = x_n - \sqrt{2}$.
 (a) Calculate x_1 , x_2 , x_3 for each method, to 6 decimal places. [4]
 (b) Show that for A, $\frac{e_{n+1}}{e_n}$ is roughly constant, but for B, e_{n+1} is roughly proportional to e_n^2 . [4]

- 16** The iteration $x_{n+1} = \frac{1}{x_n}$ starts with $x_0 = 2$. Find x_1 , x_2 , x_3 and x_4 . Explain why the iteration neither converges nor diverges, using the gradient of $g(x) = \frac{1}{x}$ at the fixed point $x = 1$. [4]

- 17** The diagram shows the iteration $x_{n+1} = 4.2 - 1.4x_n$ with $x_0 = 1.8$.



- (a) Find x_1 , x_2 , x_3 and x_4 . [3]
 (b) Find the fixed point α , show that the iteration diverges from it, and describe the shape of the diagram. [4]

- 18** The radius of a circle is measured as $r = 4.5$ cm, with a possible error of ± 0.05 cm.
 (a) Find the greatest and least possible areas, to 1 decimal place. [3]
 (b) Find the maximum percentage error in the area calculated using $r = 4.5$, to 2 significant figures. [3]

- 19** Show that the root of $x^3 - x^2 - 1 = 0$ is 1.4656 correct to 4 decimal places. [4]

- 20** The equation $x^3 - x - 3 = 0$ has a root α in the interval $[1, 2]$. The interval bisection method is used to locate α .
 (a) Show that α lies in $[1, 2]$, then carry out two bisections to find an interval of width 0.25 containing α . [4]
 (b) After 6 bisections the midpoint of the interval is taken as an estimate of α . Find the maximum possible error in this estimate. [3]
 (c) Find the least number of bisections needed so that the midpoint is within 0.0005 of α . [3]

- 21** Newton-Raphson iteration is $x_{n+1} = N(x_n)$, where $N(x) = x - \frac{f(x)}{f'(x)}$.
 (a) Show that $N'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$. [3]
 (b) Deduce that $N'(\alpha) = 0$ at a simple root α , and explain what this means for the speed of convergence. [3]

- 22** The equation $x^3 - x - 1 = 0$ has a root α in $(1, 2)$, $\alpha = 1.32471796$ to 8 decimal places.
 (a) Show that bisection needs 10 steps starting from $[1, 2]$ to locate α in an interval of width less than 0.001. [3]
 (b) Newton-Raphson with $x_0 = 1.5$ gives iterates 1.34782609, 1.32520040, 1.32471817. Find the error of each and compare the number of correct decimal places with the bisection result. [3]

23

The equation $x^3 - 6x - 4 = 0$ has a root $\alpha \approx 2.732$. Three rearrangements are (A) $x = \frac{x^3 - 4}{6}$, (B) $x = \sqrt[3]{6x + 4}$ and (C) $x = \frac{6}{x} + \frac{4}{x^2}$.

- (a) Show that (C) is a valid rearrangement of the equation. [2]
 (b) Calculate $|g'(\alpha)|$ for each rearrangement and state which one(s) will converge to α . [7]

24

For Newton-Raphson, if $e_n = x_n - \alpha$ is small, then $e_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n^2$.

- (a) For $f(x) = x^2 - 3$, show that $\frac{f''(\alpha)}{2f'(\alpha)} = \frac{1}{2\sqrt{3}} = 0.2887$. [3]
 (b) Starting with $x_0 = 2$, predict the error e_1 , and compare with the actual error of x_1 . [5]

25

For a function with a continuous second derivative, the error in the trapezium rule on $[a, b]$ with strips of width h satisfies $|\text{error}| \leq \frac{(b-a)h^2}{12} M$, where M is the maximum of $|f''(x)|$ on $[a, b]$.

- (a) Use this to find a bound for the error when $\int_0^1 e^x dx$ is estimated with 4 strips. [4]
 (b) Calculate the trapezium estimate and the actual error, and verify that it is within the bound. [4]
 (c) How many strips guarantee an error below 0.0001? [2]

26

The table shows the absolute errors $|e_n|$ for $n = 0, 1, 2, 3$ for two iterative methods P (first row) and Q (second row).

0.1	0.05	0.025	0.0125
0.1	0.01	0.0001	0.00000001

- (a) For method P, show that $\frac{|e_{n+1}|}{|e_n|}$ is constant and state the order of convergence. [3]
 (b) For method Q, show that $\frac{|e_{n+1}|}{|e_n|^2}$ is constant and state the order of convergence. [3]
 (c) Estimate the error in each method after one more iteration, and say which method you would choose. [3]

27

Newton-Raphson is used to solve $f(x) = 0$ near a simple root α . Let $e_n = x_n - \alpha$, and assume f'' is continuous.

- (a) Using Taylor expansions about α , show that $f(x_n) \approx f'(\alpha)e_n + \frac{1}{2}f''(\alpha)e_n^2$ and $f'(x_n) \approx f'(\alpha) + f''(\alpha)e_n$. [3]
 (b) Hence show that $e_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n^2$. [4]
 (c) For $f(x) = x^3 - 2$, find the constant $\frac{f''(\alpha)}{2f'(\alpha)}$ and the predicted error after one step from an error of 0.01. [3]

28

Consider the iteration $x_{n+1} = \cos x_n$ (x in radians) with any starting value x_0 in $[0, 1]$.

- (a) Show that if $x \in [0, 1]$ then $\cos x \in [\cos 1, 1] \subset [0, 1]$, so every term stays in $[0, 1]$. [2]
 (b) Show that $|g'(x)| \leq \sin 1 < 1$ on $[0, 1]$, where $g(x) = \cos x$. [3]
 (c) Deduce that the iteration converges to the unique solution of $x = \cos x$, and find the solution to 4 decimal places. [4]

29

To solve $x^2 - 2 = 0$, the iteration $x_{n+1} = x_n - \frac{x_n^2 - 2}{m}$ is used, where m is a positive constant.

- (a) Show that the iteration function $g(x) = x - \frac{x^2 - 2}{m}$ has $g'(\alpha) = 1 - \frac{2\sqrt{2}}{m}$ at $\alpha = \sqrt{2}$. [3]
- (b) Find the range of values of m for which the iteration converges. [3]
- (c) With $m = 3$ and $x_0 = 1$ find x_1, x_2, x_3 , and explain why this converges faster than $m = 6$ would, even though $m = 6$ also converges. [4]

30

Suppose $|g'(x)| \leq k < 1$ for all x in an interval containing α and the iterates, where $x_{n+1} = g(x_n)$ and $g(\alpha) = \alpha$.

- (a) Show that $|x_{n+1} - \alpha| \leq k|x_n - \alpha|$ (use the mean value theorem), and that $|x_{n+1} - x_n| \geq (1 - k)|x_n - \alpha|$. [4]
- (b) Deduce that $|x_n - \alpha| \leq \frac{k}{1 - k}|x_n - x_{n-1}|$. [2]
- (c) The iteration $x_{n+1} = \sqrt[3]{7 - 2x_n}$ with $x_0 = 1.5$ has $k = 0.3$ on the relevant interval. Use x_3 and x_4 to find a bound on $|x_4 - \alpha|$, and compare it with the actual error. [4]