

Basic Integration

1 (2 marks)

Raise each power by one and divide by the new power

M1

$$\int (6x^2 - 4x + 5) \, dx = 2x^3 - 2x^2 + 5x + c$$

A1

Answer: $2x^3 - 2x^2 + 5x + c$

2 marks

2 (3 marks)

Write in powers of x: $3x^{\frac{1}{2}} - 4x^{-2}$

M1

$$\int (3x^{\frac{1}{2}} - 4x^{-2}) \, dx = 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4 \times \frac{x^{-1}}{-1} + c$$

A1

$$= 2x^{\frac{3}{2}} + \frac{4}{x} + c$$

A1

Answer: $2x^{\frac{3}{2}} + \frac{4}{x} + c$

3 marks

3 (2 marks)

Raise the power by one and divide: $8 \times \frac{x^4}{4} + 5x$

M1

$$= 2x^4 + 5x + c$$

A1

Answer: $2x^4 + 5x + c$

2 marks

4 (2 marks)

$$6 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4 \times \frac{x^{-2}}{-2}$$

M1

$$= 4x^{\frac{3}{2}} + 2x^{-2} + c$$

A1

Answer: $4x^{\frac{3}{2}} + 2x^{-2} + c$

2 marks

5 (2 marks)

Divide each term by x^2 : $x^2 + 3x^{-2}$

M1

$$\frac{x^3}{3} - \frac{3}{x} + c$$

A1

Answer: $\frac{x^3}{3} - \frac{3}{x} + c$

2 marks

6 (2 marks)

Expand: $3x^3 - 2x^2$

M1

$$\frac{3x^4}{4} - \frac{2x^3}{3} + c$$

A1

Answer: $\frac{3x^4}{4} - \frac{2x^3}{3} + c$

2 marks

7 (4 marks)

$$y = \int (4x^3 - 3x^2 + 2) \, dx = x^4 - x^3 + 2x + c$$

M1 A1

Substitute (1, 5): $5 = 1 - 1 + 2 + c$

M1

So $c = 3$

A1

Answer: $y = x^4 - x^3 + 2x + 3$

4 marks

8 (5 marks)

$$\text{Expand the numerator: } (x + 4)(x - 2) = x^2 + 2x - 8$$

M1

$$\text{Divide each term by } x^{\frac{1}{2}}: x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 8x^{-\frac{1}{2}}$$

M1 A1

$$\text{Integrate: } \frac{2}{5}x^{\frac{5}{2}} + 2 \times \frac{2}{3}x^{\frac{3}{2}} - 8 \times 2x^{\frac{1}{2}} + c$$

M1

$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} - 16x^{\frac{1}{2}} + c$$

A1

$$\text{Answer: } \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} - 16\sqrt{x} + c$$

5 marks

9 (3 marks)

$$\text{Expand: } 2x^2 - 5x - 3$$

M1

$$\text{Integrate: } \frac{2x^3}{3} - \frac{5x^2}{2} - 3x$$

M1

$$\frac{2x^3}{3} - \frac{5x^2}{2} - 3x + c$$

A1

$$\text{Answer: } \frac{2x^3}{3} - \frac{5x^2}{2} - 3x + c$$

3 marks

10 (3 marks)

$$\text{Divide by } x^2: 6x - 2 + 5x^{-2}$$

M1 A1

$$= 3x^2 - 2x - \frac{5}{x} + c$$

A1

$$\text{Answer: } 3x^2 - 2x - \frac{5}{x} + c$$

3 marks

11 (3 marks)

$$f(x) = 2x^3 - 2x^2 + x + c$$

M1

$$f(2) = 16 - 8 + 2 + c = 10, \text{ so } c = 0$$

M1

$$f(x) = 2x^3 - 2x^2 + x$$

A1

$$\text{Answer: } f(x) = 2x^3 - 2x^2 + x$$

3 marks

12 (4 marks)

$$y = 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c$$

M1 A1

$$20 = 2 \times 8 + c, \text{ so } c = 4$$

M1

$$y = 2x^{\frac{3}{2}} + 4$$

A1

$$\text{Answer: } y = 2x^{\frac{3}{2}} + 4$$

4 marks

13 (4 marks)

$$v = \int (6t - 4) dt = 3t^2 - 4t + c$$

M1 A1

$$t = 0, v = 3 \text{ so } c = 3; \text{ at } t = 2: v = 12 - 8 + 3$$

M1

$$v = 7 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 7 \text{ m s}^{-1}$$

4 marks

14 (7 marks)

$$\text{(a) At } x = 2: 3(2)^2 + 2k - 4 = 12$$

M1

$$8 + 2k = 12, \text{ so } k = 2$$

A1

$$\text{Answer: } k = 2$$

2 marks

$$\text{(b) } y = \int (3x^2 + 2x - 4) dx = x^3 + x^2 - 4x + c$$

M1 A1

$$\text{Substitute } (2, 1): 1 = 8 + 4 - 8 + c, \text{ so } c = -3$$

M1 A1

$$\text{Hence } y = x^3 + x^2 - 4x - 3$$

A1

$$\text{Answer: } y = x^3 + x^2 - 4x - 3$$

5 marks

15 (6 marks)

$$(a) \frac{d}{dx} \left[\frac{(2x+1)^3}{3} \right] = \frac{3(2x+1)^2 \times 2}{3} = 2(2x+1)^2$$

M1 A1

This is twice the integrand, so the student's answer is incorrect

E1

Answer: derivative is $2(2x+1)^2$, not $(2x+1)^2$

3 marks

$$(b) (2x+1)^2 = 4x^2 + 4x + 1$$

M1

$$\int (4x^2 + 4x + 1) dx = \frac{4x^3}{3} + 2x^2 + x + c$$

M1 A1

$$\text{Answer: } \frac{4x^3}{3} + 2x^2 + x + c$$

3 marks

16 (5 marks)

$$\text{Expand the bracket: } x^3 - 4x^{\frac{3}{2}} + 4$$

M1

$$\text{Divide by } x^{\frac{1}{2}}: x^{\frac{5}{2}} - 4x + 4x^{-\frac{1}{2}}$$

M1 A1

$$\text{Integrate: } \frac{2}{7}x^{\frac{7}{2}} - 2x^2 + 8x^{\frac{1}{2}} + c$$

M1 A1

$$\text{Answer: } \frac{2}{7}x^{\frac{7}{2}} - 2x^2 + 8\sqrt{x} + c$$

5 marks

17 (5 marks)

$$y = x^3 - 6x^2 + 9x + c$$

M1 A1

$$\text{Through } (0, 0): c = 0, \text{ so } y = x^3 - 6x^2 + 9x$$

A1

$$\text{Factorise: } y = x(x^2 - 6x + 9) = x(x-3)^2$$

M1

$(x-3)$ is a repeated factor, so the curve meets the x-axis at $x=3$ without crossing it: it touches the x-axis at $(3, 0)$

E1

$$\text{Answer: } y = x^3 - 6x^2 + 9x = x(x-3)^2; \text{ shown}$$

5 marks

18 (5 marks)

$$y = x^3 - x^2 + kx + c$$

M1 A1

$$(0, 1) \text{ gives } c = 1$$

B1

$$(2, 9): 8 - 4 + 2k + 1 = 9$$

M1

$$2k = 4, \text{ so } k = 2$$

A1

$$\text{Answer: } k = 2$$

5 marks

19 (4 marks)

(a) The student has differentiated (halved the power and lowered it) instead of integrating: the power should increase by one

E1

Also the coefficient should be divided by the new power, not multiplied by the old one

E1

Answer: differentiated instead of integrating

2 marks

$$(b) \sqrt{x} = x^{\frac{1}{2}}, \text{ so the integral is } \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

M1

$$= \frac{2}{3}x^{\frac{3}{2}} + c$$

A1

$$\text{Answer: } \frac{2}{3}x^{\frac{3}{2}} + c$$

2 marks

20 (5 marks)

$$s = \int (3t^2 - 8t + 4) dt = t^3 - 4t^2 + 4t + c$$

M1 A1

$$t = 0, s = 5 \text{ so } c = 5$$

M1

$$t = 3: s = 27 - 36 + 12 + 5$$

M1

$$s = 8 \text{ m}$$

A1

$$\text{Answer: } 8 \text{ m}$$

5 marks

21 (9 marks)

$$f'(x) = \int (12x - 6) \, dx = 6x^2 - 6x + c \quad \text{M1 A1}$$

$$f'(1) = 6 - 6 + c = 5, \text{ so } c = 5 \quad \text{M1 A1}$$

$$f(x) = \int (6x^2 - 6x + 5) \, dx = 2x^3 - 3x^2 + 5x + d \quad \text{M1 A1}$$

$$f(2) = 16 - 12 + 10 + d = 20, \text{ so } d = 6 \quad \text{M1 A1}$$

$$f(x) = 2x^3 - 3x^2 + 5x + 6, \text{ so } f(0) = 6 \quad \text{B1}$$

$$\text{Answer: } f(x) = 2x^3 - 3x^2 + 5x + 6; f(0) = 6 \quad \text{9 marks}$$

22 (9 marks)

$$(a) V = \int (6t^{\frac{1}{2}} - t) \, dt = 6 \times \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^2}{2} + c \quad \text{M1 A1}$$

$$t = 0, V = 200 \text{ gives } c = 200, \text{ so } V = 4t^{\frac{3}{2}} - \frac{t^2}{2} + 200 \quad \text{M1 A1}$$

$$\text{Answer: } V = 4t^{\frac{3}{2}} - \frac{t^2}{2} + 200 \quad \text{4 marks}$$

$$(b) \text{ Greatest } V \text{ when } \frac{dV}{dt} = 0: \sqrt{t}(6 - \sqrt{t}) = 0, \text{ so } t = 36 \text{ (} t = 0 \text{ gives the minimum)} \quad \text{M1 A1}$$

$$V = 4 \times 216 - 648 + 200 = 416 \text{ litres} \quad \text{A1}$$

$$\text{Answer: } 416 \text{ litres} \quad \text{3 marks}$$

$$(c) 4t^{\frac{3}{2}} - \frac{t^2}{2} = 0, \text{ so } t^{\frac{3}{2}}(4 - \frac{1}{2}t^{\frac{1}{2}}) = 0 \text{ and } t^{\frac{1}{2}} = 8 \quad \text{M1}$$

$$t = 64 \text{ minutes} \quad \text{A1}$$

$$\text{Answer: } t = 64 \text{ minutes} \quad \text{2 marks}$$

23 (8 marks)

$$(a) h = \int (6 - 3t^{\frac{1}{2}}) \, dt = 6t - 2t^{\frac{3}{2}} + c \quad \text{M1 A1}$$

$$h = 0 \text{ when } t = 0, \text{ so } c = 0 \quad \text{A1}$$

$$\text{Answer: } h = 6t - 2t^{\frac{3}{2}} \quad \text{3 marks}$$

$$(b) \frac{dh}{dt} = 0: \sqrt{t} = 2, \text{ so } t = 4 \quad \text{M1 A1}$$

$$h = 24 - 2 \times 8 = 8 \text{ m} \quad \text{A1}$$

$$\text{Answer: } 8 \text{ m} \quad \text{3 marks}$$

$$(c) 6t - 2t^{\frac{3}{2}} = 2t(3 - \sqrt{t}) = 0 \quad \text{M1}$$

$$\sqrt{t} = 3, \text{ so } t = 9 \text{ (09:00)} \quad \text{A1}$$

$$\text{Answer: } t = 9 \quad \text{2 marks}$$

24 (8 marks)

$$f'(x) = 3x^2 - 4x + a \quad \text{M1 A1}$$

$$f'(1) = 0: 3 - 4 + a = 0, \text{ so } a = 1 \quad \text{M1 A1}$$

$$f(x) = x^3 - 2x^2 + x + b; f(1) = 2 \text{ gives } 1 - 2 + 1 + b = 2, \text{ so } b = 2 \quad \text{M1 A1}$$

$$f(x) = x^3 - 2x^2 + x + 2 \quad \text{A1}$$

$$f''(1) = 2 > 0, \text{ so the point is a minimum} \quad \text{E1}$$

$$\text{Answer: } f(x) = x^3 - 2x^2 + x + 2; \text{ minimum} \quad \text{8 marks}$$

25 (8 marks)

$$(a) \text{ Divide each term by } x^2: \frac{x^3}{x^2} - \frac{8}{x^2} \quad \text{M1}$$

$$= x - 8x^{-2} \quad \text{A1}$$

$$\text{Answer: shown} \quad \text{2 marks}$$

$$(b) y = \frac{x^2}{2} + 8x^{-1} + c \quad \text{M1 A1}$$

$$(2, 5): 2 + 4 + c = 5, \text{ so } c = -1 \quad \text{M1}$$

$$y = \frac{x^2}{2} + \frac{8}{x} - 1$$

A1

Answer: $y = \frac{x^2}{2} + \frac{8}{x} - 1$

4 marks

(c) Gradient at $x = 2$: $2 - 8/4 = 0$

M1

Tangent: $y = 5$

A1

Answer: $y = 5$

2 marks

26 (8 marks)

(a) $V = \frac{t^3}{2} - 6t^2 + 40t + c$

M1 A1

$t = 0, V = 12$ so $c = 12$

A1

Answer: $V = \frac{t^3}{2} - 6t^2 + 40t + 12$

3 marks

(b) $V = 32 - 96 + 160 + 12$

M1

$= 108 \text{ m}^3$

A1

Answer: 108 m^3

2 marks

(c) $\frac{dV}{dt} = \frac{3}{2}(t^2 - 8t) + 40 = \frac{3}{2}(t - 4)^2 + 16$

M1 A1

A square is non-negative, so $\frac{dV}{dt} \geq 16 > 0$ for all t

E1

Answer: $dV/dt \geq 16 > 0$

3 marks

27 (11 marks)

(a) At $x = 3$, $\frac{dy}{dx} = 0$: $9a + 3b = 0$

M1

So $b = -3a$

A1

Answer: $b = -3a$

2 marks

(b) $y = \frac{ax^3}{3} + \frac{bx^2}{2} + c$; passes through $(0, 2)$ so $c = 2$

M1 A1

Substitute $(3, -7)$ with $b = -3a$: $9a - \frac{27a}{2} + 2 = -7$

M1

$-\frac{9a}{2} = -9$, so $a = 2$ and $b = -6$

A1

$y = \frac{2}{3}x^3 - 3x^2 + 2$

A1

Answer: $a = 2, b = -6, y = \frac{2}{3}x^3 - 3x^2 + 2$

5 marks

(c) $\frac{dy}{dx} = 2x^2 - 6x = 2x(x - 3) = 0$ gives $x = 0$, so the point is $(0, 2)$

M1 A1

$\frac{d^2y}{dx^2} = 4x - 6 = -6 < 0$ at $x = 0$, so $(0, 2)$ is a maximum

M1 A1

Answer: $(0, 2)$ is a maximum

4 marks

28 (11 marks)

(a) Expand: $(x^{\frac{3}{2}} + 2x^{-1})^2 = x^3 + 4x^{\frac{1}{2}} + 4x^{-2}$

M1 A1

$f(x) = \frac{x^4}{4} + 4 \times \frac{2}{3}x^{\frac{3}{2}} + 4 \times \frac{x^{-1}}{-1} + c$

M1 A1

$= \frac{x^4}{4} + \frac{8}{3}x^{\frac{3}{2}} - \frac{4}{x} + c$

A1

$f(1) = 0$: $\frac{1}{4} + \frac{8}{3} - 4 + c = 0$, so $c = \frac{13}{12}$

M1 A1

Answer: $f(x) = \frac{x^4}{4} + \frac{8}{3}x^{\frac{3}{2}} - \frac{4}{x} + \frac{13}{12}$

7 marks

(b) $f(4) = 64 + \frac{8}{3} \times 8 - 1 + \frac{13}{12}$

M1

$= \frac{1025}{12}$

A1

Answer: $\frac{1025}{12}$

2 marks

(c) $f'(x)$ is a square, so $f'(x) > 0$ for all $x > 0$: f is strictly increasing

E1

A strictly increasing function takes the value 0 once only, which happens at $x = 1$

E1

Answer: f is strictly increasing, and $f(1) = 0$

2 marks

29 (12 marks)(a) Stationary points need $3x^2 - 6x + k = 0$

M1

Discriminant: $36 - 12k < 0$ when $k > 3$

M1 A1

So there are no real roots and no stationary points

E1

Answer: shown

4 marks

(b) $y = x^3 - 3x^2 + 3x + c$

M1 A1

Substitute $(1, -1)$: $1 - 3 + 3 + c = -1$, so $c = -2$

M1

 $y = x^3 - 3x^2 + 3x - 2$

A1

Answer: $y = x^3 - 3x^2 + 3x - 2$

4 marks

(c) $y = (x - 1)^3 - 1$

M1 A1

 $y = 0$ gives $(x - 1)^3 = 1$, so $x = 2$ only (a cube takes each value once)

M1 A1

Answer: $x = 2$ only

4 marks

30 (12 marks)(a) $f''(x) = 6x + 4x^{-3}$, so $f'(x) = 3x^2 - 2x^{-2} + a$

M1 A1

 $f'(1) = 3 - 2 + a = 4$, so $a = 3$ and $f'(x) = 3x^2 - \frac{2}{x^2} + 3$

A1

Answer: $f'(x) = 3x^2 - \frac{2}{x^2} + 3$

3 marks

(b) $f(x) = x^3 + 2x^{-1} + 3x + b$

M1 A1

 $f(1) = 1 + 2 + 3 + b = 4$, so $b = -2$

M1

 $f(2) = 8 + 1 + 6 - 2 = 13$

A1

Answer: $f(x) = x^3 + \frac{2}{x} + 3x - 2$; $f(2) = 13$

4 marks

(c) $f'(x) = 0$: $3x^4 + 3x^2 - 2 = 0$ (multiply by x^2)

M1

A quadratic in x^2 : $x^2 = \frac{-3 \pm \sqrt{9 + 24}}{6} = \frac{-3 \pm \sqrt{33}}{6}$

M1 A1

 x^2 must be positive, so only $x^2 = \frac{-3 + \sqrt{33}}{6}$ is valid: one stationary point

E1

$$x = \sqrt{\frac{-3 + \sqrt{33}}{6}}$$

A1

Answer: $x = \sqrt{\frac{-3 + \sqrt{33}}{6}}$

5 marks

Definite Integration**1 (2 marks)**

$$\int_1^3 (2x + 1) \, dx = [x^2 + x]_1^3$$

M1

$$= (9 + 3) - (1 + 1) = 10$$

A1

Answer: 10

2 marks

2 (2 marks)

$$[x^3 - x^2]_0^2 = (8 - 4) - 0$$

M1

$$= 4$$

A1

Answer: 4

2 marks

3 (2 marks)

$$[2x^2 + x]_0^3 = (18 + 3) - 0$$

$$= 21$$

M1

A1

Answer: 21

2 marks

4 (2 marks)

$$[\frac{x^3}{3} - x^2]_{-1}^2 = (\frac{8}{3} - 4) - (-\frac{1}{3} - 1)$$

$$= -\frac{4}{3} + \frac{4}{3} = 0$$

M1

A1

Answer: 0

2 marks

5 (2 marks)

$$3x^{-\frac{1}{2}} \text{ integrates to } 6x^{\frac{1}{2}}$$

$$[6\sqrt{x}]_1^4 = 12 - 6 = 6$$

M1

A1

Answer: 6

2 marks

6 (2 marks)

$$[\frac{x^4}{4} - 2x^2]_0^2 = (4 - 8) - 0$$

$$= -4$$

M1

A1

Answer: -4

2 marks

7 (4 marks)

$$\text{Write as powers: } 3x^{\frac{1}{2}} + 2x^{-2}$$

M1

$$\text{Integral} = [2x^{\frac{3}{2}} - \frac{2}{x}]_1^4$$

M1 A1

$$= (16 - \frac{1}{2}) - (2 - 2) = \frac{31}{2}$$

A1

Answer: $\frac{31}{2}$

4 marks

8 (4 marks)

$$[x^2 + 3x]_1^a = (a^2 + 3a) - 4$$

M1 A1

$$a^2 + 3a - 4 = 14, \text{ so } a^2 + 3a - 18 = 0$$

M1

$$(a + 6)(a - 3) = 0, \text{ and } a > 1 \text{ so } a = 3$$

A1

Answer: a = 3

4 marks

9 (4 marks)

$$\text{Divide: } x + 2x^{-2}$$

M1

$$[\frac{x^2}{2} - \frac{2}{x}]_1^3 = (\frac{9}{2} - \frac{2}{3}) - (\frac{1}{2} - 2)$$

M1 A1

$$= \frac{16}{3}$$

A1

Answer: $\frac{16}{3}$

4 marks

10 (4 marks)

$$\text{Divide: } x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

M1

$$[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}]_1^9 = (18 + 6) - (\frac{2}{3} + 2)$$

M1 A1

$$= \frac{64}{3}$$

A1

Answer: $\frac{64}{3}$

4 marks

11 (4 marks)

$$[x^3 + 2x]_0^k = k^3 + 2k$$

M1

$$k^3 + 2k - 12 = 0$$

A1

$$(k - 2)(k^2 + 2k + 6) = 0; \text{ the quadratic has negative discriminant, so } k = 2$$

M1 A1

Answer: $k = 2$

4 marks

12 (4 marks)

$$(a) \int_3^5 f(x) \, dx = 7 - 4$$

M1

$$= 3$$

A1

Answer: 3

2 marks

$$(b) 2 \times 7 + [3x]_1^5$$

M1

$$= 14 + 12 = 26$$

A1

Answer: 26

2 marks

13 (4 marks)

$$\text{Expand: } x(x^2 - 4x + 4) = x^3 - 4x^2 + 4x$$

M1

$$\left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 \right]_0^2$$

M1

$$= 4 - \frac{32}{3} + 8$$

M1

$$= \frac{4}{3}$$

A1

Answer: $\frac{4}{3}$

4 marks

14 (6 marks)

$$\text{Expand: } \frac{(x^2 - 1)^2}{x^2} = \frac{x^4 - 2x^2 + 1}{x^2} = x^2 - 2 + x^{-2}$$

M1 A1

$$\int_1^2 (x^2 - 2 + x^{-2}) \, dx = \left[\frac{x^3}{3} - 2x - \frac{1}{x} \right]_1^2$$

M1 A1

$$= \left(\frac{8}{3} - 4 - \frac{1}{2} \right) - \left(\frac{1}{3} - 2 - 1 \right)$$

M1

$$= -\frac{11}{6} + \frac{8}{3} = \frac{5}{6}$$

A1

Answer: $\frac{5}{6}$

6 marks

15 (7 marks)

$$(a) \text{ Region B lies below the axis, so its integral is negative: } 7 - 4 = 3$$

B1

Answer: 3

1 mark

$$(b) \int_0^6 f(x) \, dx = 7 - 4 + 2 = 5$$

M1

$$\int_0^6 (2f(x) + 3) \, dx = 2 \times 5 + 3 \times 6$$

M1

$$= 28$$

A1

Answer: 28

3 marks

$$(c) \int_5^2 f(x) \, dx = -\int_2^5 f(x) \, dx = -(-4) = 4$$

M1 A1

$$\text{Total area} = 4 + 2 = 6$$

B1

Answer: 4; area = 6

3 marks

16 (5 marks)Write as $x^{\frac{1}{3}} + x^{-\frac{1}{3}}$

M1

$$\left[\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}}\right]_1^8$$

M1 A1

$$= \left(\frac{3}{4} \times 16 + \frac{3}{2} \times 4\right) - \left(\frac{3}{4} + \frac{3}{2}\right)$$

M1

$$= 18 - \frac{9}{4} = \frac{63}{4}$$

A1

Answer: $\frac{63}{4}$

5 marks

17 (4 marks)

$$\text{Integrate: } \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_{-3}^3$$

M1 A1

$$\text{Upper: } \frac{81}{4} + 18 + \frac{9}{2}; \text{ lower: } \frac{81}{4} - 18 + \frac{9}{2}$$

M1

$$\text{Difference} = 36$$

A1

Answer: 36

4 marks

18 (4 marks)

$$\text{Increase} = \int_4^9 12t^{\frac{1}{2}} dt$$

M1

$$= [8t^{\frac{3}{2}}]_4^9$$

M1 A1

$$= 8 \times 27 - 8 \times 8 = 152 \text{ thousand}$$

A1

Answer: 152 thousand

4 marks

19 (4 marks)

$$\left[-\frac{1}{x}\right]_1^a = -\frac{1}{a} + 1$$

M1 A1

$$1 - \frac{1}{a} = \frac{1}{3}, \text{ so } \frac{1}{a} = \frac{2}{3}$$

M1

$$a = \frac{3}{2}$$

A1

Answer: $a = \frac{3}{2}$

4 marks

20 (6 marks)

$$\text{First: } \left[\frac{px^2}{2} + qx\right]_0^1 = \frac{p}{2} + q = 5$$

M1 A1

$$\text{Second: } \left[\frac{px^2}{2} + qx\right]_1^3 = 4p + 2q = 16$$

M1 A1

$$\text{From the first, } p + 2q = 10; \text{ subtract: } 3p = 6$$

M1

$$p = 2, q = 4$$

A1

Answer: $p = 2, q = 4$

6 marks

21 (8 marks)**(a)** The integrand $\frac{1}{x^2}$ is positive wherever it is defined, so any area under it must be positive

E1

 $\frac{1}{x^2}$ is undefined at $x = 0$, which lies inside the interval, so the integral cannot be found this way

E1

Answer: integrand is positive and undefined at $x = 0$

2 marks

$$\text{(b)} \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1)$$

M1

$$= \frac{1}{2}$$

A1

Answer: $\frac{1}{2}$

2 marks

$$\text{(c)} \left[-\frac{1}{x}\right]_{-2}^{-1} = 1 - \frac{1}{2} = \frac{1}{2}$$

M1 A1

$\frac{1}{x^2}$ is an even function, so its graph is symmetrical about the y-axis and the areas match

E1 E1

Answer: $\frac{1}{2}$

4 marks

22 (9 marks)

(a) $\frac{dq}{dt} = 12 - 6t = 0$ gives $t = 2$

M1 A1

$q = 24 - 12 = 12$ litres per minute

A1

Answer: 12 litres per minute

3 marks

(b) Volume = $\int_0^4 (12t - 3t^2) dt = [6t^2 - t^3]_0^4$

M1 A1

= $96 - 64 = 32$ litres

A1

Answer: 32 litres

3 marks

(c) Volume to time T is $6T^2 - T^3$; at $T = 3$ this is $54 - 27 = 27$

B1

$6T^2 - T^3 = 27$ gives $T^3 - 6T^2 + 27 = 0$, which factorises as $(T - 3)(T^2 - 3T - 9) = 0$

M1

$T^2 - 3T - 9 = 0$ gives $T = \frac{3 \pm \sqrt{45}}{2}$, i.e. $T \approx 4.85$ or $T \approx -1.85$, both outside $0 < T < 4$

A1

Answer: $T = 3$ only

3 marks

23 (8 marks)

(a) Mean = $\frac{1}{3} \int_1^4 x^2 dx$

M1

= $\frac{1}{3} [\frac{x^3}{3}]_1^4 = \frac{1}{3} \times \frac{63}{3}$

M1 A1

= 7

A1

Answer: 7

4 marks

(b) Mean = $\frac{1}{k} [x^3 - 2x^2]_0^k = k^2 - 2k$

M1 A1

$k^2 - 2k = 8$, so $(k - 4)(k + 2) = 0$

M1

$k = 4$ (since $k > 0$)

A1

Answer: $k^2 - 2k$; $k = 4$

4 marks

24 (9 marks)

(a) $t^2 - 4t + 3 = (t - 1)(t - 3) = 0$

M1

$t = 1$ and $t = 3$

A1

Answer: $t = 1$, $t = 3$

2 marks

(b) $\int_0^4 (t^2 - 4t + 3) dt = [\frac{t^3}{3} - 2t^2 + 3t]_0^4$

M1

= $\frac{64}{3} - 32 + 12 = \frac{4}{3}$ m

M1 A1

Answer: $\frac{4}{3}$ m

3 marks

(c) Split at the times when $v = 0$: the integrals over $[0, 1]$, $[1, 3]$, $[3, 4]$ are $\frac{4}{3}$, $-\frac{4}{3}$, $\frac{4}{3}$

M1 A1

Distance = $\frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ m

M1 A1

Answer: 4 m

4 marks

25 (7 marks)

(a) $[12x - x^3]_{-a}^a = (12a - a^3) - (-12a + a^3)$

M1 A1

= $24a - 2a^3$

A1

Answer: shown

3 marks

(b) $I'(a) = 24 - 6a^2 = 0$, so $a = 2$

M1 A1

$I''(a) = -12a < 0$, so a maximum

M1

$I(2) = 48 - 16 = 32$

A1

Answer: $a = 2$, $I = 32$

4 marks

26 (7 marks)

$$(a) [t^3 + t]_1^x = (x^3 + x) - (1 + 1)$$

M1 A1

$$= x^3 + x - 2$$

A1

Answer: shown

3 marks

(b) $F'(x) = 3x^2 + 1 > 0$, so F is strictly increasing and $F(x) = 8$ has at most one solution

M1 E1

$$x^3 + x - 2 = 8 \text{ gives } x^3 + x - 10 = 0$$

M1

$$(x - 2)(x^2 + 2x + 5) = 0 \text{ and the quadratic has no real roots, so } x = 2$$

A1

Answer: $F' > 0$; $x = 2$

4 marks

27 (8 marks)

$$(a) [2x^2 - \frac{6}{x}]_1^k = (2k^2 - \frac{6}{k}) - (2 - 6)$$

M1 A1

$$2k^2 - \frac{6}{k} + 4 = 12$$

M1

$$\text{Multiply by } k \text{ and rearrange: } 2k^3 - 6 + 4k = 12k, \text{ so } k^3 - 4k - 3 = 0$$

A1

Answer: shown

4 marks

(b) $f(-1) = -1 + 4 - 3 = 0$, so $(k + 1)$ is a factor

M1

$$k^3 - 4k - 3 = (k + 1)(k^2 - k - 3)$$

A1

$$k = \frac{1 \pm \sqrt{13}}{2}$$

M1

$$k > 1, \text{ so } k = \frac{1 + \sqrt{13}}{2}$$

A1

$$\text{Answer: } k = \frac{1 + \sqrt{13}}{2}$$

4 marks

28 (10 marks)

$$(a) \int_{-a}^a (px^3 + qx^2 + rx + s) dx = [\frac{px^4}{4} + \frac{qx^3}{3} + \frac{rx^2}{2} + sx]_{-a}^a$$

M1 A1

The terms in x^4 and x^2 take equal values at $x = a$ and $x = -a$, so they cancel

M1

$$\text{The terms in } x^3 \text{ and } x \text{ double: } \frac{qa^3}{3} - (-\frac{qa^3}{3}) = \frac{2qa^3}{3}, \text{ and } sa - (-sa) = 2sa$$

M1

$$\text{Total} = \frac{2qa^3}{3} + 2sa = 2a(\frac{qa^2}{3} + s)$$

A1

Answer: shown

5 marks

$$(b) p = 5, q = -3, r = 4, s = 7, a = 2$$

M1

$$2 \times 2 \times (-\frac{3 \times 4}{3} + 7) = 4 \times 3 = 12$$

A1

Answer: 12

2 marks

$$(c) 2a(2a^2 - 4) = 0, \text{ so } 4a^3 - 8a = 0$$

M1 A1

$$a^2 = 2, \text{ so } a = \sqrt{2}$$

A1

$$\text{Answer: } a = \sqrt{2}$$

3 marks

29 (10 marks)

$$(a) I(k) = [\frac{x^3}{3} - 2x^2 + 3x]_0^k = \frac{k^3}{3} - 2k^2 + 3k$$

M1 A1

$$= \frac{k}{3}(k^2 - 6k + 9)$$

M1

$$= \frac{k(k - 3)^2}{3}$$

A1

Answer: shown

4 marks

$$(b) I(k) = 0 \text{ gives } k = 3 \text{ only } (k > 0)$$

M1 A1

$$k > 0 \text{ and } (k - 3)^2 \geq 0, \text{ so } I(k) \geq 0$$

E1

Answer: $k = 3$

3 marks

(c) f is positive on $(0, 1)$, negative on $(1, 3)$, positive for $x > 3$

B1

The area below the x -axis between $x = 1$ and 3 is exactly cancelled by the area above for $0 < x <$

E1 E1

1 when $k = 3$ (each equal to $\frac{4}{3}$)

Answer: net area is zero at $k = 3$

3 marks

30 (8 marks)

(a) $\left[\frac{x^{n+1}}{n+1} + \frac{x^{2n+1}}{2n+1} \right]_0^1$

M1 A1

Lower limit gives 0, upper gives the stated terms

A1

Answer: shown

3 marks

(b) $\frac{1}{n+1} + \frac{1}{2n+1} = \frac{11}{28}$

M1

Common denominator: $28(3n+2) = 11(n+1)(2n+1)$

M1 A1

$22n^2 - 51n - 45 = 0, (n-3)(22n+15) = 0$

M1

$n = 3$ (n is positive)

A1

Answer: $n = 3$

5 marks

Area Under and Between Curves

1 (2 marks)

Area = $\int_0^2 3x^2 \, dx = [x^3]_0^2$

M1

= 8

A1

Answer: 8

2 marks

2 (3 marks)

Area = $\int_1^3 (x^2 + 1) \, dx = \left[\frac{x^3}{3} + x \right]_1^3$

M1 A1

= $(9 + 3) - \left(\frac{1}{3} + 1\right) = \frac{32}{3}$

A1

Answer: $\frac{32}{3}$

3 marks

3 (2 marks)

Area = $\int_1^3 4x \, dx = [2x^2]_1^3$

M1

= $18 - 2 = 16$

A1

Answer: 16

2 marks

4 (2 marks)

Area = $\int_0^3 (x^2 + 2) \, dx = \left[\frac{x^3}{3} + 2x \right]_0^3$

M1

= $9 + 6 = 15$

A1

Answer: 15

2 marks

5 (2 marks)

Area = $\int_0^4 x^{\frac{1}{2}} \, dx = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^4$

M1

= $\frac{2}{3} \times 8 = \frac{16}{3}$

A1

Answer: $\frac{16}{3}$

2 marks

6 (2 marks)

Area = $\int_1^2 x^{-2} \, dx = \left[-\frac{1}{x} \right]_1^2$

M1

= $-\frac{1}{2} + 1 = \frac{1}{2}$

A1

Answer: $\frac{1}{2}$

2 marks

7 (4 marks)Curve meets the x-axis at $x = 0$ and $x = 4$

B1

$$\int_0^4 (x^2 - 4x) \, dx = \left[\frac{x^3}{3} - 2x^2 \right]_0^4 = \frac{64}{3} - 32 = -\frac{32}{3}$$

M1 A1

The region is below the axis, so the area is $\frac{32}{3}$

A1

Answer: $\frac{32}{3}$

4 marks

8 (4 marks) $x^2 = 2x$ gives $x = 0$ or $x = 2$, so P is (2, 4)

M1 A1

$$\text{Area} = \int_0^2 (2x - x^2) \, dx = \left[x^2 - \frac{x^3}{3} \right]_0^2$$

M1

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

A1

Answer: P(2, 4); area $\frac{4}{3}$

4 marks

9 (4 marks)

$$\int_1^4 (x^2 - 5x + 4) \, dx = \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_1^4$$

M1 A1

$$= \left(\frac{64}{3} - 40 + 16 \right) - \left(\frac{1}{3} - \frac{5}{2} + 4 \right) = -\frac{9}{2}$$

M1

The region is below the x-axis, so area = $\frac{9}{2}$

A1

Answer: $\frac{9}{2}$

4 marks

10 (4 marks) $x(6 - x) = 2x$ gives $x = 0$ or $x = 4$

M1 A1

$$\text{Area} = \int_0^4 (4x - x^2) \, dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4$$

M1

$$= 32 - \frac{64}{3} = \frac{32}{3}$$

A1

Answer: $\frac{32}{3}$

4 marks

11 (4 marks)Rearrange: $x = \sqrt{y}$

B1

$$\text{Area} = \int_0^4 \sqrt{y} \, dy = \left[\frac{2}{3} y^{\frac{3}{2}} \right]_0^4$$

M1 A1

$$= \frac{16}{3}$$

A1

Answer: $\frac{16}{3}$

4 marks

12 (4 marks)

$$\text{Area} = \int_0^k 6x^2 \, dx = \left[2x^3 \right]_0^k = 2k^3$$

M1 A1

$$2k^3 = 16, \text{ so } k^3 = 8$$

M1

$$k = 2$$

A1

Answer: $k = 2$

4 marks

13 (4 marks)The curve crosses the axis at $x = -2, 0, 2$

B1

$$\int_{-2}^0 (x^3 - 4x) \, dx = \left[\frac{x^4}{4} - 2x^2 \right]_{-2}^0 = 0 - (4 - 8) = 4$$

M1 A1

By symmetry (odd function) the other region also has area 4, total 8

A1

Answer: 8

4 marks

14 (7 marks)Expand: $y = x^3 - x^2 - 2x$; intercepts at $x = -1, 0, 2$

B1 M1

$$\int_{-1}^0 y \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 = 0 - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) = \frac{5}{12}$$

M1 A1

$$\int_0^2 y \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 = 4 - \frac{8}{3} - 4 = -\frac{8}{3}$$

M1 A1

$$\text{Total area} = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

A1

Answer: $\frac{37}{12}$

7 marks

15 (7 marks)(a) $9 - x^2 = x + 3$, so $x^2 + x - 6 = 0$

M1

 $(x + 3)(x - 2) = 0$, so A(-3, 0) and B(2, 5)

A1 A1

Answer: A(-3, 0), B(2, 5)

3 marks

$$(b) \text{Area} = \int_{-3}^2 ((9 - x^2) - (x + 3)) \, dx = \int_{-3}^2 (6 - x - x^2) \, dx$$

M1

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-3}^2$$

M1 A1

$$= \left(\frac{22}{3} \right) - \left(-\frac{27}{2} \right) = \frac{125}{6}$$

A1

Answer: $\frac{125}{6}$

4 marks

16 (6 marks) $x^2 = 2x + 3$, so $x^2 - 2x - 3 = (x - 3)(x + 1) = 0$

M1

A(-1, 1) and B(3, 9)

A1

$$\text{Area} = \int_{-1}^3 (2x + 3 - x^2) \, dx = \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^3$$

M1 A1

$$= 9 - \left(-\frac{5}{3} \right) = \frac{32}{3}$$

M1 A1

Answer: A(-1, 1), B(3, 9); area $\frac{32}{3}$

6 marks

17 (6 marks)

$$\frac{dy}{dx} = 2x = 4 \text{ at } x = 2$$

M1 A1

Tangent: $y - 4 = 4(x - 2)$, so $y = 4x - 4$

A1

$$\text{Area} = \int_0^2 (x^2 - 4x + 4) \, dx = \left[\frac{(x - 2)^3}{3} \right]_0^2$$

M1 A1

$$= 0 - \left(-\frac{8}{3} \right) = \frac{8}{3}$$

A1

Answer: $y = 4x - 4$; area $\frac{8}{3}$

6 marks

18 (6 marks) $x^3 - 9x = x(x - 3)(x + 3)$, so the curve meets the axis at $x = -3, 0, 3$

M1 A1

$$\int_{-3}^0 (x^3 - 9x) \, dx = \left[\frac{x^4}{4} - \frac{9x^2}{2} \right]_{-3}^0 = 0 - \left(\frac{81}{4} - \frac{81}{2} \right) = \frac{81}{4}$$

M1 A1

The curve is odd, so the second region has the same area; total = $\frac{81}{2}$

M1 A1

Answer: $\frac{81}{2}$

6 marks

19 (6 marks)

The curve meets the line where $x = \pm\sqrt{k}$

B1

$$\text{Area} = \int_{-\sqrt{k}}^{\sqrt{k}} (k - x^2) \, dx = 2 \times \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}}$$

M1 A1

$$= 2\left(k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}}\right) = \frac{4}{3}k^{\frac{3}{2}}$$

A1

$$\frac{4}{3}k^{\frac{3}{2}} = 36, \text{ so } k^{\frac{3}{2}} = 27 \text{ and } k = 9$$

M1 A1

Answer: $k = 9$

6 marks

20 (6 marks)

$$\text{Area} = \int_{-4}^4 \left(4 - \frac{x^2}{4}\right) \, dx$$

M1

$$= 2\left[4x - \frac{x^3}{12}\right]_0^4 = 2\left(16 - \frac{16}{3}\right)$$

M1 A1

$$= \frac{64}{3} \, \text{m}^2$$

A1

$$\text{Volume} = 50 \times \frac{64}{3} = 1066.7 \approx 1100 \, \text{m}^3 \text{ (2 s.f.)}$$

M1 A1

Answer: $\frac{64}{3} \, \text{m}^2$; $1100 \, \text{m}^3$

6 marks

21 (8 marks)

(a) Gradient = $\frac{2-8}{2-1} = -6$

M1 A1

$y - 8 = -6(x - 1)$, so $y = 14 - 6x$

A1

Answer: $y = 14 - 6x$

3 marks

(b) Area = $\int_1^2 (14 - 6x - 8x^{-2}) \, dx$

M1 A1

$$= \left[14x - 3x^2 + \frac{8}{x}\right]_1^2$$

M1 A1

$$= (28 - 12 + 4) - (14 - 3 + 8) = 20 - 19 = 1$$

A1

Answer: 1

5 marks

22 (9 marks)

(a) Area = $\int_0^a (ax - x^2) \, dx = \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a$

M1 A1

$$= \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}$$

A1

Answer: shown

3 marks

(b) $\frac{a^3}{6} = 36$, so $a^3 = 216$

M1

$a = 6$

A1

Answer: $a = 6$

2 marks

(c) $6x - x^2 = 2x$ gives $x = 0$ or $x = 4$

M1

Area between line and curve = $\int_0^4 (4x - x^2) \, dx = 32 - \frac{64}{3} = \frac{32}{3}$

M1 A1

Other part = $36 - \frac{32}{3} = \frac{76}{3}$, so the ratio is $32 : 76 = 8 : 19$

A1

Answer: 8 : 19

4 marks

23 (7 marks)

$$x^3 = x \text{ gives } x(x - 1)(x + 1) = 0$$

M1

Points $(-1, -1)$, $(0, 0)$, $(1, 1)$

A1

$$\text{For } 0 \leq x \leq 1 \text{ the line is above: } \int_0^1 (x - x^3) \, dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1$$

M1 A1

$$= \frac{1}{4}$$

A1

For $-1 \leq x \leq 0$ the curve is above the line; by symmetry about the origin the area is also $\frac{1}{4}$

M1

$$\text{Total area} = \frac{1}{2}$$

A1

Answer: $(-1, -1)$, $(0, 0)$, $(1, 1)$; area $\frac{1}{2}$

7 marks

24 (7 marks)

$$x(4 - x) = 3, \text{ so } x^2 - 4x + 3 = 0, \text{ giving } x = 1 \text{ and } x = 3$$

M1 A1

$$\text{Area} = \int_1^3 (4x - x^2 - 3) \, dx$$

M1

$$= \left[2x^2 - \frac{x^3}{3} - 3x \right]_1^3$$

M1 A1

$$= (18 - 9 - 9) - \left(2 - \frac{1}{3} - 3 \right) = \frac{4}{3}$$

M1 A1

Answer: $\frac{4}{3}$

7 marks

25 (9 marks)

$$\text{(a) Area of R} = \int_0^2 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^2$$

M1 A1

$$= \frac{8}{3}$$

A1

Answer: $\frac{8}{3}$

3 marks

$$\text{(b) } x = \sqrt{y}, \text{ so area of S} = \int_0^4 y^{\frac{1}{2}} \, dy$$

M1 A1

$$= \left[\frac{2}{3} y^{\frac{3}{2}} \right]_0^4 = \frac{2}{3} \times 8 = \frac{16}{3}$$

M1 A1

Answer: $\frac{16}{3}$

4 marks

$$\text{(c) } \frac{8}{3} + \frac{16}{3} = 8 = 2 \times 4, \text{ the area of the rectangle}$$

M1

Ratio R : S = 1 : 2

A1

Answer: 1 : 2

2 marks

26 (5 marks)

$$\text{Total area} = \int_0^2 3x^2 \, dx = [x^3]_0^2 = 8$$

M1 A1

Area from 0 to k is k^3

M1

$$k^3 = 4$$

M1

$$k = \sqrt[3]{4}$$

A1

Answer: $k = \sqrt[3]{4} \approx 1.59$

5 marks

27 (10 marks)

$$\text{(a) } y = x(x^2 - 6x + 8) = x(x - 2)(x - 4)$$

M1

P and Q have x-coordinates 2 and 4

A1

Answer: $x = 2$ and $x = 4$

2 marks

$$\text{(b) } \int_0^2 y \, dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 4 - 16 + 16 = 4$$

M1 A1

$$\int_2^4 y \, dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_2^4 = (64 - 128 + 64) - 4 = -4$$

M1 A1

The second region is below the axis, so its area is 4, equal to the first. Total area = 8

Answer: each area is 4; total 8

A1

5 marks

(c) $f(2 + h) = (2 + h)h(h - 2) = h(h^2 - 4)$

M1

$f(2 - h) = (2 - h)(-h)(-h - 2) = h(2 - h)(2 + h) = h(4 - h^2) = -f(2 + h)$

A1

So the curve has rotational symmetry of order 2 about (2, 0); the regions are congruent, so their areas are equal

E1

Answer: shown

3 marks

28 (12 marks)

(a) $\int_0^4 (4x - x^2) dx = [2x^2 - \frac{x^3}{3}]_0^4$

M1 A1

$= 32 - \frac{64}{3} = \frac{32}{3}$

A1

Answer: $\frac{32}{3}$

3 marks

(b) $4x - x^2 = mx$ gives $x(4 - m - x) = 0$

M1

So $x = 0$ or $x = 4 - m$

A1

Answer: shown

2 marks

(c) Area of R = $\int_0^{4-m} ((4 - m)x - x^2) dx$

M1

$= [\frac{(4 - m)x^2}{2} - \frac{x^3}{3}]_0^{4-m} = \frac{(4 - m)^3}{2} - \frac{(4 - m)^3}{3} = \frac{(4 - m)^3}{6}$

M1 A1

Answer: shown

3 marks

(d) $\frac{(4 - m)^3}{6} = \frac{1}{2} \times \frac{32}{3}$, so $(4 - m)^3 = 32$

M1 A1

$4 - m = \sqrt[3]{32} = 2\sqrt[3]{4}$, so $m = 4 - 2\sqrt[3]{4}$

M1

$m = 0.825$ (3 s.f.)

A1

Answer: $m = 4 - 2\sqrt[3]{4} \approx 0.825$

4 marks

29 (12 marks)

(a) Gradient = $\frac{b^2 - a^2}{b - a} = a + b$

M1

$y - a^2 = (a + b)(x - a)$, so $y = (a + b)x - a^2 - ab + a^2 = (a + b)x - ab$

M1 A1

Answer: shown

3 marks

(b) Area = $\int_a^b ((a + b)x - ab - x^2) dx$

M1

$= [\frac{(a + b)x^2}{2} - abx - \frac{x^3}{3}]_a^b$

M1 A1

$= \frac{(a + b)(b^2 - a^2)}{2} - ab(b - a) - \frac{b^3 - a^3}{3} = (b - a)[\frac{(a + b)^2}{2} - ab - \frac{a^2 + ab + b^2}{3}]$

M1

$= (b - a) \times \frac{(b - a)^2}{6} = \frac{(b - a)^3}{6}$

A1

Answer: shown

5 marks

(c) $x^2 = x + 2$ gives $(x - 2)(x + 1) = 0$, so $x = -1$ and 2

M1 A1

The result holds for any $a < b$, so with $a = -1$ and $b = 2$: area = $\frac{3^3}{6}$

M1

$= \frac{9}{2}$

A1

Answer: $\frac{9}{2}$

4 marks

30 (12 marks)

(a) $x^2 = \frac{16}{x^2}$ gives $x^4 = 16$, so $x = 2$ ($x > 0$), $y = 4$

M1 A1

Answer: shown

2 marks

(b) $\int_0^2 x^2 dx = [\frac{x^3}{3}]_0^2 = \frac{8}{3}$

M1 A1

$$\int_2^4 16x^{-2} dx = \left[-\frac{16}{x}\right]_2^4 = -4 + 8 = 4$$

M1 A1

$$\text{Area} = \frac{8}{3} + 4 = \frac{20}{3}$$

A1

$$\text{Answer: } \frac{20}{3}$$

5 marks

(c) Half the area is $\frac{10}{3}$; so $\frac{8}{3} + \int_2^k 16x^{-2} dx = \frac{10}{3}$

M1

$$8 - \frac{16}{k} = \frac{2}{3}$$

M1 A1

$$\frac{16}{k} = \frac{22}{3}, \text{ so } k = \frac{24}{11}$$

M1 A1

$$\text{Answer: } k = \frac{24}{11}$$

5 marks

Trigonometric Integration

1 (2 marks)

Use $\int \cos x dx = \sin x$ and $\int \sin x dx = -\cos x$

M1

$$= 4\sin x + 3\cos x + c$$

A1

$$\text{Answer: } 4\sin x + 3\cos x + c$$

2 marks

2 (3 marks)

$$\int \sin 3x dx = -\frac{1}{3}\cos 3x$$

M1

$$\int 6\cos 2x dx = 6 \times \frac{1}{2}\sin 2x = 3\sin 2x$$

A1

$$\text{Result: } -\frac{1}{3}\cos 3x - 3\sin 2x + c$$

A1

$$\text{Answer: } -\frac{1}{3}\cos 3x - 3\sin 2x + c$$

3 marks

3 (2 marks)

Use the standard results $\frac{d}{dx}(\sec x) = \sec x \tan x$ and $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

M1

$$= \sec x - 4\cot x + c$$

A1

$$\text{Answer: } \sec x - 4\cot x + c$$

2 marks

4 (2 marks)

The integral of $\sin kx$ is $-\frac{1}{k}\cos kx$, with $k = \frac{1}{2}$

M1

$$= -16\cos\left(\frac{x}{2}\right) + c$$

A1

$$\text{Answer: } -16\cos\left(\frac{x}{2}\right) + c$$

2 marks

5 (2 marks)

$$[-\cos x + \sin x]_0^{\frac{\pi}{2}} = (0 + 1) - (-1 + 0)$$

M1

$$= 2$$

A1

$$\text{Answer: } 2$$

2 marks

6 (2 marks)

The integral of $\sec^2 x$ is $\tan x$

M1

$$= 5x + 2\tan x + c$$

A1

$$\text{Answer: } 5x + 2\tan x + c$$

2 marks

7 (4 marks)

$$\begin{aligned} \text{Area} &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin 2x \, dx = \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} && \text{M1 A1} \\ &= -\frac{1}{2}\cos \frac{2\pi}{3} + \frac{1}{2}\cos 0 = -\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2} && \text{M1} \\ &= \frac{3}{4} && \text{A1} \end{aligned}$$

Answer: $\frac{3}{4}$

4 marks

8 (4 marks)

$$\begin{aligned} \int \sec^2 2x \, dx &= \frac{1}{2}\tan 2x && \text{B1} \\ \left[\sin x + \frac{1}{2}\tan 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} &&& \text{M1} \\ &= \frac{1}{2} + \frac{1}{2}\tan \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} && \text{M1 A1} \end{aligned}$$

Answer: $\frac{1+\sqrt{3}}{2}$

4 marks

9 (5 marks)

$$\begin{aligned} \cot^2 x &= \operatorname{cosec}^2 x - 1 && \text{M1} \\ \int 3(\operatorname{cosec}^2 x - 1) \, dx &= 3[-\cot x - x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} && \text{M1 A1} \\ &= 3\left(-\frac{1}{\sqrt{3}} - \frac{\pi}{3}\right) - (-\sqrt{3} - \frac{\pi}{6}) = 3\left[\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right] && \text{M1} \\ &= 2\sqrt{3} - \frac{\pi}{2} && \text{A1} \end{aligned}$$

Answer: $2\sqrt{3} - \frac{\pi}{2}$

5 marks

10 (4 marks)

$$\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos 2x) && \text{M1 A1} \\ \int &= \frac{x}{2} - \frac{\sin 2x}{4} + c && \text{M1 A1} \\ \text{Answer: } &\frac{x}{2} - \frac{\sin 2x}{4} + c && \text{4 marks} \end{aligned}$$

11 (4 marks)

$$\begin{aligned} [2\sin(2x - \frac{\pi}{3})]_{\frac{\pi}{6}}^{\frac{\pi}{3}} &&& \text{M1 A1} \\ &= 2\sin \frac{2\pi}{3} - 2\sin 0 = 2 \times \frac{\sqrt{3}}{2} && \text{M1} \\ &= \sqrt{3} && \text{A1} \end{aligned}$$

Answer: $\sqrt{3}$

4 marks

12 (2 marks)

$$\begin{aligned} \tan^2 x &= \sec^2 x - 1 && \text{M1} \\ \int (\sec^2 x - 1) \, dx &= \tan x - x + c && \text{A1} \\ \text{Answer: } &\tan x - x + c && \text{2 marks} \end{aligned}$$

13 (4 marks)

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

M1

$$\left[\frac{x}{2} + \frac{\sin 2x}{4} \right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= \frac{\pi}{4}$$

A1

Answer: $\frac{\pi}{4}$

4 marks

14 (6 marks)

(a) $\cos 2x = 1 - 2\sin^2 x$

M1

So $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

A1

Answer: shown

2 marks

(b) $\int_0^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos 2x) dx = \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\frac{\pi}{3}}$

M1 A1

$$= \frac{\pi}{6} - \frac{1}{4} \times \frac{\sqrt{3}}{2}$$

M1

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

A1

Answer: $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

4 marks

15 (7 marks)

(a) $\sin 2x = \sin x$ gives $2\sin x \cos x - \sin x = 0$, so $\sin x(2\cos x - 1) = 0$

M1 A1

$\sin x = 0$ gives $x = 0$; $\cos x = \frac{1}{2}$ gives $x = \frac{\pi}{3}$ in the interval

A1

Answer: a = $\frac{\pi}{3}$

3 marks

(b) Area = $\int_0^{\frac{\pi}{3}} (\sin 2x - \sin x) dx$

M1

$$= \left[-\frac{1}{2}\cos 2x + \cos x \right]_0^{\frac{\pi}{3}}$$

M1 A1

$$= \left(\frac{1}{4} + \frac{1}{2} \right) - \left(-\frac{1}{2} + 1 \right) = \frac{1}{4}$$

A1

Answer: $\frac{1}{4}$

4 marks

16 (6 marks)

Expand: $1 + 2\cos x + \cos^2 x$

M1

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, so the integrand is $\frac{3}{2} + 2\cos x + \frac{1}{2}\cos 2x$

M1 A1

$$\left[\frac{3x}{2} + 2\sin x + \frac{\sin 2x}{4} \right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= \frac{3\pi}{4} + 2$$

A1

Answer: $\frac{3\pi}{4} + 2$

6 marks

17 (5 marks)

$\tan^2 2x = \sec^2 2x - 1$

M1

$$\int (\sec^2 2x - 1) dx = \left[\frac{\tan 2x}{2} - x \right]_0^{\frac{\pi}{6}}$$

M1 A1

$$= \left(\frac{\tan(\frac{\pi}{3})}{2} - \frac{\pi}{6} \right) - 0$$

M1

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

A1

Answer: shown

5 marks

18 (6 marks)

$$\text{With } A = 3x, B = x: \sin 3x \cos x = \frac{1}{2}(\sin 4x + \sin 2x)$$

M1 A1

$$\int \frac{1}{2}(\sin 4x + \sin 2x) dx = \frac{1}{2}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= \frac{1}{2}\left[\left(-\frac{1}{4} + \frac{1}{2}\right) - \left(-\frac{1}{4} - \frac{1}{2}\right)\right] = \frac{1}{2}\left[\frac{1}{4} + \frac{3}{4}\right] = \frac{1}{2}$$

M1 A1

$$\text{Answer: } \frac{1}{2}$$

6 marks

19 (6 marks)

$$\text{The curves meet where } \tan x = 1, \text{ i.e. } x = \frac{\pi}{4}$$

M1 A1

$$\text{Area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx = [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

M1 A1

$$= (\sqrt{2}/2 + \sqrt{2}/2) - (0 + 1) = \sqrt{2} - 1$$

M1 A1

$$\text{Answer: } \sqrt{2} - 1$$

6 marks

20 (5 marks)

$$\text{The integral of } \operatorname{cosec}^2 x \text{ is } -\cot x$$

M1 A1

$$[-\cot x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\cot \frac{\pi}{3} + \cot \frac{\pi}{6}$$

M1

$$= -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{2}{\sqrt{3}}$$

A1

$$= \frac{2\sqrt{3}}{3}$$

A1

$$\text{Answer: shown}$$

5 marks

21 (8 marks)

$$(a) \frac{d}{dx} \left[\frac{\sin^3 x}{3} \right] = \sin^2 x \cos x$$

M1

This is not $\cos^2 x$, so the answer is incorrect

A1

$$\text{Answer: derivative is } \sin^2 x \cos x$$

2 marks

$$(b) \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

M1

$$\int \frac{1}{2}(1 + \cos 2x) dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

M1 A1

$$\text{Answer: } \frac{x}{2} + \frac{\sin 2x}{4} + c$$

3 marks

$$(c) 4 \times \left[\frac{x}{2} + \frac{\sin 2x}{4} \right] = [2x + \sin 2x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

M1

$$= (\pi + 0) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

M1

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

A1

$$\text{Answer: } \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

3 marks

22 (8 marks)

$$(a) V^2 = 144\sin^2(100\pi t)$$

M1

$$\text{Using } \sin^2 A = \frac{1}{2}(1 - \cos 2A): V^2 = 72(1 - \cos 200\pi t)$$

A1

$$\text{Answer: shown}$$

2 marks

$$(b) \int_0^{0.02} 72(1 - \cos 200\pi t) dt = 72\left[t - \frac{\sin 200\pi t}{200\pi}\right]_0^{0.02}$$

M1 A1

$$= 72\left(0.02 - \frac{\sin 4\pi}{200\pi}\right) = 1.44$$

M1

$$\text{Mean} = \frac{1.44}{0.02} = 72$$

A1

Answer: 72**4 marks**

$$(c) \text{ RMS} = \sqrt{72} = 6\sqrt{2}$$

$$= 8.49 \text{ volts}$$

M1**A1****Answer: 8.49 volts****2 marks****23 (7 marks)**

$$\sin^4 x = \left(\frac{1}{2}(1 - \cos 2x)\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

M1 A1

$$\cos^2 2x = \frac{1}{2}(1 + \cos 4x), \text{ so } \sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

M1 A1

$$\left[\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= \frac{3\pi}{16}$$

A1**Answer: $\frac{3\pi}{16}$** **7 marks****24 (9 marks)**

$$(a) \text{ Expand: } \sin^2 x + 2\sin x \cos x + \cos^2 x$$

M1

$$= 1 + \sin 2x \text{ using } \sin^2 x + \cos^2 x = 1 \text{ and } \sin 2x = 2\sin x \cos x$$

M1 A1**Answer: shown****3 marks**

$$(b) \int (1 + \sin 2x) dx = \left[x - \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= \left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right) = \frac{\pi}{2} + 1$$

M1 A1**Answer: $\frac{\pi}{2} + 1$** **4 marks**

$$(c) \text{ Mean} = \frac{2}{\pi} \times \left(\frac{\pi}{2} + 1\right)$$

M1

$$= 1 + \frac{2}{\pi}$$

A1**Answer: $1 + \frac{2}{\pi}$** **2 marks****25 (9 marks)**

$$(a) \cos x = \cos\left(2 \times \frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$$

M1

$$\text{so } 1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$$

A1**Answer: shown****2 marks**

$$(b) \frac{1}{1 + \cos x} = \frac{1}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

M1 A1

$$\int \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx = \frac{1}{2} \times 2\tan\left(\frac{x}{2}\right) = \tan\left(\frac{x}{2}\right)$$

M1

$$\left[\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}} = \tan\frac{\pi}{4} - \tan 0 = 1$$

A1**Answer: 1****4 marks**

$$(c) \text{ Split: } \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} dx$$

M1

$$\int \frac{\sin x}{1 + \cos x} dx = -\ln(1 + \cos x), \text{ so the second integral is } \left[-\ln(1 + \cos x)\right]_0^{\frac{\pi}{2}} = -\ln 1 + \ln 2 = \ln 2$$

M1

$$\text{Total} = 1 + \ln 2$$

A1**Answer: $1 + \ln 2$** **3 marks**

26 (7 marks)

$$4\sin^2\left(\frac{x}{2}\right) = 2(1 - \cos x)$$

M1 A1

$$y = 2x - 2\sin x + c$$

M1 A1

$$(\pi, 0): 0 = 2\pi - 0 + c, \text{ so } c = -2\pi$$

M1

$$\text{At } x = \frac{\pi}{2}: y = \pi - 2 - 2\pi$$

M1

$$y = -\pi - 2$$

A1

$$\text{Answer: } y = -\pi - 2$$

7 marks

27 (11 marks)

$$(a) \text{ Rearrange: } 4\sin^3 x = 3\sin x - \sin 3x$$

M1

Divide by 4

A1

Answer: shown

2 marks

$$(b) \int \frac{1}{4}(3\sin x - \sin 3x) dx$$

M1

$$= \frac{1}{4}(-3\cos x + \frac{1}{3}\cos 3x) + c$$

M1 A1

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + c$$

A1

$$\text{Answer: } -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + c$$

4 marks

$$(c) \text{ At } x = \frac{\pi}{3}: -\frac{3}{4} \times \frac{1}{2} + \frac{1}{12} \times (-1) = -\frac{3}{8} - \frac{1}{12} = -\frac{11}{24}$$

M1 A1

$$\text{At } x = 0: -\frac{3}{4} + \frac{1}{12} = -\frac{2}{3} = -\frac{16}{24}, \text{ so integral} = -\frac{11}{24} + \frac{16}{24} = \frac{5}{24}$$

A1

$$\text{Answer: } \frac{5}{24}$$

3 marks

$$(d) \text{ At } x = \pi: \frac{3}{4} - \frac{1}{12} = \frac{2}{3}; \text{ at } x = 0: -\frac{2}{3}$$

M1

$$\text{Integral} = \frac{2}{3} - (-\frac{2}{3}) = \frac{4}{3}$$

A1

Answer: shown

2 marks

28 (12 marks)

$$(a) \sin 2x = 2\sin x \cos x, \text{ so } y = 2\sin x - 2\sin x \cos x = 2\sin x(1 - \cos x)$$

M1 A1

On $0 \leq x \leq \pi$, $\sin x \geq 0$ and $1 - \cos x \geq 0$, so $y \geq 0$

E1

Answer: shown

3 marks

$$(b) \text{ Area} = \int_0^\pi (2\sin x - \sin 2x) dx = [-2\cos x + \frac{1}{2}\cos 2x]_0^\pi$$

M1 A1

$$= (2 + \frac{1}{2}) - (-2 + \frac{1}{2})$$

M1

$$= 4$$

A1

Answer: 4

4 marks

$$(c) \frac{dy}{dx} = 2\cos x - 2\cos 2x = 0, \text{ so } \cos x = 2\cos^2 x - 1$$

M1 A1

$$2\cos^2 x - \cos x - 1 = 0 \text{ gives } (2\cos x + 1)(\cos x - 1) = 0$$

M1

$$\cos x = -\frac{1}{2} \text{ gives } x = \frac{2\pi}{3} \text{ (} \cos x = 1 \text{ gives } x = 0, \text{ the minimum)}$$

A1

$$y = 2\sin \frac{2\pi}{3} - \sin \frac{4\pi}{3} = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

A1

$$\text{Answer: } y = \frac{3\sqrt{3}}{2}$$

5 marks

29 (12 marks)

$$(a) \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$

M1

$$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$$

M1 A1

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x = 4\cos^3 x - 3\cos x$$

A1

Answer: proved

4 marks

(b) Rearrange: $4\cos^3 x = \cos 3x + 3\cos x$

M1

$$\int \frac{1}{4}(\cos 3x + 3\cos x) dx = \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= \frac{1}{4}\left(-\frac{1}{3} + 3\right) = \frac{2}{3}$$

A1

Answer: $\frac{2}{3}$

4 marks

(c) For $0 < \cos x < 1$, $\cos^3 x < \cos x$, so $y = \cos x$ is above

E1

$$\int_0^{\frac{\pi}{2}} \cos x dx = 1$$

B1

$$\text{Area} = 1 - \frac{2}{3} = \frac{1}{3}$$

M1 A1

Answer: $\frac{1}{3}$

4 marks

30 (12 marks)

$$\text{(a)} \sin 2x \sin 3x = \frac{1}{2}(\cos x - \cos 5x)$$

M1 A1

$$\int \frac{1}{2}(\cos x - \cos 5x) dx = \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi}$$

M1

 $\sin \pi = 0$, $\sin 5\pi = 0$ and $\sin 0 = 0$, so the value is 0

A1

Answer: shown

4 marks

$$\text{(b)} \sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

M1

$$\text{Integral} = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi}$$

M1 A1

 $m-n$ and $m+n$ are non-zero integers, so each sine is $\sin(k\pi) = 0$ and $\sin 0 = 0$; hence the integral is 0

E1 A1

Answer: proved

5 marks

$$\text{(c)} \sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

M1

$$\int = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_0^{\pi} = \frac{\pi}{2}$$

M1 A1

Answer: $\frac{\pi}{2}$

3 marks

Exponential and Logarithmic Integration**1 (2 marks)**

$$\int e^{4x} dx = \frac{1}{4}e^{4x} \text{ and } \int \frac{5}{x} dx = 5\ln|x|$$

M1

$$= \frac{1}{4}e^{4x} + 5\ln|x| + c$$

A1

Answer: $\frac{1}{4}e^{4x} + 5\ln|x| + c$

2 marks

2 (2 marks)

$$\text{Use } \int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b|$$

M1

$$= \frac{6}{3}\ln|3x-1| + c = 2\ln|3x-1| + c$$

A1

Answer: $2\ln|3x-1| + c$

2 marks

3 (2 marks)The integral of e^x is e^x ; of $\frac{1}{x}$ is $\ln|x|$

M1

$$= 3e^x + 2\ln|x| + c$$

A1

Answer: $3e^x + 2\ln|x| + c$

2 marks

4 (2 marks)

The integral of e^{kx} is $\frac{1}{k}e^{kx}$, with $k = -2$

M1

$$= -\frac{1}{2}e^{-2x} + c$$

A1

Answer: $-\frac{1}{2}e^{-2x} + c$

2 marks

5 (2 marks)

$$\left[\frac{1}{3}e^{3x}\right]_0^1$$

M1

$$= \frac{1}{3}(e^3 - 1)$$

A1

Answer: $\frac{1}{3}(e^3 - 1)$

2 marks

6 (2 marks)

The integral of e^{kx} is $\frac{1}{k}e^{kx}$, with $k = \frac{1}{3}$

M1

$$= 36e^{\frac{x}{3}} + c$$

A1

Answer: $36e^{\frac{x}{3}} + c$

2 marks

7 (4 marks)

$$\int_0^{\ln 3} 4e^{2x} dx = [2e^{2x}]_0^{\ln 3}$$

M1 A1

$$e^{2\ln 3} = e^{\ln 9} = 9, \text{ so value} = 2 \times 9 - 2 \times 1$$

M1

$$= 16$$

A1

Answer: 16

4 marks

8 (4 marks)

$$\int_1^3 \frac{5}{2x-1} dx = \left[\frac{5}{2}\ln|2x-1|\right]_1^3$$

M1 A1

$$= \frac{5}{2}\ln 5 - \frac{5}{2}\ln 1$$

M1

$$= \frac{5}{2}\ln 5, \text{ since } \ln 1 = 0$$

A1

Answer: shown

4 marks

9 (4 marks)

$$[e^x - e^{-x}]_0^{\ln 2}$$

M1 A1

$$= \left(2 - \frac{1}{2}\right) - (1 - 1) = \frac{3}{2}$$

M1 A1

Answer: $\frac{3}{2}$

4 marks

10 (3 marks)

Expand: $e^{2x} + 2e^x + 1$

M1

$$= \frac{1}{2}e^{2x} + 2e^x + x + c$$

M1 A1

Answer: $\frac{1}{2}e^{2x} + 2e^x + x + c$

3 marks

11 (4 marks)

$$\left[3\ln x + \frac{2}{x}\right]_1^2$$

M1 A1

$$= (3\ln 2 + 1) - (0 + 2) = 3\ln 2 - 1$$

M1 A1

Answer: $3\ln 2 - 1$

4 marks

12 (4 marks)

$$\begin{aligned} \text{The numerator is } 2 \times \text{the derivative of the denominator, so the integral is } [2\ln(x^3 + 2)]_0^1 \\ = 2\ln 3 - 2\ln 2 \\ = 2\ln \frac{3}{2} \end{aligned}$$

M1 A1

M1

A1

Answer: shown

4 marks

13 (4 marks)

$$\begin{aligned} y &= e^{2x} + 3\ln x + c \\ (1, 5): 5 &= e^2 + 0 + c, \text{ so } c = 5 - e^2 \\ y &= e^{2x} + 3\ln x + 5 - e^2 \\ \text{Answer: } y &= e^{2x} + 3\ln x + 5 - e^2 \end{aligned}$$

M1 A1

M1

A1

4 marks

14 (6 marks)

$$\begin{aligned} \text{(a) Area} &= \int_1^k \frac{3}{x} dx = [3\ln x]_1^k \\ &= 3\ln k - 3\ln 1 = 3\ln k \end{aligned}$$

M1 A1

A1

Answer: shown

3 marks

$$\begin{aligned} \text{(b) } 3\ln k &= 6\ln 2, \text{ so } \ln k = 2\ln 2 = \ln 4 \\ k &= 4 \end{aligned}$$

M1 A1

A1

Answer: k = 4

3 marks

15 (6 marks)

$$\begin{aligned} \frac{(x+1)^2}{x} &= \frac{x^2 + 2x + 1}{x} = x + 2 + \frac{1}{x} \\ \int_1^2 \left(x + 2 + \frac{1}{x}\right) dx &= \left[\frac{x^2}{2} + 2x + \ln x\right]_1^2 \\ &= (2 + 4 + \ln 2) - \left(\frac{1}{2} + 2 + \ln 1\right) \\ &= \frac{7}{2} + \ln 2 \end{aligned}$$

M1 A1

M1 A1

M1

A1

Answer: shown

6 marks

16 (4 marks)

$$\begin{aligned} \text{Divide each term by } e^x: e^x + 3e^{-x} \\ = e^x - 3e^{-x} + c \\ \text{Answer: } e^x - 3e^{-x} + c \end{aligned}$$

M1 A1

M1 A1

4 marks

17 (5 marks)

$$\begin{aligned} \frac{x+1}{x-1} &= 1 + \frac{2}{x-1} \\ [x + 2\ln(x-1)]_2^5 \\ &= (5 + 2\ln 4) - (2 + 2\ln 1) \\ &= 3 + 4\ln 2 \end{aligned}$$

M1

M1 A1

M1

A1

Answer: shown

5 marks

18 (5 marks)

$$\begin{aligned} [e^{2x}]_0^a &= e^{2a} - 1 \\ e^{2a} &= 16 \\ 2a &= \ln 16, \text{ so } a = \ln 4 \\ \text{Answer: } a &= \ln 4 \end{aligned}$$

M1 A1

M1

M1 A1

5 marks

19 (5 marks)

$$N = \int -24e^{-0.3t} dt = 80e^{-0.3t} + c$$

M1 A1

$$t = 0, N = 100: c = 20, \text{ so } N = 80e^{-0.3t} + 20$$

M1 A1

$$\text{As } t \rightarrow \infty, e^{-0.3t} \rightarrow 0, \text{ so } N \rightarrow 20 \text{ thousand}$$

B1

$$\text{Answer: } N = 80e^{-0.3t} + 20; N \rightarrow 20 \text{ thousand}$$

5 marks

20 (4 marks)

$$\text{Area} = \int_0^1 (e^x - e^{-x}) dx = [e^x + e^{-x}]_0^1$$

M1 A1

$$= (e + e^{-1}) - (1 + 1)$$

M1

$$= e + \frac{1}{e} - 2$$

A1

$$\text{Answer: } e + \frac{1}{e} - 2$$

4 marks

21 (8 marks)

$$\text{(a) Volume} = \int_0^{10} 40e^{-0.1t} dt = [-400e^{-0.1t}]_0^{10}$$

M1 A1

$$= 400(1 - e^{-1}) = 253 \text{ litres}$$

A1

$$\text{Answer: 253 litres}$$

3 marks

$$\text{(b) Volume lost up to time } T = 400(1 - e^{-0.1T})$$

M1

$$e^{-0.1T} > 0 \text{ for all } T, \text{ so the volume is less than 400}$$

E1

$$\text{Answer: less than 400 litres}$$

2 marks

$$\text{(c) } 400(1 - e^{-0.1T}) = 300, \text{ so } e^{-0.1T} = \frac{1}{4}$$

M1 A1

$$T = 10 \ln 4 = 13.9 \text{ minutes}$$

A1

$$\text{Answer: 13.9 minutes}$$

3 marks

22 (7 marks)

$$\text{(a) } \frac{1}{x} \text{ is undefined at } x = 0, \text{ which lies between } -1 \text{ and } 2$$

E1

The graph has an asymptote there, so the integral cannot be evaluated by substituting the limits

E1

$$\text{Answer: } \frac{1}{x} \text{ is undefined at } x = 0$$

2 marks

$$\text{(b) } \int_{-4}^{-1} \frac{1}{x} dx = [\ln|x|]_{-4}^{-1}$$

M1

$$= \ln 1 - \ln 4 = -\ln 4$$

A1

$$\text{Answer: } -\ln 4$$

2 marks

$$\text{(c) For } x < 0, \frac{1}{x} < 0, \text{ so the curve lies below the } x\text{-axis and the integral is negative}$$

E1

$$\text{Area is the positive value } \ln 4 = 2 \ln 2$$

M1 A1

$$\text{Answer: area} = 2 \ln 2$$

3 marks

23 (7 marks)

$$8 - e^{2a} = 0, \text{ so } e^{2a} = 8 \text{ and } 2a = \ln 8 = 3 \ln 2$$

M1 A1

$$a = \frac{3}{2} \ln 2$$

A1

$$\text{Area} = [8x - \frac{1}{2}e^{2x}]_0^a$$

M1 A1

$$= (12 \ln 2 - \frac{1}{2} \times 8) - (0 - \frac{1}{2})$$

M1

$$= 12 \ln 2 - \frac{7}{2}$$

A1

$$\text{Answer: } a = \frac{3}{2} \ln 2; \text{ shown}$$

7 marks

24 (7 marks)

$$(a) \text{ Total} = \int_0^4 (50 + 30e^{-0.5t}) dt = [50t - 60e^{-0.5t}]_0^4$$

M1 A1

$$= (200 - 60e^{-2}) - (0 - 60)$$

M1

$$= 260 - 60e^{-2} \text{ thousand litres}$$

A1

$$\text{Answer: } 260 - 60e^{-2}$$

4 marks

$$(b) 50 + 30e^{-0.5t} = 55, \text{ so } e^{-0.5t} = \frac{1}{6}$$

M1

$$-0.5t = -\ln 6, \text{ so } t = 2\ln 6 \approx 3.58 \text{ hours}$$

M1 A1

$$\text{Answer: } t = 2\ln 6 \approx 3.58$$

3 marks

25 (7 marks)

$$e^x = 3e^{-x} \text{ gives } e^{2x} = 3$$

M1 A1

$$2x = \ln 3, \text{ so } a = \frac{1}{2}\ln 3$$

A1

$$\text{Area} = \int_0^a (3e^{-x} - e^x) dx = [-3e^{-x} - e^x]_0^a$$

M1 A1

$$e^a = \sqrt{3}, e^{-a} = \frac{1}{\sqrt{3}}: (-\sqrt{3} - \sqrt{3}) - (-3 - 1)$$

M1

$$= 4 - 2\sqrt{3}$$

A1

$$\text{Answer: } a = \frac{1}{2}\ln 3; \text{ area } 4 - 2\sqrt{3}$$

7 marks

26 (7 marks)

$$(a) \frac{d}{dx}[\ln|1 - x|] = \frac{1}{1 - x} \times (-1)$$

M1 A1

$$= -\frac{1}{1 - x}, \text{ which is the negative of the integrand}$$

E1

$$\text{Answer: derivative is } -1/(1 - x)$$

3 marks

$$(b) \int \frac{1}{1 - x} dx = -\ln|1 - x|$$

M1 A1

$$[-\ln|1 - x|]_{-1}^0 = -\ln 1 + \ln 2$$

M1

$$= \ln 2$$

A1

$$\text{Answer: } \ln 2$$

4 marks

27 (11 marks)

$$(a) e^{2x} - 4e^x = e^x(e^x - 4) = 0$$

M1

$$e^x > 0, \text{ so } e^x = 4 \text{ and } x = \ln 4; \text{ the point is } (\ln 4, 0)$$

A1

$$\text{Answer: } (\ln 4, 0)$$

2 marks

$$(b) \frac{dy}{dx} = 2e^{2x} - 4e^x = 2e^x(e^x - 2)$$

M1 A1

$$\frac{dy}{dx} = 0 \text{ gives } e^x = 2, \text{ so } x = \ln 2$$

M1

$$y = 4 - 8 = -4, \text{ so } P \text{ is } (\ln 2, -4)$$

A1

$$\text{Answer: } P(\ln 2, -4)$$

4 marks

$$(c) \text{ Area} = -\int_0^{\ln 4} (e^{2x} - 4e^x) dx = -[\frac{1}{2}e^{2x} - 4e^x]_0^{\ln 4}$$

M1 A1

$$[]: (\frac{1}{2} \times 16 - 4 \times 4) - (\frac{1}{2} - 4) = -8 + \frac{7}{2} = -\frac{9}{2}$$

M1 A1

$$\text{Area is the positive value } \frac{9}{2}$$

A1

$$\text{Answer: } \frac{9}{2}$$

5 marks

28 (12 marks)

$$(a) \frac{4}{x} = 5 - x \text{ gives } 4 = 5x - x^2, \text{ so } x^2 - 5x + 4 = 0$$

M1 A1

$$(x - 1)(x - 4) = 0, \text{ so } x = 1 \text{ or } x = 4$$

A1

$$\text{Answer: shown}$$

3 marks

$$(b) \text{ Area} = \int_1^4 \left(5 - x - \frac{4}{x}\right) dx \quad \text{M1}$$

$$= \left[5x - \frac{x^2}{2} - 4\ln x\right]_1^4 \quad \text{M1 A1}$$

$$= (20 - 8 - 4\ln 4) - \left(5 - \frac{1}{2} - 0\right) = \frac{15}{2} - 4\ln 4 \quad \text{M1}$$

$$4\ln 4 = 8\ln 2, \text{ so area} = \frac{15}{2} - 8\ln 2 \quad \text{A1}$$

Answer: shown 5 marks

$$(c) \text{ Area under the line from } x = 1 \text{ to } 4 \text{ is a trapezium: } \frac{1}{2}(4 + 1) \times 3 = \frac{15}{2} \quad \text{M1}$$

$$\text{Area under the curve} = \text{trapezium} - R = \frac{15}{2} - \left(\frac{15}{2} - 8\ln 2\right) = 8\ln 2 \quad \text{M1 A1}$$

$$\text{Directly: } [4\ln x]_1^4 = 4\ln 4 = 8\ln 2 \quad \text{A1}$$

Answer: 8ln 2 4 marks

29 (12 marks)

$$(a) \frac{dy}{dx} = e^x, \text{ so the gradient at } P \text{ is } e^a \quad \text{M1 A1}$$

$$y - e^a = e^a(x - a) \quad \text{M1}$$

$$y = e^a(x - a + 1) \quad \text{A1}$$

Answer: shown 4 marks

$$(b) x = 0, y = 0: 0 = e^a(1 - a) \quad \text{M1}$$

$$e^a > 0 \text{ for all } a, \text{ so } 1 - a = 0 \text{ and } a = 1 \quad \text{M1 E1}$$

Answer: a = 1 3 marks

$$(c) i: y = ex \quad \text{B1}$$

$$\text{Area} = \int_0^1 (e^x - ex) dx = \left[e^x - \frac{ex^2}{2}\right]_0^1 \quad \text{M1 A1}$$

$$= \left(e - \frac{e}{2}\right) - 1 = \frac{e}{2} - 1 \quad \text{M1 A1}$$

$$\text{Answer: } \frac{e}{2} - 1 \quad \text{5 marks}$$

30 (9 marks)

$$(a) [\ln x]_1^n = \ln n - \ln 1 \quad \text{M1}$$

$$= \ln n \quad \text{A1}$$

Answer: ln n 2 marks

(b) The curve lies below the top of each rectangle on its interval ($1/x$ decreasing), so area under curve < total area of rectangles M1 E1

$$\text{Rectangles for } x \text{ from } 1 \text{ to } 5 \text{ have area } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}; \text{ the area under the curve is } \ln 5 \quad \text{A1}$$

$$\text{The same argument for } x \text{ from } 1 \text{ to } n \text{ gives } \ln n < \sum_{r=1}^{n-1} \frac{1}{r} \quad \text{E1}$$

Answer: shown 4 marks

$$(c) \text{ Take } n = 100 \text{ in (b): the sum of } \frac{1}{r} \text{ for } r = 1 \text{ to } 99 \text{ exceeds } \ln 100 \quad \text{M1}$$

$$\ln 100 = 2\ln 10 = 4.605... > 4.6 \quad \text{M1 A1}$$

Answer: > ln 100 = 4.605 3 marks

Integration by Substitution

1 (3 marks)

$$u = 3x - 2 \text{ gives } \frac{d u}{d x} = 3, \text{ so } d x = \frac{1}{3} d u$$

M1

$$\int \frac{1}{3} u^4 d u = \frac{u^5}{15} + c$$

M1

$$= \frac{(3x - 2)^5}{15} + c$$

A1

$$\text{Answer: } \frac{(3x - 2)^5}{15} + c$$

3 marks

2 (3 marks)

$$\frac{d u}{d x} = 2x, \text{ so } 2x d x = d u$$

M1

$$\int u^3 d u = \frac{u^4}{4} + c$$

A1

$$= \frac{(x^2 + 1)^4}{4} + c$$

A1

$$\text{Answer: } \frac{(x^2 + 1)^4}{4} + c$$

3 marks

3 (2 marks)

$$u = 2x - 1, \text{ so } d x = \frac{1}{2} d u: 3 \int u^{-3} d u$$

M1

$$= -\frac{3}{2u^2} = -\frac{3}{2(2x - 1)^2} + c$$

A1

$$\text{Answer: } -\frac{3}{2(2x - 1)^2} + c$$

2 marks

4 (2 marks)

$$d u = 3x^2 d x, \text{ so the integral is } \int u^4 d u$$

M1

$$= \frac{u^5}{5} = \frac{(x^3 + 1)^5}{5} + c$$

A1

$$\text{Answer: } \frac{(x^3 + 1)^5}{5} + c$$

2 marks

5 (2 marks)

$$d u = 2x d x, \text{ so } x d x = \frac{1}{2} d u$$

M1

$$\frac{1}{2} \int e^u d u = \frac{1}{2} e^{x^2} + c$$

A1

$$\text{Answer: } \frac{1}{2} e^{x^2} + c$$

2 marks

6 (2 marks)

$$d u = 2x d x; x = 0 \text{ gives } u = 1 \text{ and } x = 1 \text{ gives } u = 2$$

M1

$$\int_1^2 u^2 d u = \left[\frac{u^3}{3} \right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

A1

$$\text{Answer: } \frac{7}{3}$$

2 marks

7 (4 marks)

$$\frac{du}{dx} = 2x, \text{ so } x \, dx = \frac{1}{2} \, du; \text{ limits: } x = 0 \text{ gives } u = 1, x = \sqrt{3} \text{ gives } u = 4$$

M1 A1

$$\text{Area} = \int_1^4 \frac{1}{2} u^{\frac{1}{2}} \, du = \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^4$$

M1

$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

A1

$$\text{Answer: } \frac{7}{3}$$

4 marks

8 (4 marks)

$$\frac{du}{dx} = \cos x; \text{ when } x = 0, u = 1 \text{ and when } x = \frac{\pi}{2}, u = 2$$

M1 A1

$$\int_1^2 u^{-2} \, du = \left[-\frac{1}{u} \right]_1^2$$

M1

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2}$$

4 marks

9 (4 marks)

$$u = 2x - 1, \text{ so } x = \frac{u+1}{2} \text{ and } dx = \frac{1}{2} \, du$$

M1

$$\int \frac{2(u+1)}{u} \times \frac{1}{2} \, du = \int \left(1 + \frac{1}{u}\right) \, du$$

M1 A1

$$= u + \ln|u| + c = 2x - 1 + \ln|2x - 1| + c$$

A1

$$\text{Answer: } 2x - 1 + \ln|2x - 1| + c$$

4 marks

10 (4 marks)

$$du = \cos x \, dx; \text{ limits } 0 \rightarrow 0 \text{ and } \frac{\pi}{2} \rightarrow 1$$

M1 A1

$$\int_0^1 u^3 \, du = \left[\frac{u^4}{4} \right]_0^1 = \frac{1}{4}$$

M1 A1

$$\text{Answer: } \frac{1}{4}$$

4 marks

11 (4 marks)

$$du = 2x \, dx, \text{ so } x \, dx = \frac{1}{2} \, du$$

M1

$$\frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + c$$

M1 A1

$$= \frac{1}{2} \ln(x^2 + 1) + c$$

A1

$$\text{Answer: } \frac{1}{2} \ln(x^2 + 1) + c$$

4 marks

12 (4 marks)

$$x = u - 1, \, dx = du; \text{ limits } u = 1 \text{ to } 4$$

M1

$$\int_1^4 \frac{u-1}{\sqrt{u}} \, du = \int_1^4 (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \, du$$

M1

$$= \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^4$$

A1

$$= \left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - 2 \right) = \frac{8}{3}$$

A1

$$\text{Answer: shown}$$

4 marks

13 (4 marks)

$$du = e^x \, dx; \text{ limits } 2 \rightarrow 3$$

M1 A1

$$\int_2^3 \frac{1}{u} \, du = [\ln u]_2^3 = \ln 3 - \ln 2 = \ln \frac{3}{2}$$

M1 A1

$$\text{Answer: } \ln \frac{3}{2}$$

4 marks

14 (5 marks)

$$\frac{du}{dx} = 2x, \text{ so } 6x \, dx = 3 \, du$$

M1 A1

$$x = 0 \text{ gives } u = 4 \text{ and } x = 2 \text{ gives } u = 8$$

B1

$$\int_4^8 \frac{3}{u} \, du = [3 \ln u]_4^8$$

M1

$$= 3 \ln 8 - 3 \ln 4 = 3 \ln \frac{8}{4} = 3 \ln 2$$

A1

Answer: shown

5 marks

15 (6 marks)

$$\frac{du}{dx} = e^x, \text{ so } dx = \frac{du}{e^x} = \frac{du}{u-1}$$

M1

$$\frac{e^{2x}}{e^x + 1} \, dx = \frac{(u-1)^2}{u} \times \frac{1}{u-1} \, du = \frac{u-1}{u} \, du$$

M1 A1

$$\text{Limits: } x = 0 \text{ gives } u = 2, \, x = \ln 3 \text{ gives } u = 4$$

B1

$$\int_2^4 \left(1 - \frac{1}{u}\right) \, du = [u - \ln u]_2^4$$

M1

$$= (4 - \ln 4) - (2 - \ln 2) = 2 - 2 \ln 2 + \ln 2 = 2 - \ln 2$$

A1

Answer: shown

6 marks

16 (5 marks)

$$x = 1 - u, \, dx = -du; \, x = 0 \text{ gives } u = 1, \, x = 1 \text{ gives } u = 0$$

M1 A1

$$\int_1^0 (1-u)u^5(-du) = \int_0^1 (u^5 - u^6) \, du$$

M1

$$= \left[\frac{u^6}{6} - \frac{u^7}{7} \right]_0^1$$

A1

$$= \frac{1}{6} - \frac{1}{7} = \frac{1}{42}$$

A1

Answer: shown

5 marks

17 (5 marks)

$$\tan x = \frac{\sin x}{\cos x}; \, du = -\sin x \, dx$$

M1 A1

$$\text{Limits: } x = 0 \text{ gives } u = 1; \, x = \frac{\pi}{3} \text{ gives } u = \frac{1}{2}$$

B1

$$\int -\frac{1}{u} \, du = [-\ln u]_{\frac{1}{2}}^1$$

M1

$$= -\ln \frac{1}{2} + \ln 1 = \ln 2$$

A1

Answer: shown

5 marks

18 (5 marks)

$$du = \frac{1}{2\sqrt{x}} \, dx, \text{ so } \frac{dx}{\sqrt{x}} = 2 \, du; \text{ limits } u = 2 \text{ to } 3$$

M1 A1

$$2 \int_2^3 u^{-2} \, du = 2 \left[-\frac{1}{u} \right]_2^3$$

M1 A1

$$= 2 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}$$

A1

Answer: $\frac{1}{3}$

5 marks

19 (5 marks)

$$d u = 2 \sin x \cos x \, d x$$

M1 A1

$$\text{Limits: } 0 \rightarrow 1 \text{ and } \frac{\pi}{2} \rightarrow 2$$

B1

$$\frac{1}{2} \int_1^2 \frac{1}{u} \, d u = \frac{1}{2} [\ln u]_1^2$$

M1

$$= \frac{1}{2} \ln 2$$

A1

$$\text{Answer: } \frac{1}{2} \ln 2$$

5 marks

20 (6 marks)

$$x = u^2, \, d x = 2u \, d u; \text{ limits } u = 0 \text{ to } 2$$

M1 A1

$$\int_0^2 \frac{2u}{1+u} \, d u = 2 \int_0^2 \left(1 - \frac{1}{1+u}\right) \, d u$$

M1 A1

$$= 2[u - \ln(1+u)]_0^2$$

M1

$$= 2(2 - \ln 3) = 4 - 2 \ln 3$$

A1

$$\text{Answer: shown}$$

6 marks

21 (8 marks)

$$\text{(a) } u = 2x + 1 \text{ gives } x = \frac{u-1}{2} \text{ and } d x = \frac{1}{2} \, d u$$

M1

$$\text{Limits: } x = 0 \text{ gives } u = 1, \, x = 4 \text{ gives } u = 9$$

B1

$$\text{Area} = \int_1^9 \frac{u-1}{2\sqrt{u}} \times \frac{1}{2} \, d u$$

M1

$$= \frac{1}{4} \int_1^9 (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \, d u$$

A1

$$\text{Answer: shown}$$

4 marks

$$\text{(b) } \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^9$$

M1 A1

$$= \frac{1}{4} \left[(18 - 6) - \left(\frac{2}{3} - 2 \right) \right] = \frac{1}{4} \times \frac{40}{3}$$

M1

$$= \frac{10}{3}$$

A1

$$\text{Answer: } \frac{10}{3}$$

4 marks

22 (9 marks)

$$\text{(a) } \cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x$$

M1

$$d u = \cos x \, d x; \, x = 0 \text{ gives } u = 0 \text{ and } x = \frac{\pi}{2} \text{ gives } u = 1$$

M1 A1

$$\text{So the integral is } \int_0^1 (1 - u^2) \, d u$$

A1

$$\text{Answer: shown}$$

4 marks

$$\text{(b) } \left[u - \frac{u^3}{3} \right]_0^1 = 1 - \frac{1}{3}$$

M1

$$= \frac{2}{3}$$

A1

$$\text{Answer: } \frac{2}{3}$$

2 marks

$$\text{(c) } \cos^5 x \, d x = (1 - u^2)^2 \, d u$$

M1

$$\int_0^1 (1 - 2u^2 + u^4) \, d u = \left[u - \frac{2u^3}{3} + \frac{u^5}{5} \right]_0^1$$

M1

$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

A1

$$\text{Answer: } \frac{8}{15}$$

3 marks

23 (8 marks)

$$d u = -2x d x; x^2 = 1 - u; \text{ limits } x = 0 \rightarrow u = 1, x = 1 \rightarrow u = 0$$

M1 A1

$$x^3 d x = x^2 \cdot x d x = -\frac{1}{2}(1 - u) d u$$

M1 A1

$$\int_1^0 -\frac{1}{2}(1 - u)u^{\frac{1}{2}} d u = \frac{1}{2} \int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}}) d u$$

M1

$$= \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_0^1 = \frac{1}{2} \left(\frac{2}{3} - \frac{2}{5} \right)$$

M1 A1

$$= \frac{2}{15}$$

A1

Answer: $\frac{2}{15}$

8 marks

24 (7 marks)

$$d x = 2 \cos \theta d \theta; x = 0 \text{ gives } \theta = 0 \text{ and } x = 2 \text{ gives } \theta = \frac{\pi}{2}$$

M1 A1

$$\sqrt{4 - 4 \sin^2 \theta} = 2 \cos \theta \text{ (} \cos \theta \geq 0 \text{ on this range)}$$

M1 A1

$$\text{Area} = \int_0^{\frac{\pi}{2}} 4 \cos^2 \theta d \theta = \int_0^{\frac{\pi}{2}} 2(1 + \cos 2\theta) d \theta$$

M1

$$= [2\theta + \sin 2\theta]_0^{\frac{\pi}{2}} = \pi$$

A1

$$R \text{ is a quarter of a circle of radius 2: } \frac{1}{4} \pi (2)^2 = \pi$$

E1

Answer: π

7 marks

25 (6 marks)

$$d u = e^x d x = u d x, \text{ so } d x = \frac{d u}{u}; \text{ limits } u = 1 \text{ to } 2$$

M1 A1

$$\int_1^2 \frac{1}{u(u+1)} d u; \text{ partial fractions } \frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}$$

M1 A1

$$[\ln u - \ln(u+1)]_1^2 = (\ln 2 - \ln 3) - (0 - \ln 2)$$

M1

$$= \ln \frac{4}{3}$$

A1

Answer: $\ln \frac{4}{3}$

6 marks

26 (8 marks)

$$(a) \frac{d y}{d x} = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2}(1 - 2x^2) = 0$$

M1 A1

$$x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}} e^{-1/2}$$

M1 A1

Answer: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} e^{-1/2})$

4 marks

$$(b) d u = 2x d x; \text{ limits } 0 \rightarrow 4$$

M1

$$\frac{1}{2} \int_0^4 e^{-u} d u = \frac{1}{2} [-e^{-u}]_0^4$$

M1 A1

$$= \frac{1}{2}(1 - e^{-4})$$

A1

Answer: $\frac{1}{2}(1 - e^{-4})$

4 marks

27 (12 marks)

$$(a) \frac{d y}{d x} = \frac{(\frac{1}{x})^x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

M1 A1

$$\frac{d y}{d x} = 0 \text{ gives } \ln x = 1, \text{ so } x = e$$

M1

$$y = \frac{1}{e}, \text{ so } M \text{ is } (e, \frac{1}{e})$$

A1

Answer: $M(e, \frac{1}{e})$

4 marks

(b) $u = \ln x$ gives $du = \frac{1}{x} dx$, so $\frac{\ln x}{x} dx = u du$

M1 A1

Limits: $x = 1$ gives $u = 0$ and $x = e^2$ gives $u = 2$

B1

$$\int_0^2 u du = \left[\frac{u^2}{2}\right]_0^2 = 2$$

M1 A1

Answer: 2

5 marks

(c) Same substitution with upper limit $u = k$: $\int_0^k u du = \frac{k^2}{2}$

M1 A1

$$\frac{k^2}{2} = 8 \text{ gives } k = 4$$

A1

Answer: $\frac{k^2}{2}$; $k = 4$

3 marks

28 (12 marks)

(a) $u = a - x$ gives $du = -dx$; $x = 0$ gives $u = a$, $x = a$ gives $u = 0$

M1 A1

$$\int_0^a f(x) dx = \int_a^0 f(a - u)(-du)$$

M1

$$= \int_0^a f(a - u) du, \text{ and } u \text{ is a dummy variable, so this equals } \int_0^a f(a - x) dx$$

M1 A1

Answer: proved

5 marks

(b) Take $a = \frac{\pi}{2}$: $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$

M1 A1

$$\text{So } I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$$

A1

Answer: shown

3 marks

(c) Add the two expressions: $2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x} dx$

M1

$$= \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

M1 A1

$$I = \frac{\pi}{4}$$

A1

Answer: $\frac{\pi}{4}$

4 marks

29 (11 marks)

(a) $dx = \sec^2 \theta d\theta$; $x = 0$ gives $\theta = 0$ and $x = 1$ gives $\theta = \frac{\pi}{4}$

M1 A1

$$1 + x^2 = 1 + \tan^2 \theta = \sec^2 \theta$$

M1

$$I = \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int_0^{\frac{\pi}{4}} \cos^2 \theta d\theta$$

A1

Answer: shown

4 marks

(b) $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$

M1

$$\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}$$

M1 A1

$$= \frac{\pi}{8} + \frac{1}{4}$$

A1

Answer: $\frac{\pi}{8} + \frac{1}{4}$

4 marks

(c) $\frac{x^2}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{1}{(1+x^2)^2}$

M1

$$\int_0^1 \frac{1}{1+x^2} dx = [\arctan x]_0^1 = \frac{\pi}{4}$$

B1

$$\frac{\pi}{4} - \left(\frac{\pi}{8} + \frac{1}{4}\right) = \frac{\pi}{8} - \frac{1}{4}$$

A1

Answer: $\frac{\pi}{8} - \frac{1}{4}$

3 marks

30 (12 marks)

$$(a) I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\tan^2 x + 1) dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x dx$$

M1 A1

Let $u = \tan x$; $du = \sec^2 x dx$; limits $0 \rightarrow 1$

M1

$$\int_0^1 u^{n-2} du = \left[\frac{u^{n-1}}{n-1} \right]_0^1 = \frac{1}{n-1}$$

M1 A1

Answer: shown

5 marks

$$(b) I_0 = \frac{\pi}{4}$$

B1

$$I_2 = \frac{1}{1} - I_0 = 1 - \frac{\pi}{4}$$

M1 A1

$$I_4 = \frac{1}{3} - I_2 = \frac{\pi}{4} - \frac{2}{3}$$

M1 A1

$$\text{Answer: } I_4 = \frac{\pi}{4} - \frac{2}{3}$$

5 marks

$$(c) \tan^4 x > 0 \text{ for } 0 < x \leq \frac{\pi}{4}, \text{ so } I_4 > 0$$

E1

$$\frac{\pi}{4} - \frac{2}{3} > 0 \text{ gives } \pi > \frac{8}{3}$$

E1

Answer: $\pi > 8/3$

2 marks

Integration by Parts**1 (3 marks)**

$$u = x, \frac{dv}{dx} = e^x, \text{ so } \frac{du}{dx} = 1 \text{ and } v = e^x$$

M1

$$\int x e^x dx = x e^x - \int e^x dx$$

A1

$$= x e^x - e^x + c = (x - 1)e^x + c$$

A1

Answer: $(x - 1)e^x + c$

3 marks

2 (3 marks)

$$u = x, \frac{dv}{dx} = \cos x, \text{ so } v = \sin x$$

M1

$$\int x \cos x dx = x \sin x - \int \sin x dx$$

A1

$$= x \sin x + \cos x + c$$

A1

Answer: $x \sin x + \cos x + c$

3 marks

3 (2 marks)

$$u = x, \frac{dv}{dx} = e^{2x}; v = \frac{1}{2}e^{2x}$$

M1

$$= \frac{x}{2}e^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$$

A1

$$\text{Answer: } \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + c$$

2 marks

4 (2 marks)

$$u = x, \frac{dv}{dx} = \sin x; v = -\cos x$$

M1

$$= -x \cos x + \int \cos x dx = -x \cos x + \sin x + c$$

A1

Answer: $-x \cos x + \sin x + c$

2 marks

5 (2 marks)

$$u = 3x, \frac{dv}{dx} = e^{-x}; v = -e^{-x}$$

M1

$$= -3xe^{-x} + 3 \int e^{-x} dx = -3xe^{-x} - 3e^{-x} + c$$

A1

Answer: $-3(x + 1)e^{-x} + c$

2 marks

6 (2 marks)

$$u = x, \frac{d v}{d x} = e^{\frac{x}{2}}; v = 2e^{\frac{x}{2}} \quad \text{M1}$$

$$= 2xe^{\frac{x}{2}} - 2 \int e^{\frac{x}{2}} d x = 2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c \quad \text{A1}$$

$$\text{Answer: } 2(x - 2)e^{\frac{x}{2}} + c \quad \text{2 marks}$$

7 (4 marks)

$$u = x, \frac{d v}{d x} = \sin 2x, \text{ so } v = -\frac{1}{2} \cos 2x \quad \text{M1 A1}$$

$$\int x \sin 2x d x = -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x d x \quad \text{M1}$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + c \quad \text{A1}$$

$$\text{Answer: } -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + c \quad \text{4 marks}$$

8 (4 marks)

$$u = \ln x, \frac{d v}{d x} = x, \text{ so } \frac{d u}{d x} = \frac{1}{x} \text{ and } v = \frac{x^2}{2} \quad \text{M1 A1}$$

$$\int x \ln x d x = \frac{x^2}{2} \ln x - \int \frac{x}{2} d x \quad \text{M1}$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c \quad \text{A1}$$

$$\text{Answer: } \frac{x^2}{2} \ln x - \frac{x^2}{4} + c \quad \text{4 marks}$$

9 (4 marks)

$$[x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x d x \quad \text{M1 A1}$$

$$= \frac{\pi}{2} - [-\cos x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1 \quad \text{M1 A1}$$

$$\text{Answer: } \frac{\pi}{2} - 1 \quad \text{4 marks}$$

10 (4 marks)

$$u = \ln x, \frac{d v}{d x} = x^2; v = \frac{x^3}{3} \quad \text{M1 A1}$$

$$= \left[\frac{x^3}{3} \ln x \right]_1^e - \frac{1}{3} \int_1^e x^2 d x \quad \text{M1}$$

$$= \frac{e^3}{3} - \frac{1}{9} (e^3 - 1) = \frac{2e^3 + 1}{9} \quad \text{A1}$$

$$\text{Answer: } \frac{2e^3 + 1}{9} \quad \text{4 marks}$$

11 (4 marks)

$$u = x, \frac{d v}{d x} = \cos 3x; v = \frac{1}{3} \sin 3x \quad \text{M1 A1}$$

$$= \frac{x}{3} \sin 3x - \frac{1}{3} \int \sin 3x d x \quad \text{M1}$$

$$= \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + c \quad \text{A1}$$

$$\text{Answer: } \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + c \quad \text{4 marks}$$

12 (4 marks)

$$[-xe^{-x}]_0^2 + \int_0^2 e^{-x} dx$$

M1 A1

$$= -2e^{-2} + [-e^{-x}]_0^2 = -2e^{-2} - e^{-2} + 1$$

M1

$$= 1 - 3e^{-2}$$

A1

Answer: shown

4 marks

13 (4 marks)

$$u = \ln x, \frac{dv}{dx} = x^{-2}; v = -x^{-1}$$

M1 A1

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + c$$

M1 A1

$$\text{Answer: } -\frac{\ln x}{x} - \frac{1}{x} + c$$

4 marks

14 (5 marks)

$$u = 2x + 1, \frac{dv}{dx} = e^x, \text{ so } \frac{du}{dx} = 2 \text{ and } v = e^x$$

M1 A1

$$\int_0^1 (2x + 1)e^x dx = [(2x + 1)e^x]_0^1 - \int_0^1 2e^x dx$$

M1

$$= (3e - 1) - [2e^x]_0^1$$

A1

$$= 3e - 1 - 2e + 2 = e + 1$$

A1

Answer: shown

5 marks

15 (7 marks)

$$(a) u = \ln x, \frac{dv}{dx} = 1, \text{ so } \frac{du}{dx} = \frac{1}{x} \text{ and } v = x$$

M1 A1

$$\int \ln x dx = x \ln x - \int 1 dx = x \ln x - x + c$$

A1

Answer: $x \ln x - x + c$

3 marks

$$(b) u = (\ln x)^2, \frac{dv}{dx} = 1; \frac{du}{dx} = \frac{2 \ln x}{x}, v = x$$

M1

$$\int_1^e (\ln x)^2 dx = [x(\ln x)^2]_1^e - 2 \int_1^e \ln x dx$$

M1 A1

$$= e - 2[x \ln x - x]_1^e = e - 2(0 - (-1)) = e - 2$$

A1

Answer: shown

4 marks

16 (5 marks)

$$u = x^2, \frac{dv}{dx} = e^x; x^2 e^x - 2 \int x e^x dx$$

M1 A1

$$\text{Parts again: } \int x e^x dx = x e^x - e^x$$

M1 A1

$$= x^2 e^x - 2x e^x + 2e^x + c$$

A1

Answer: $e^x(x^2 - 2x + 2) + c$

5 marks

17 (5 marks)

$$[x^2 \sin x]_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} x \sin x dx$$

M1 A1

$$\int_0^{\frac{\pi}{2}} x \sin x dx = [-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx = 0 + 1 = 1$$

M1 A1

$$= \frac{\pi^2}{4} - 2$$

A1

Answer: shown

5 marks

18 (6 marks)

$$u = x, \frac{dv}{dx} = (x+1)^{\frac{1}{2}}; v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

M1 A1

$$\left[\frac{2x}{3}(x+1)^{\frac{3}{2}}\right]_0^3 - \frac{2}{3} \int_0^3 (x+1)^{\frac{3}{2}} dx$$

M1

$$= 16 - \frac{2}{3} \left[\frac{2}{5}(x+1)^{\frac{5}{2}}\right]_0^3 = 16 - \frac{4}{15}(32-1)$$

M1 A1

$$= \frac{116}{15}$$

A1

$$\text{Answer: } \frac{116}{15}$$

6 marks

19 (6 marks)

$$u = \ln x, \frac{dv}{dx} = x^{-\frac{1}{2}}; v = 2x^{\frac{1}{2}}$$

M1 A1

$$[2\sqrt{x} \ln x]_1^3 - 2 \int_1^3 x^{-\frac{1}{2}} dx$$

M1 A1

$$= 2\sqrt{3} \ln 3 - 2[2\sqrt{x}]_1^3 = 2\sqrt{3} \ln 3 - 4\sqrt{3} + 4$$

M1 A1

Answer: shown

6 marks

20 (5 marks)

$$\text{Integrate by parts with } u = \ln x, \frac{dv}{dx} = 1: \int \ln x \, dx = x \ln x - x$$

M1 A1

$$[x \ln x - x]_1^a = a \ln a - a + 1$$

M1

$$a \ln a - a + 1 = 1, \text{ so } a(\ln a - 1) = 0$$

M1

$$a > 1, \text{ so } \ln a = 1 \text{ and } a = e$$

A1

Answer: a = e

5 marks

21 (9 marks)

$$(a) u = x^2, \frac{dv}{dx} = e^{-x}: \int x^2 e^{-x} \, dx = -x^2 e^{-x} + \int 2x e^{-x} \, dx$$

M1 A1

$$\int 2x e^{-x} \, dx = -2x e^{-x} - 2e^{-x}, \text{ so the result is } -(x^2 + 2x + 2)e^{-x} + c$$

M1 A1

$$\text{Answer: } -(x^2 + 2x + 2)e^{-x} + c$$

4 marks

$$(b) [-(x^2 + 2x + 2)e^{-x}]_0^1 = -5e^{-1} + 2$$

M1

$$\text{Area} = 2 - \frac{5}{e}$$

A1

$$\text{Answer: } 2 - \frac{5}{e}$$

2 marks

$$(c) \frac{dy}{dx} = 2x e^{-x} - x^2 e^{-x} = x(2-x)e^{-x}$$

M1 A1

$$\frac{dy}{dx} = 0 \text{ for } x > 0 \text{ gives } x = 2, \text{ so } S \text{ is } (2, 4e^{-2})$$

A1

$$\text{Answer: } S(2, 4e^{-2})$$

3 marks

22 (8 marks)

$$(a) \text{ First pass } (u = \sin x, \frac{dv}{dx} = e^x): I = e^x \sin x - \int e^x \cos x \, dx$$

M1 A1

$$\text{Second pass } (u = \cos x): \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx = e^x \cos x + I$$

M1 A1

$$\text{So } I = e^x \sin x - e^x \cos x - I, \text{ giving } 2I = e^x(\sin x - \cos x)$$

M1

$$I = \frac{1}{2}e^x(\sin x - \cos x) + c$$

A1

Answer: shown

6 marks

$$(b) \text{ Area} = \left[\frac{1}{2}e^x(\sin x - \cos x)\right]_0^\pi = \frac{1}{2}e^\pi(0+1) - \frac{1}{2}(0-1)$$

M1

$$= \frac{1}{2}(e^\pi + 1) = 12.1$$

A1

$$\text{Answer: } \frac{1}{2}(e^\pi + 1) \approx 12.1$$

2 marks

23 (7 marks)

$$u = x, \frac{d v}{d x} = \sec^2 x: \frac{d u}{d x} = 1, v = \tan x$$

M1 A1

$$= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, d x$$

M1

$$\int \tan x \, d x = -\ln|\cos x|$$

B1

$$= \frac{\pi}{4} - [-\ln(\cos x)]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \ln \sqrt{2}$$

M1 A1

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

A1

Answer: shown

7 marks

24 (8 marks)

$$(a) \frac{d y}{d x} = e^x + (x - 2)e^x = (x - 1)e^x = 0$$

M1 A1

$$x = 1, y = -e$$

A1

Answer: M(1, -e)

3 marks

$$(b) \int_0^2 (x - 2)e^x \, d x = [(x - 2)e^x]_0^2 - \int_0^2 e^x \, d x$$

M1 A1

$$= (0 + 2) - (e^2 - 1) = 3 - e^2$$

M1 A1

$$R \text{ is below the } x\text{-axis, so area} = e^2 - 3$$

A1

Answer: $e^2 - 3$

5 marks

25 (7 marks)

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \text{ so the integral is } \frac{1}{2} \int_0^{\frac{\pi}{2}} (x + x \cos 2x) \, d x$$

M1 A1

$$\int_0^{\frac{\pi}{2}} x \, d x = \frac{\pi^2}{8}$$

B1

$$\int_0^{\frac{\pi}{2}} x \cos 2x \, d x = \left[\frac{x \sin 2x}{2} \right]_0^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2x \, d x$$

M1 A1

$$= 0 - \frac{1}{2} \times 1 = -\frac{1}{2}$$

M1

$$\text{Total} = \frac{1}{2} \left(\frac{\pi^2}{8} - \frac{1}{2} \right) = \frac{\pi^2}{16} - \frac{1}{4}$$

A1

$$\text{Answer: } \frac{\pi^2}{16} - \frac{1}{4}$$

7 marks

26 (6 marks)

$$x = u^2, \, d x = 2u \, d u; \text{ limits } u = 0 \text{ to } 1$$

M1 A1

$$\text{Integral} = 2 \int_0^1 u e^u \, d u$$

M1

$$= 2([u e^u]_0^1 - \int_0^1 e^u \, d u)$$

M1 A1

$$= 2(e - (e - 1)) = 2$$

A1

Answer: shown

6 marks

27 (10 marks)

$$(a) u = x^n, \frac{d v}{d x} = e^x: \frac{d u}{d x} = n x^{n-1}, v = e^x$$

M1 A1

$$I_n = [x^n e^x]_0^1 - n \int_0^1 x^{n-1} e^x \, d x$$

M1

$$= e - 0 - n I_{n-1}$$

A1

Answer: shown

4 marks

$$(b) I_0 = \int_0^1 e^x \, d x = e - 1$$

B1

$$I_1 = e - I_0 = 1; I_2 = e - 2I_1 = e - 2$$

M1 A1

$$I_3 = e - 3(e - 2) = 6 - 2e$$

A1

Answer: $I_3 = 6 - 2e$ **4 marks****(c)** $x^3 e^x > 0$ for $0 < x \leq 1$, so the integral is positive**E1** $6 - 2e > 0$ gives $e < 3$ **E1****Answer: $e < 3$** **2 marks****28 (11 marks)****(a)** $\frac{dy}{dx} = \ln x + 1 = 0$ gives $x = \frac{1}{e}$, $y = -\frac{1}{e}$ **M1 A1** $\frac{d^2y}{dx^2} = \frac{1}{x} > 0$, so the point $(1/e, -1/e)$ is a minimum**M1 A1****Answer: minimum at $(\frac{1}{e}, -\frac{1}{e})$** **4 marks****(b)** $u = \ln x$, $\frac{dv}{dx} = x$: $v = \frac{x^2}{2}$ **M1**

$$\int_1^e x \ln x \, dx = \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x}{2} \, dx$$

M1 A1

$$= \frac{e^2}{2} - \left[\frac{x^2}{4} \right]_1^e = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{1}{4}(e^2 + 1)$$

A1**Answer: shown****4 marks****(c)** $x = x \ln x$ gives $\ln x = 1$, so A is (e, e) ; area of R = $\int_1^e (x - x \ln x) \, dx$ **M1**

$$\int_1^e x \, dx = \frac{1}{2}(e^2 - 1), \text{ so area} = \frac{1}{2}(e^2 - 1) - \frac{1}{4}(e^2 + 1)$$

M1

$$= \frac{1}{4}(e^2 - 3)$$

A1**Answer: $\frac{1}{4}(e^2 - 3)$** **3 marks****29 (11 marks)****(a)** Write $\sin^n x = \sin^{n-1} x \cdot \sin x$; $u = \sin^{n-1} x$, $\frac{dv}{dx} = \sin x$: $v = -\cos x$ **M1 A1**

$$J_n = [-\sin^{n-1} x \cos x]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$$

M1 A1The bracket is 0; $\cos^2 x = 1 - \sin^2 x$ gives $J_n = (n-1)(J_{n-2} - J_n)$ **M1**

$$nJ_n = (n-1)J_{n-2}$$

A1**Answer: shown****6 marks****(b)** $J_1 = 1$ and $J_0 = \frac{\pi}{2}$ **B1**

$$J_3 = \frac{2}{3}J_1 = \frac{2}{3}, J_5 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

M1 A1

$$J_2 = \frac{1}{2}J_0 = \frac{\pi}{4}, J_4 = \frac{3}{4}J_2 = \frac{3\pi}{16}, J_6 = \frac{5}{6}J_4 = \frac{5\pi}{32}$$

M1 A1**Answer: $J_5 = \frac{8}{15}$, $J_6 = \frac{5\pi}{32}$** **5 marks****30 (10 marks)****(a)** First pass ($u = \cos 3x$, $\frac{dv}{dx} = e^{2x}$): $K = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx$ **M1 A1**Second pass ($u = \sin 3x$): $\int e^{2x} \sin 3x \, dx = \frac{1}{2}e^{2x} \sin 3x - \frac{3}{2}K$ **M1 A1**

$$K = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x - \frac{9}{4}K$$

M1

$$\frac{13}{4}K = \frac{e^{2x}}{4}(2\cos 3x + 3\sin 3x), \text{ so } K = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + c$$

A1**Answer: shown****6 marks****(b)** $y \geq 0$ on $0 \leq x \leq \frac{\pi}{6}$ because $\cos 3x \geq 0$ **B1**

$$\text{Area} = \frac{1}{13}[e^{2x}(2\cos 3x + 3\sin 3x)]_0^{\frac{\pi}{6}} = \frac{1}{13}(3e^{\frac{\pi}{3}} - 2)$$

M1 A1

$$= 0.504 \text{ (3 s.f.)}$$

A1

Answer: $\frac{1}{13}(3e^{\frac{\pi}{3}} - 2) \approx 0.504$

4 marks

Integration Using Partial Fractions

1 (3 marks)

$$\frac{x+5}{(x+1)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+3)}, \text{ so } x+5 \equiv A(x+3) + B(x+1)$$

M1

$$x = -1: 4 = 2A, \text{ so } A = 2$$

A1

$$x = -3: 2 = -2B, \text{ so } B = -1$$

A1

$$\text{Hence } \frac{x+5}{(x+1)(x+3)} \equiv -\frac{1}{(x+3)} + \frac{2}{(x+1)}$$

Answer: $-\frac{1}{(x+3)} + \frac{2}{(x+1)}$

3 marks

2 (5 marks)

$$\frac{2}{(x-1)(x+1)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+1)}, \text{ so } 2 \equiv A(x+1) + B(x-1)$$

M1

$$x = 1: 2 = 2A, \text{ so } A = 1$$

A1

$$x = -1: 2 = -2B, \text{ so } B = -1$$

A1

$$\int \frac{2}{(x-1)(x+1)} dx = \ln|x-1| - \ln|x+1| + c$$

M1 A1

Answer: $\ln|x-1| - \ln|x+1| + c$

5 marks

3 (5 marks)

$$\frac{5x+13}{(x+2)(x+3)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+3)}, \text{ so } 5x+13 \equiv A(x+3) + B(x+2)$$

M1

$$x = -2: 3 = A$$

A1

$$x = -3: -2 = -B, \text{ so } B = 2$$

A1

$$\int \frac{5x+13}{(x+2)(x+3)} dx = 2\ln|x+3| + 3\ln|x+2| + c$$

M1 A1

Answer: $2\ln|x+3| + 3\ln|x+2| + c$

5 marks

4 (3 marks)

$$\frac{9x+1}{(2x+1)(x-3)} \equiv \frac{A}{(2x+1)} + \frac{B}{(x-3)}, \text{ so } 9x+1 \equiv A(x-3) + B(2x+1)$$

M1

$$x = -\frac{1}{2}: -\frac{7}{2} = -\frac{7}{2}A, \text{ so } A = 1$$

A1

$$x = 3: 28 = 7B, \text{ so } B = 4$$

A1

$$\text{Hence } \frac{9x+1}{(2x+1)(x-3)} \equiv \frac{1}{(2x+1)} + \frac{4}{(x-3)}$$

Answer: $\frac{1}{(2x+1)} + \frac{4}{(x-3)}$

3 marks

5 (3 marks)

$$\frac{5x+8}{x(x+4)} \equiv \frac{A}{x} + \frac{B}{(x+4)}, \text{ so } 5x+8 \equiv A(x+4) + Bx$$

M1

$$x = 0: 8 = 4A, \text{ so } A = 2$$

A1

$$x = -4: -12 = -4B, \text{ so } B = 3$$

A1

$$\text{Hence } \frac{5x+8}{x(x+4)} \equiv \frac{2}{x} + \frac{3}{(x+4)}$$

Answer: $\frac{2}{x} + \frac{3}{(x+4)}$

3 marks

6 (5 marks)

$$x^2 - 9 = (x - 3)(x + 3): \frac{6}{(x - 3)(x + 3)} \equiv \frac{A}{(x - 3)} + \frac{B}{(x + 3)}, \text{ so } 6 \equiv A(x + 3) + B(x - 3) \quad \text{M1}$$

$$x = 3: 6 = 6A, \text{ so } A = 1 \quad \text{A1}$$

$$x = -3: 6 = -6B, \text{ so } B = -1 \quad \text{A1}$$

$$\int \left(\frac{1}{x - 3} - \frac{1}{x + 3} \right) dx = \ln|x - 3| - \ln|x + 3| \quad \text{M1}$$

$$= \ln \left| \frac{x - 3}{x + 3} \right| + c \quad \text{A1}$$

Answer: $\ln \left| \frac{x - 3}{x + 3} \right| + c$ 5 marks

7 (5 marks)

$$\frac{7x + 4}{(2x - 1)(x + 2)} \equiv \frac{A}{(2x - 1)} + \frac{B}{(x + 2)}, \text{ so } 7x + 4 \equiv A(x + 2) + B(2x - 1) \quad \text{M1}$$

$$x = \frac{1}{2}: \frac{15}{2} = \frac{5}{2}A, \text{ so } A = 3 \quad \text{A1}$$

$$x = -2: -10 = -5B, \text{ so } B = 2 \quad \text{A1}$$

$$\int \frac{7x + 4}{(2x - 1)(x + 2)} dx = 2\ln|x + 2| + \frac{3}{2}\ln|2x - 1| + c \quad \text{M1 A1}$$

Answer: $2\ln|x + 2| + \frac{3}{2}\ln|2x - 1| + c$ 5 marks

8 (7 marks)

$$\frac{3x + 5}{(x + 1)(x + 2)} \equiv \frac{A}{(x + 1)} + \frac{B}{(x + 2)}, \text{ so } 3x + 5 \equiv A(x + 2) + B(x + 1) \quad \text{M1}$$

$$x = -1: 2 = A \quad \text{A1}$$

$$x = -2: -1 = -B, \text{ so } B = 1 \quad \text{A1}$$

$$[\ln|x + 2| + 2\ln|x + 1|]_0^1 \quad \text{M1 A1}$$

$$= \ln \frac{3}{2} + 2\ln 2 \quad \text{M1}$$

$$= \ln 2 + \ln 3$$

$$= \ln 6 \quad \text{A1}$$

Answer: shown 7 marks

9 (7 marks)

$$\frac{5x + 3}{(x - 1)(x + 3)} \equiv \frac{A}{(x - 1)} + \frac{B}{(x + 3)}, \text{ so } 5x + 3 \equiv A(x + 3) + B(x - 1) \quad \text{M1}$$

$$x = 1: 8 = 4A, \text{ so } A = 2 \quad \text{A1}$$

$$x = -3: -12 = -4B, \text{ so } B = 3 \quad \text{A1}$$

$$[2\ln|x - 1| + 3\ln|x + 3|]_2^4 \quad \text{M1 A1}$$

$$= 2\ln 3 + 3\ln \frac{7}{5} \quad \text{M1}$$

$$= 2\ln 3 - 3\ln 5 + 3\ln 7 \quad \text{A1}$$

Answer: $2\ln 3 - 3\ln 5 + 3\ln 7$ 7 marks

10 (4 marks)

$$\frac{-7x - 2}{x(x + 2)(x - 1)} \equiv \frac{A}{x} + \frac{B}{(x + 2)} + \frac{C}{(x - 1)}, \text{ so } -7x - 2 \equiv A(x + 2)(x - 1) + Bx(x - 1) + Cx(x + 2) \quad \text{M1}$$

$$x = 0: -2 = -2A, \text{ so } A = 1 \quad \text{A1}$$

$$x = -2: 12 = 6B, \text{ so } B = 2 \quad \text{A1}$$

$$x = 1: -9 = 3C, \text{ so } C = -3 \quad \text{A1}$$

$$\text{Hence } \frac{-7x - 2}{x(x + 2)(x - 1)} \equiv \frac{1}{x} - \frac{3}{(x - 1)} + \frac{2}{(x + 2)}$$

Answer: $\frac{1}{x} - \frac{3}{(x - 1)} + \frac{2}{(x + 2)}$ 4 marks

11 (5 marks)

$$(a) \frac{5x+10}{(x-1)(x+4)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+4)}, \text{ so } 5x+10 \equiv A(x+4) + B(x-1) \quad \text{M1}$$

$$x = 1: 15 = 5A, \text{ so } A = 3 \quad \text{A1}$$

$$x = -4: -10 = -5B, \text{ so } B = 2 \quad \text{A1}$$

$$\text{Hence } \frac{5x+10}{(x-1)(x+4)} \equiv \frac{2}{(x+4)} + \frac{3}{(x-1)}$$

$$\text{Answer: } \frac{2}{(x+4)} + \frac{3}{(x-1)} \quad \text{3 marks}$$

$$(b) \text{ Integrate each term: } 3\ln|x-1| + 2\ln|x+4| + c \quad \text{M1 A1}$$

$$\text{Answer: } 3\ln|x-1| + 2\ln|x+4| + c \quad \text{2 marks}$$

12 (5 marks)

$$\frac{6}{x(x-3)} \equiv \frac{A}{x} + \frac{B}{(x-3)}, \text{ so } 6 \equiv A(x-3) + Bx \quad \text{M1}$$

$$x = 0: 6 = -3A, \text{ so } A = -2 \quad \text{A1}$$

$$x = 3: 6 = 3B, \text{ so } B = 2 \quad \text{A1}$$

$$\int \frac{6}{x(x-3)} dx = -2\ln|x| + 2\ln|x-3| + c \quad \text{M1 A1}$$

$$\text{Answer: } -2\ln|x| + 2\ln|x-3| + c \quad \text{5 marks}$$

13 (4 marks)

$$\frac{4x^2+8x}{(x+1)^2(x-1)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}, \text{ so } 4x^2+8x \equiv A(x+1)(x-1) + B(x-1) + C(x+1)^2 \quad \text{M1}$$

$$x = -1: -4 = -2B, \text{ so } B = 2 \quad \text{A1}$$

$$x = 1: 12 = 4C, \text{ so } C = 3 \quad \text{A1}$$

$$\text{Coefficient of } x^2: 4 = A + 3, \text{ so } A = 1 \quad \text{A1}$$

$$\text{Hence } \frac{4x^2+8x}{(x+1)^2(x-1)} \equiv \frac{1}{(x+1)} + \frac{2}{(x+1)^2} + \frac{3}{(x-1)}$$

$$\text{Answer: } \frac{1}{(x+1)} + \frac{2}{(x+1)^2} + \frac{3}{(x-1)} \quad \text{4 marks}$$

14 (6 marks)

$$\frac{6x^2+19x+11}{(x+2)^2(x-1)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x-1)}, \text{ so } 6x^2+19x+11 \equiv A(x+2)(x-1) + B(x-1) + C(x+2)^2 \quad \text{M1}$$

$$x = -2: -3 = -3B, \text{ so } B = 1 \quad \text{A1}$$

$$x = 1: 36 = 9C, \text{ so } C = 4 \quad \text{A1}$$

$$\text{Coefficient of } x^2: 6 = A + 4, \text{ so } A = 2 \quad \text{A1}$$

$$\int \frac{6x^2+19x+11}{(x+2)^2(x-1)} dx = -\frac{1}{(x+2)} + 2\ln|x+2| + 4\ln|x-1| + c \quad \text{M1 A1}$$

$$\text{Answer: } -\frac{1}{(x+2)} + 2\ln|x+2| + 4\ln|x-1| + c \quad \text{6 marks}$$

15 (8 marks)

$$\frac{4x^2 + 8x}{(x+1)^2(x-1)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}, \text{ so } 4x^2 + 8x \equiv A(x+1)(x-1) + B(x-1) + C(x+1)^2 \quad \text{M1}$$

$$x = -1: -4 = -2B, \text{ so } B = 2 \quad \text{A1}$$

$$x = 1: 12 = 4C, \text{ so } C = 3 \quad \text{A1}$$

$$\text{Coefficient of } x^2: 4 = A + 3, \text{ so } A = 1 \quad \text{A1}$$

$$[\ln|x+1| - \frac{2}{(x+1)} + 3\ln|x-1|]_2^3 \quad \text{M1 A1}$$

$$= \ln \frac{4}{3} + \frac{1}{6} + 3\ln 2 \quad \text{M1}$$

$$= 5\ln 2 - \ln 3 + \frac{1}{6} \quad \text{A1}$$

$$\text{Answer: } 5\ln 2 - \ln 3 + \frac{1}{6} \quad \text{8 marks}$$

16 (7 marks)

$$\text{Divide: } \frac{x^2 + 8x + 9}{x^2 + 3x + 2} = 1 + \frac{5x + 7}{x^2 + 3x + 2} \quad \text{M1 A1}$$

$$\frac{5x + 7}{(x+1)(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+2)}, \text{ so } 5x + 7 \equiv A(x+2) + B(x+1) \quad \text{M1}$$

$$x = -1: 2 = A \quad \text{A1}$$

$$x = -2: -3 = -B, \text{ so } B = 3 \quad \text{A1}$$

$$\int \frac{x^2 + 8x + 9}{(x+1)(x+2)} dx = x + 2\ln|x+1| + 3\ln|x+2| + c \quad \text{M1 A1}$$

$$\text{Answer: } x + 2\ln|x+1| + 3\ln|x+2| + c \quad \text{7 marks}$$

17 (6 marks)

$$\frac{4x^2 - 13x - 12}{x(x+2)(x-3)} \equiv \frac{A}{x} + \frac{B}{(x+2)} + \frac{C}{(x-3)}, \text{ so } 4x^2 - 13x - 12 \equiv A(x+2)(x-3) + Bx(x-3) + Cx(x+2) \quad \text{M1}$$

$$x = 0: -12 = -6A, \text{ so } A = 2 \quad \text{A1}$$

$$x = -2: 30 = 10B, \text{ so } B = 3 \quad \text{A1}$$

$$x = 3: -15 = 15C, \text{ so } C = -1 \quad \text{A1}$$

$$\int \frac{4x^2 - 13x - 12}{x(x+2)(x-3)} dx = -\ln|x-3| + 2\ln|x| + 3\ln|x+2| + c \quad \text{M1 A1}$$

$$\text{Answer: } -\ln|x-3| + 2\ln|x| + 3\ln|x+2| + c \quad \text{6 marks}$$

18 (8 marks)

$$\frac{2}{x(x+1)(x+2)} \equiv \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+2)}, \text{ so } 2 \equiv A(x+1)(x+2) + Bx(x+2) + Cx(x+1) \quad \text{M1}$$

$$x = 0: 2 = 2A, \text{ so } A = 1 \quad \text{A1}$$

$$x = -1: 2 = -B, \text{ so } B = -2 \quad \text{A1}$$

$$x = -2: 2 = 2C, \text{ so } C = 1 \quad \text{A1}$$

$$[\ln|x| + \ln|x+2| - 2\ln|x+1|]_1^2 \quad \text{M1 A1}$$

$$= \ln 2 + \ln \frac{4}{3} - 2\ln \frac{3}{2} \quad \text{M1}$$

$$= 5\ln 2 - 3\ln 3$$

$$= \ln \frac{32}{27} \quad \text{A1}$$

$$\text{Answer: shown} \quad \text{8 marks}$$

19 (8 marks)

$$\frac{x^2}{x^2 - 1} = 1 + \frac{1}{x^2 - 1}$$

M1 A1

$$\frac{1}{(x-1)(x+1)} \equiv \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

M1 A1

$$\left[x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| \right]_2^3$$

M1 A1

$$= \left(3 + \frac{1}{2} \ln 2 - \frac{1}{2} \ln 4 \right) - \left(2 + 0 - \frac{1}{2} \ln 3 \right)$$

M1

$$= 1 + \frac{1}{2} \ln \frac{3}{2}$$

A1

$$\text{Answer: } 1 + \frac{1}{2} \ln \frac{3}{2}$$

8 marks

20 (9 marks)

$$\text{Divide: } \frac{2x^2 + 11x + 14}{x^2 + 4x + 3} = 2 + \frac{3x + 8}{x^2 + 4x + 3}$$

M1 A1

$$\frac{3x + 8}{(x+1)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+3)}, \text{ so } 3x + 8 \equiv A(x+3) + B(x+1)$$

M1

$$x = -1: 5 = 2A, \text{ so } A = \frac{5}{2}$$

A1

$$x = -3: -1 = -2B, \text{ so } B = \frac{1}{2}$$

A1

$$\left[2x + \frac{5}{2} \ln|x+1| + \frac{1}{2} \ln|x+3| \right]_0^1$$

M1 A1

$$= 2 + \frac{5}{2} \ln 2 + \frac{1}{2} \ln 4 - \frac{1}{2} \ln 3$$

M1

$$= 2 + \frac{7}{2} \ln 2 - \frac{1}{2} \ln 3$$

A1

$$\text{Answer: } 2 + \frac{7}{2} \ln 2 - \frac{1}{2} \ln 3$$

9 marks

21 (7 marks)

$$\text{Divide: } \frac{x^3 + 4x^2 + 6x - 5}{x^2 + x - 2} = x + 3 + \frac{5x + 1}{x^2 + x - 2}$$

M1 A1

$$\frac{5x + 1}{(x-1)(x+2)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+2)}, \text{ so } 5x + 1 \equiv A(x+2) + B(x-1)$$

M1

$$x = 1: 6 = 3A, \text{ so } A = 2$$

A1

$$x = -2: -9 = -3B, \text{ so } B = 3$$

A1

$$\int \frac{x^3 + 4x^2 + 6x - 5}{(x-1)(x+2)} dx = \frac{x^2}{2} + 3x + 2 \ln|x-1| + 3 \ln|x+2| + c$$

M1 A1

$$\text{Answer: } \frac{x^2}{2} + 3x + 2 \ln|x-1| + 3 \ln|x+2| + c$$

7 marks

22 (9 marks)

$$\text{(a) } \frac{4x^2 + 3x - 4}{x(x+1)(x-2)} \equiv \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x-2)}, \text{ so } 4x^2 + 3x - 4 \equiv A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

M1

$$x = 0: -4 = -2A, \text{ so } A = 2$$

A1

$$x = -1: -3 = 3B, \text{ so } B = -1$$

A1

$$x = 2: 18 = 6C, \text{ so } C = 3$$

A1

$$\text{Answer: } A = 2, B = -1, C = 3$$

4 marks

$$\text{(b) } [2 \ln|x| - \ln|x+1| + 3 \ln|x-2|]_3^4$$

M1 A1

$$= (2 \ln 4 - \ln 5 + 3 \ln 2) - (2 \ln 3 - \ln 4 + 0)$$

M1

$$= 7 \ln 2 + 2 \ln 2 - 2 \ln 3 - \ln 5 = 9 \ln 2 - 2 \ln 3 - \ln 5$$

M1 A1

$$\text{Answer: } 9 \ln 2 - 2 \ln 3 - \ln 5$$

5 marks

23 (7 marks)

$$(a) \frac{4}{(x+1)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+3)}, \text{ so } 4 \equiv A(x+3) + B(x+1) \quad \text{M1}$$

$$x = -1: 4 = 2A, \text{ so } A = 2 \quad \text{A1}$$

$$x = -3: 4 = -2B, \text{ so } B = -2 \quad \text{A1}$$

$$\text{Answer: } \frac{2}{(x+1)} - \frac{2}{(x+3)} \quad \text{3 marks}$$

$$(b) \text{ Area} = \int_0^1 \left(\frac{2}{x+1} - \frac{2}{x+3} \right) dx = [2\ln(x+1) - 2\ln(x+3)]_0^1 \quad \text{M1 A1}$$

$$= (2\ln 2 - 2\ln 4) - (0 - 2\ln 3) \quad \text{M1}$$

$$= 2\ln 3 - 2\ln 2 = 2\ln \frac{3}{2} \quad \text{A1}$$

Answer: shown 4 marks

24 (6 marks)

$$\frac{3}{(x+1)(x+4)} \equiv \frac{1}{x+1} - \frac{1}{x+4} \quad \text{M1 A1}$$

$$[\ln(x+1) - \ln(x+4)]_0^k = \ln(k+1) - \ln(k+4) + \ln 4 \quad \text{M1 A1}$$

$$\ln \frac{4(k+1)}{k+4} = \ln 2, \text{ so } 4(k+1) = 2(k+4) \quad \text{M1}$$

$$2k = 4, k = 2 \quad \text{A1}$$

Answer: k = 2 6 marks

25 (9 marks)

$$\text{Divide: } \frac{2x^3 + 12x^2 + 21x + 13}{x^3 + 4x^2 + 5x + 2} = 2 + \frac{4x^2 + 11x + 9}{x^3 + 4x^2 + 5x + 2} \quad \text{M1}$$

$$\frac{4x^2 + 11x + 9}{(x+1)^2(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}, \text{ so } 4x^2 + 11x + 9 \equiv A(x+1)(x+2) + B(x+2) + C(x+1)^2 \quad \text{M1}$$

$$C(x+1)^2$$

$$x = -1: 2 = B, \text{ so } B = 2 \quad \text{A1}$$

$$x = -2: 3 = C, \text{ so } C = 3 \quad \text{A1}$$

$$\text{Coefficient of } x^2: 4 = A + 3, \text{ so } A = 1 \quad \text{A1}$$

$$[2x + \ln|x+1| - \frac{2}{(x+1)} + 3\ln|x+2|]_0^1 \quad \text{M1 A1}$$

$$= 2 + \ln 2 + 1 + 3\ln \frac{3}{2} \quad \text{M1}$$

$$= -2\ln 2 + 3\ln 3 + 3 \quad \text{A1}$$

Answer: $-2\ln 2 + 3\ln 3 + 3$ 9 marks

26 (8 marks)

$$(a) \frac{1}{x+1} + \frac{1}{x+2} = \frac{2x+3}{(x+1)(x+2)} \quad \text{M1 A1}$$

The numerator is $2x+3$, not 1 E1

Answer: numerator would be $2x+3$ 3 marks

$$(b) 1 \equiv A(x+2) + B(x+1); x = -1 \text{ gives } A = 1; x = -2 \text{ gives } B = -1 \quad \text{M1 A1}$$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}, \text{ so the integral is } \ln|x+1| - \ln|x+2| + c \quad \text{M1 A1}$$

$$= \ln \left| \frac{x+1}{x+2} \right| + c \quad \text{A1}$$

Answer: $\ln \left| \frac{x+1}{x+2} \right| + c$ 5 marks

27 (12 marks)

$$(a) 1 \equiv A(x+2) + Bx; x = 0 \text{ gives } A = \frac{1}{2}; x = -2 \text{ gives } B = -\frac{1}{2} \quad \text{M1 A1}$$

$$\frac{1}{x(x+2)} \equiv \frac{1}{2} \left(\frac{1}{x} - \frac{1}{x+2} \right) \quad \text{A1}$$

Answer: $\frac{1}{2} \left(\frac{1}{x} - \frac{1}{x+2} \right)$ 3 marks

$$(b) \frac{1}{r(r+2)} = \frac{1}{2} \left(\frac{1}{r} - \frac{1}{r+2} \right); \text{ the terms telescope}$$

M1 A1

$$S_n = \frac{1}{2} \left[\left(1 + \frac{1}{2} \right) - \left(\frac{1}{n+1} + \frac{1}{n+2} \right) \right]$$

M1 A1

$$= \frac{3}{4} - \frac{1}{2} \left(\frac{1}{n+1} + \frac{1}{n+2} \right)$$

A1

Answer: shown

5 marks

$$(c) \frac{1}{2} [\ln x - \ln(x+2)]_1^n = \frac{1}{2} \left(\ln \frac{n}{n+2} + \ln 3 \right)$$

M1 A1

$$= \frac{1}{2} \ln \frac{3n}{n+2}$$

A1

$$\frac{3n}{n+2} < 3 \text{ for all } n, \text{ so the value is less than } \frac{1}{2} \ln 3$$

E1

$$\text{Answer: } \frac{1}{2} \ln \frac{3n}{n+2} < \frac{1}{2} \ln 3$$

4 marks

28 (9 marks)

$$f(0) = \frac{q}{2} = \frac{7}{2}, \text{ so } q = 7$$

M1 A1

$$\frac{px+7}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}; x = -1 \text{ gives } A = 7 - p; x = -2 \text{ gives } B = 2p - 7$$

M1 A1

$$\int_0^2 f(x) dx = [A \ln(x+1) + B \ln(x+2)]_0^2 = A \ln 3 + B \ln 2$$

M1 A1

$$\ln 72 = 2 \ln 3 + 3 \ln 2$$

B1

$\ln 2$ and $\ln 3$ are independent over the rationals, so $A = 2$ and $B = 3$

M1

$$7 - p = 2 \text{ gives } p = 5 \text{ (check: } 2p - 7 = 3)$$

A1

Answer: $p = 5, q = 7$

9 marks

29 (9 marks)

$$(a) (x+1)(x+2) = 6, \text{ so } x^2 + 3x - 4 = 0$$

M1

$$(x+4)(x-1) = 0, \text{ and } x > 0 \text{ on the diagram, so } x = 1$$

A1

Answer: shown

2 marks

$$(b) 6 \equiv A(x+2) + B(x+1); x = -1 \text{ gives } A = 6; x = -2 \text{ gives } B = -6$$

M1 A1

$$\frac{6}{(x+1)(x+2)} \equiv \frac{6}{x+1} - \frac{6}{x+2}$$

A1

$$\text{Answer: } \frac{6}{x+1} - \frac{6}{x+2}$$

3 marks

$$(c) \text{Area} = \int_0^1 \left(\frac{6}{x+1} - \frac{6}{x+2} - 1 \right) dx = [6 \ln(x+1) - 6 \ln(x+2) - x]_0^1$$

M1 A1

$$= (6 \ln 2 - 6 \ln 3 - 1) - (0 - 6 \ln 2)$$

M1

$$= 12 \ln 2 - 6 \ln 3 - 1 = 6 \ln \frac{4}{3} - 1$$

A1

Answer: shown

4 marks

30 (7 marks)

$$\text{Let } u = \sin x: du = \cos x dx; \text{ limits } 0 \rightarrow 0 \text{ and } \frac{\pi}{2} \rightarrow 1$$

M1 A1

$$\text{Integral} = \int_0^1 \frac{1}{(1+u)(2+u)} du$$

A1

$$\frac{1}{(1+u)(2+u)} \equiv \frac{1}{1+u} - \frac{1}{2+u} \text{ (A = 1 at } u = -1, B = -1 \text{ at } u = -2)$$

M1 A1

$$[\ln(1+u) - \ln(2+u)]_0^1 = (\ln 2 - \ln 3) - (0 - \ln 2)$$

M1

$$= 2 \ln 2 - \ln 3 = \ln \frac{4}{3}$$

A1

Answer: shown

7 marks

Parametric Integration

1 (3 marks)

$$\frac{dx}{dt} = 2$$

$$\text{Area} = \int_0^3 y \frac{dx}{dt} dt = \int_0^3 2t^2 dt$$

M1

$$= \left[\frac{2t^3}{3} \right]_0^3 = 18$$

M1 A1

Answer: 18

3 marks

2 (3 marks)

$$\frac{dx}{dt} = 2t$$

$$\text{Area} = \int_1^3 y \frac{dx}{dt} dt = \int_1^3 8t^2 dt$$

M1

$$= \left[\frac{8t^3}{3} \right]_1^3 = \frac{208}{3}$$

M1 A1

Answer: $\frac{208}{3}$

3 marks

3 (3 marks)

$$x = 2 \text{ gives } t = 1 \text{ and } x = 4 \text{ gives } t = 3; \frac{dx}{dt} = 1$$

B1

$$\text{Area} = \int_1^3 t^2 dt = \left[\frac{t^3}{3} \right]_1^3$$

M1

$$= 9 - \frac{1}{3} = \frac{26}{3}$$

A1

Answer: $\frac{26}{3}$

3 marks

4 (4 marks)

$$x = 1 \text{ gives } t = 0 \text{ and } x = e \text{ gives } t = 1$$

B1

$$\frac{dx}{dt} = e^t$$

$$\text{Area} = \int_0^1 y \frac{dx}{dt} dt = \int_0^1 1 dt$$

M1

$$= [t]_0^1 = 1$$

M1 A1

Answer: 1

4 marks

5 (3 marks)

$$\frac{dx}{dt} = 3t^2$$

$$\text{Area} = \int_1^2 y \frac{dx}{dt} dt = \int_1^2 3t^3 dt$$

M1

$$= \left[\frac{3t^4}{4} \right]_1^2 = \frac{45}{4}$$

M1 A1

Answer: $\frac{45}{4}$

3 marks

6 (3 marks)

$$x = 0 \text{ gives } t = 0, x = 6 \text{ gives } t = 2; \frac{dx}{dt} = 3$$

B1

$$\text{Area} = \int_0^2 (2 + t^2) \times 3 dt = \int_0^2 (6 + 3t^2) dt$$

M1

$$= [6t + t^3]_0^2 = 12 + 8 = 20$$

A1

Answer: 20

3 marks

7 (5 marks)

$$x = 0 \text{ when } t = \frac{\pi}{2}; x = 1 \text{ when } t = 0$$

B1

$$\frac{dx}{dt} = -\sin t$$

B1

$$\text{Area} = \int_{\frac{\pi}{2}}^0 \sin t (-\sin t) dt = \int_0^{\frac{\pi}{2}} \sin^2 t dt \text{ (reversing the limits)}$$

M1

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t), \text{ so the integral is } \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{\frac{\pi}{2}}$$

M1

$$= \frac{\pi}{4}$$

A1

$$\text{Answer: } \frac{\pi}{4}$$

5 marks

8 (6 marks)

$$x = 2 \text{ gives } t = 1; x = 9 \text{ gives } t = 2$$

B1

$$\frac{dx}{dt} = 3t^2$$

B1

$$\text{Area} = \int_1^2 y \frac{dx}{dt} dt = \int_1^2 6t^3 dt$$

M1 A1

$$= \left[\frac{3t^4}{2} \right]_1^2 = \frac{45}{2}$$

M1 A1

$$\text{Answer: } \frac{45}{2}$$

6 marks

9 (5 marks)

$$\frac{dx}{dt} = 2t$$

B1

$$\text{Area} = \int_0^3 y \frac{dx}{dt} dt = \int_0^3 -2t^3 + 6t^2 dt$$

M1 A1

$$= \left[-\frac{t^4}{2} + 2t^3 \right]_0^3 = \frac{27}{2}$$

M1 A1

$$\text{Answer: } \frac{27}{2}$$

5 marks

10 (5 marks)

$$\frac{dx}{dt} = 2e^{2t}$$

B1

$$\text{Area} = \int_0^{\ln 2} y \frac{dx}{dt} dt = \int_0^{\ln 2} 2e^{3t} dt$$

M1 A1

$$= \left[\frac{2e^{3t}}{3} \right]_0^{\ln 2} = \frac{14}{3}$$

M1 A1

$$\text{Answer: } \frac{14}{3}$$

5 marks

11 (6 marks)

$$\frac{dx}{dt} = 2\cos t$$

B1

$$\text{Area} = \int_0^{\frac{\pi}{2}} 4\cos t \times 2\cos t dt = \int_0^{\frac{\pi}{2}} 8\cos^2 t dt$$

M1 A1

$$= \int_0^{\frac{\pi}{2}} 4(1 + \cos 2t) dt = [4t + 2\sin 2t]_0^{\frac{\pi}{2}}$$

M1

$$= 2\pi$$

A1

$$\text{Answer: } 2\pi$$

6 marks

12 (5 marks)

$$\frac{dx}{dt} = 2$$

B1

$$\text{Area} = \int_1^e y \frac{dx}{dt} dt = \int_1^e \frac{6}{t} dt$$

M1 A1

$$= [6 \ln t]_1^e = 6$$

M1 A1

Answer: 6

5 marks

13 (6 marks)

$$\frac{dx}{dt} = 2t$$

B1

$$\text{Area} = \int_1^e 2t \ln t dt$$

M1

$$\text{Parts with } u = \ln t, \frac{dv}{dt} = 2t: [t^2 \ln t]_1^e - \int_1^e t dt$$

M1 A1

$$= e^2 - \left[\frac{t^2}{2}\right]_1^e = e^2 - \frac{e^2}{2} + \frac{1}{2}$$

M1

$$= \frac{1}{2}(e^2 + 1)$$

A1

Answer: $\frac{1}{2}(e^2 + 1)$

6 marks

14 (5 marks)

$$\frac{dx}{dt} = \sec^2 t$$

B1

$$x = 0 \text{ gives } t = 0; x = 1 \text{ gives } t = \frac{\pi}{4}$$

B1

$$y \frac{dx}{dt} = \cos^2 t \times \frac{1}{\cos^2 t} = 1$$

M1

$$\text{Area} = \int_0^{\frac{\pi}{4}} 1 dt = \frac{\pi}{4}$$

M1 A1

Answer: shown

5 marks

15 (6 marks)

$$\frac{dx}{dt} = 4 \cos t$$

B1

$$\text{Area} = \int_0^{\frac{\pi}{2}} \sin 2t \times 4 \cos t dt = \int_0^{\frac{\pi}{2}} 8 \sin t \cos^2 t dt$$

M1 A1

$$= \left[-\frac{8 \cos^3 t}{3}\right]_0^{\frac{\pi}{2}}$$

M1 A1

$$= 0 + \frac{8}{3} = \frac{8}{3}$$

A1

Answer: $\frac{8}{3}$

6 marks

16 (5 marks)

$$y = 1 \text{ gives } t = 0; y = 5 \text{ gives } t = 2; \frac{dy}{dt} = 2$$

B1

$$\text{Area} = \int_1^5 x dy = \int_0^2 t^2 \times 2 dt$$

M1 A1

$$= \left[\frac{2t^3}{3}\right]_0^2$$

M1

$$= \frac{16}{3}$$

A1

Answer: $\frac{16}{3}$

5 marks

17 (5 marks)**(a)** An area cannot be negative

E1

 $x = \frac{1}{2}$ corresponds to $t = 2$ and $x = 1$ to $t = 1$: the limits should run from $t = 2$ to $t = 1$, not 1 to 2

E1

Answer: limits reversed

2 marks

(b) Area = $\int_2^1 (t^2 + 1)(-\frac{1}{t^2}) dt = \int_1^2 (1 + \frac{1}{t^2}) dt$

M1

$$= [t - \frac{1}{t}]_1^2 = \frac{3}{2}$$

M1 A1

Answer: $\frac{3}{2}$

3 marks

18 (5 marks)Upper half: $0 \leq t \leq \pi$, where x runs from 2 ($t = 0$) to -2 ($t = \pi$)

B1

$$\frac{dx}{dt} = -2\sin t$$

B1

Area = $\int_{\pi}^0 3\sin t(-2\sin t) dt = 6 \int_0^{\pi} \sin^2 t dt$

M1

$$= 3 \int_0^{\pi} (1 - \cos 2t) dt = 3[t - \frac{\sin 2t}{2}]_0^{\pi}$$

M1

$$= 3\pi$$

A1

Answer: 3π

5 marks

19 (9 marks)**(a)** $x = 4$ gives $t = 1$ and $x = 10$ gives $t = 3$

B1

$$\frac{dx}{dt} = 3$$

B1

Area = $\int_1^3 y \frac{dx}{dt} dt = \int_1^3 \frac{18}{t^2} \times 3 dt = \int_1^3 \frac{54}{t^2} dt$

M1

Answer: shown

3 marks

(b) $\int_1^3 54t^{-2} dt = [-\frac{54}{t}]_1^3$

M1

$$= -18 + 54 = 36$$

A1

Answer: 36

2 marks

(c) $t = \frac{x-1}{3}$, so $t^2 = \frac{(x-1)^2}{9}$ and $y = \frac{18 \times 9}{(x-1)^2} = \frac{162}{(x-1)^2}$

M1 A1

$$\int_4^{10} 162(x-1)^{-2} dx = [-\frac{162}{x-1}]_4^{10}$$

M1

$$= -18 + 54 = 36, \text{ as before}$$

A1

Answer: shown; 36

4 marks

20 (8 marks)**(a)** $y = 0$ gives $t = 0$, so P is (9, 0)

B1

 $x = 0$ gives $t = 3$, so $y = 12$ and Q is (0, 12)

B1

Answer: P(9, 0), Q(0, 12)

2 marks

(b) $\frac{dx}{dt} = -2t$

B1

Area = $\int_0^9 y dx = \int_3^0 t(t+1)(-2t) dt = \int_0^3 (2t^3 + 2t^2) dt$

M1 A1

$$= [\frac{t^4}{2} + \frac{2t^3}{3}]_0^3$$

M1 A1

$$= \frac{81}{2} + 18 = \frac{117}{2}$$

A1

Answer: $\frac{117}{2}$

6 marks

21 (7 marks)

$$\frac{dx}{dt} = 2t$$

B1

$$\text{Area} = \int_0^1 2t^2 e^{-t} dt$$

M1 A1

$$\text{Parts: } \int_0^1 t^2 e^{-t} dt = [-t^2 e^{-t}]_0^1 + 2 \int_0^1 t e^{-t} dt$$

M1 A1

$$\int_0^1 t e^{-t} dt = [-t e^{-t}]_0^1 + \int_0^1 e^{-t} dt = 1 - 2e^{-1}$$

M1

$$\int_0^1 t^2 e^{-t} dt = -e^{-1} + 2(1 - 2e^{-1}) = 2 - 5e^{-1}, \text{ so area} = 4 - 10e^{-1}$$

A1

Answer: $4 - 10e^{-1}$

7 marks

22 (7 marks)

$$x = 3 \text{ when } t = 0 \text{ and } x = 0 \text{ when } t = \frac{\pi}{2}$$

B1

$$\frac{dx}{dt} = -3\sin t$$

B1

$$\text{Area} = \int_{\frac{\pi}{2}}^0 2\sin 2t(-3\sin t) dt \text{ over } t \text{ from } \frac{\pi}{2} \text{ to } 0 = \int_0^{\frac{\pi}{2}} 12\sin^2 t \cos t dt$$

M1 A1

$$\sin 2t = 2\sin t \cos t; \text{ integral} = [4\sin^3 t]_0^{\frac{\pi}{2}}$$

M1 A1

$$= 4$$

A1

Answer: 4

7 marks

23 (7 marks)

$$y = 0: t = 0 \text{ or } t^2 = 6, \text{ so } A(1, 0) \text{ and } B(7, 0)$$

M1 A1

$$\frac{dx}{dt} = 2t; \text{ Area} = \int_0^{\sqrt{6}} t(6 - t^2) \times 2t dt = \int_0^{\sqrt{6}} (12t^2 - 2t^4) dt$$

M1 A1

$$= [4t^3 - \frac{2t^5}{5}]_0^{\sqrt{6}}$$

M1

$$= 24\sqrt{6} - \frac{72\sqrt{6}}{5} = \frac{48\sqrt{6}}{5}$$

M1 A1

Answer: A(1, 0), B(7, 0); area $\frac{48\sqrt{6}}{5}$

7 marks

24 (7 marks)

$$P \text{ is } (4, 4), \text{ so } l \text{ has equation } y = x$$

M1 A1

$$\text{Area under } C \text{ from } O \text{ to } P = \int_0^2 2t \times 2t dt = [\frac{4t^3}{3}]_0^2 = \frac{32}{3}$$

M1 A1

$$\text{Area under } l = \text{triangle with base 4 and height 4} = 8$$

M1

$$C \text{ lies above } l \text{ between } O \text{ and } P \text{ (e.g. } t = 1 \text{ gives } (1, 2) \text{ with } 2 > 1), \text{ so area} = \frac{32}{3} - 8 = \frac{8}{3}$$

M1 A1

Answer: $\frac{8}{3}$

7 marks

25 (6 marks)

$$x = 1 \text{ gives } t = 0; x = 3 \text{ gives } t = \ln 3; \frac{dx}{dt} = e^t$$

B1 B1

$$\text{Area} = \int_0^{\ln 3} \frac{e^t}{1 + e^t} dt$$

M1

$$= [\ln(1 + e^t)]_0^{\ln 3}$$

M1 A1

$$= \ln 4 - \ln 2 = \ln 2$$

A1

Answer: shown

6 marks

26 (10 marks)

(a) $2\sin \theta = 1$, so $\theta = \frac{\pi}{6}$

B1

$x = 6\cos \frac{\pi}{6} = 3\sqrt{3}$

B1

Answer: $\theta = \frac{\pi}{6}$; $x = 3\sqrt{3}$

2 marks

(b) $\frac{dy}{d\theta} = 2\cos \theta$

B1

Area = $\int_1^2 x \, dy$; $y = 1$ gives $\theta = \frac{\pi}{6}$ and $y = 2$ gives $\theta = \frac{\pi}{2}$

M1

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 6\cos \theta \times 2\cos \theta \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12\cos^2 \theta \, d\theta$

A1

Answer: shown

3 marks

(c) $12\cos^2 \theta = 6(1 + \cos 2\theta)$

M1

$[6\theta + 3\sin 2\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

M1 A1

$= (3\pi + 0) - (\pi + \frac{3\sqrt{3}}{2})$

M1

$= 2\pi - \frac{3\sqrt{3}}{2}$

A1

Answer: $2\pi - \frac{3\sqrt{3}}{2}$

5 marks

27 (9 marks)

(a) $y = 0$ for $t = \pm\sqrt{3}$ (and $t = 0$); $t = \sqrt{3}$ gives $x = 3$

M1

A is (3, 0)

A1

Answer: (3, 0)

2 marks

(b) Upper half: $y > 0$ for $-\sqrt{3} < t < 0$, and x decreases from 3 to 0 as t goes from $-\sqrt{3}$ to 0

B1

$\frac{dx}{dt} = 2t$; the area of the upper half is the magnitude of $\int_{-\sqrt{3}}^0 y \frac{dx}{dt} \, dt = \int_{-\sqrt{3}}^0 2t^2(t^2 - 3) \, dt$

M1 A1

$= [\frac{2t^5}{5} - 2t^3]_{-\sqrt{3}}^0 = 0 - (-\frac{18\sqrt{3}}{5} + 6\sqrt{3}) = -\frac{12\sqrt{3}}{5}$

M1 A1

The magnitude $\frac{12\sqrt{3}}{5}$ is the upper half; by symmetry in the x -axis ($y(-t) = -y(t)$) the loop has

M1 A1

area $\frac{24\sqrt{3}}{5}$

Answer: shown

7 marks

28 (11 marks)

(a) $\frac{dx}{dt} = -a \sin t$; $t = 0$ gives $x = a$ and $t = \frac{\pi}{2}$ gives $x = 0$

B1 B1

Area = $\int_{\frac{\pi}{2}}^0 b \sin t (-a \sin t) \, dt = ab \int_0^{\frac{\pi}{2}} \sin^2 t \, dt$ (limits reversed)

M1 A1

$= \frac{ab}{2} [t - \frac{\sin 2t}{2}]_0^{\frac{\pi}{2}}$

M1

$= \frac{\pi a b}{4}$

A1

Answer: shown

6 marks

(b) E is symmetric in both axes, so the total is $4 \times \frac{\pi a b}{4} = \pi a b$

M1 A1

$a = 5$, $b = 3$: 15π

A1

Answer: $\pi a b$; 15π

3 marks

(c) $a = b = r$ gives $x = r \cos t$, $y = r \sin t$, a circle of radius r

M1

Area = πr^2

A1

Answer: πr^2

2 marks

29 (12 marks)

(a) $\frac{dx}{dt} = -3\cos^2 t \sin t$; $t = 0$ gives $x = 1$ and $t = \frac{\pi}{2}$ gives $x = 0$

M1 A1

Area = $\int_{\frac{\pi}{2}}^0 \sin^3 t (-3\cos^2 t \sin t) dt$ (limits reversed) = $3 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt$

M1 A1

Answer: shown

4 marks

(b) $\sin^4 t \cos^2 t = (\sin t \cos t)^2 \sin^2 t = \frac{1}{4} \sin^2 2t \sin^2 t$

M1 A1

$\sin^2 t = \frac{1}{2}(1 - \cos 2t)$, so the product is $\frac{1}{8} \sin^2 2t (1 - \cos 2t)$

A1

Answer: shown

3 marks

(c) $\int_0^{\frac{\pi}{2}} \sin^2 2t dt = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4t) dt = \frac{\pi}{4}$

M1 A1

$\int_0^{\frac{\pi}{2}} \sin^2 2t \cos 2t dt = \left[\frac{\sin^3 2t}{6} \right]_0^{\frac{\pi}{2}} = 0$

M1 A1

Area = $3 \times \frac{1}{8} \left(\frac{\pi}{4} - 0 \right) = \frac{3\pi}{32}$

A1

Answer: $\frac{3\pi}{32}$

5 marks

30 (11 marks)

(a) $\frac{dx}{dt} = a(1 - \cos t) \geq 0$, so x increases with t

M1 A1

$t = 0$ and $t = 2\pi$ give the ends of the arch ($x = 0$ and $x = 2\pi a$), so the limits are 0 and 2π

E1

Answer: $\frac{dx}{dt} = a(1 - \cos t)$

3 marks

(b) Area = $a^2 \int_0^{2\pi} (1 - \cos t)^2 dt = a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) dt$

M1 A1

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$, so the integrand is $\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t$

M1 A1

$a^2 \left[\frac{3t}{2} - 2\sin t + \frac{\sin 2t}{4} \right]_0^{2\pi} = a^2 \times 3\pi$

M1 A1

Answer: shown

6 marks

(c) Circle area = πa^2

B1

Ratio = $3\pi a^2 : \pi a^2 = 3 : 1$

A1

Answer: 3 : 1

2 marks

Differential Equations**1 (2 marks)**

$y = 3e^{2x}$, so $\frac{dy}{dx} = 6e^{2x}$

M1

$= 2 \times 3e^{2x} = 2y$

A1

Answer: shown

2 marks

2 (2 marks)

Integrate both sides with respect to x

M1

$y = 2x^2 + c$

A1

Answer: $y = 2x^2 + c$

2 marks

3 (2 marks)

Separate: $\int \frac{1}{y} dy = \int 3 dx$, so $\ln|y| = 3x + c$

M1

$y = Ae^{3x}$; $y = 2$ at $x = 0$ gives $A = 2$, so $y = 2e^{3x}$

A1

Answer: $y = 2e^{3x}$

2 marks

4 (2 marks)

$$\text{Separate: } \int y \, dy = \int x \, dx$$

M1

$$\frac{y^2}{2} = \frac{x^2}{2} + c, \text{ so } y^2 = x^2 + k$$

A1

$$\text{Answer: } y^2 = x^2 + k$$

2 marks

5 (2 marks)

$$\text{Rate of increase is } \frac{dP}{dt}$$

M1

$$\frac{dP}{dt} = kP$$

A1

$$\text{Answer: } \frac{dP}{dt} = kP$$

2 marks

6 (2 marks)

$$\int \frac{1}{y} \, dy = \int \frac{1}{x} \, dx$$

M1

$$\ln y = \ln x + c, \text{ so } y = Ax$$

A1

$$\text{Answer: } y = Ax$$

2 marks

7 (4 marks)

$$\text{Separate: } \int y^2 \, dy = \int x \, dx$$

M1

$$\frac{y^3}{3} = \frac{x^2}{2} + c$$

M1 A1

$$y^3 = \frac{3x^2}{2} + k$$

A1

$$\text{Answer: } y^3 = \frac{3x^2}{2} + k$$

4 marks

8 (4 marks)

$$\int \frac{1}{y^2} \, dy = \int 3 \, dx$$

M1

$$-\frac{1}{y} = 3x + c$$

A1

$$x = 0, y = 1: c = -1, \text{ so } \frac{1}{y} = 1 - 3x$$

M1

$$y = \frac{1}{1 - 3x}$$

A1

$$\text{Answer: } y = \frac{1}{1 - 3x}$$

4 marks

9 (4 marks)

$$\int \frac{1}{y} \, dy = \int \cos x \, dx$$

M1

$$\ln y = \sin x + c$$

A1

$$x = 0, y = 2: c = \ln 2$$

M1

$$y = 2e^{\sin x}$$

A1

$$\text{Answer: } y = 2e^{\sin x}$$

4 marks

10 (4 marks)

$$\text{Separate: } \int e^y \, dy = \int x \, dx$$

M1

$$e^y = \frac{x^2}{2} + c$$

A1

$$x = 0, y = 0: c = 1$$

M1

$$y = \ln\left(1 + \frac{x^2}{2}\right)$$

A1

$$\text{Answer: } y = \ln\left(1 + \frac{x^2}{2}\right)$$

4 marks

11 (3 marks)

$$\frac{dy}{dx} = 2e^x e^{-y}, \text{ so } \int e^y dy = \int 2e^x dx$$

M1

$$e^y = 2e^x + C$$

A1

$$y = \ln(2e^x + C)$$

A1

$$\text{Answer: } y = \ln(2e^x + C)$$

3 marks

12 (4 marks)

$$P = Ae^{0.04t}, \text{ and } P = 500\,000 \text{ at } t = 0 \text{ gives } A = 500\,000$$

M1 A1

$$\text{Double: } e^{0.04t} = 2, \text{ so } t = \frac{\ln 2}{0.04}$$

M1

$$t = 17.3 \text{ years}$$

A1

$$\text{Answer: } P = 500\,000e^{0.04t}; 17.3 \text{ years}$$

4 marks

13 (3 marks)

$$\int \frac{1}{y-1} dy = \int \frac{1}{x} dx$$

M1

$$\ln(y-1) = \ln x + c$$

A1

$$y-1 = Ax, \text{ so } y = 1 + Ax$$

A1

$$\text{Answer: } y = 1 + Ax$$

3 marks

14 (5 marks)

$$\int y dy = \int \frac{x}{1+x^2} dx$$

M1

$$\frac{y^2}{2} = \frac{1}{2} \ln(1+x^2) + c$$

M1 A1

$$x = 0, y = 2: c = 2, \text{ so } y^2 = \ln(1+x^2) + 4$$

M1

$$y = \sqrt{\ln(1+x^2) + 4}$$

A1

$$\text{Answer: } y = \sqrt{\ln(1+x^2) + 4}$$

5 marks

15 (7 marks)

$$\int \frac{1}{\theta-20} d\theta = \int -k dt, \text{ so } \theta - 20 = Ae^{-kt}$$

M1 A1

$$t = 0, \theta = 90: A = 70$$

B1

$$t = 10: 40 = 70e^{-10k}, \text{ so } k = \frac{1}{10} \ln \frac{7}{4} = 0.0560$$

M1 A1

$$\theta = 30: 10 = 70e^{-kt}, \text{ so } t = \frac{\ln 7}{k} = 34.8 \text{ minutes}$$

M1 A1

$$\text{Answer: } k = 0.0560; 34.8 \text{ minutes}$$

7 marks

16 (6 marks)

$$\int V^{-\frac{1}{2}} dV = \int -k dt$$

M1

$$2\sqrt{V} = -kt + c; t = 0: c = 20$$

M1 A1

$$t = 5, V = 64: 16 = -5k + 20, \text{ so } k = \frac{4}{5}$$

M1 A1

$$V = 0: 0 = -\frac{4}{5}t + 20, \text{ so } t = 25 \text{ minutes}$$

A1

$$\text{Answer: } 25 \text{ minutes}$$

6 marks

17 (5 marks)

$$\int \frac{1}{y} dy = \int \tan x dx$$

M1

$$\ln y = -\ln|\cos x| + c$$

M1 A1

$$y = A \sec x$$

A1

$$x = 0, y = 3: A = 3, \text{ so } y = 3 \sec x$$

A1

$$\text{Answer: } y = 3 \sec x$$

5 marks

18 (7 marks)

$$(a) m = m_0 e^{-kt} \quad \text{M1}$$

$$t = 5: \frac{1}{2}m_0 = m_0 e^{-5k}, \text{ so } 5k = \ln 2 \quad \text{M1 A1}$$

Answer: shown 3 marks

$$(b) m/m_0 = e^{-12\ln 2/5} = 0.1895, \text{ so } 19.0\% \text{ (3 s.f.)} \quad \text{M1 A1}$$

$$e^{-kt} = 0.1: t = \frac{5\ln 10}{\ln 2} = 16.6 \text{ years} \quad \text{M1 A1}$$

Answer: 19.0%; 16.6 years 4 marks

19 (5 marks)

$$\int y \, dy = \int -x \, dx \quad \text{M1}$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \text{ so } x^2 + y^2 = k \quad \text{M1 A1}$$

$$(3, 4): k = 9 + 16 = 25, \text{ so } C \text{ is } x^2 + y^2 = 25 \quad \text{A1}$$

$$\text{Gradient} = -\frac{3}{4} \quad \text{A1}$$

Answer: $x^2 + y^2 = 25$; gradient $-\frac{3}{4}$ 5 marks

20 (7 marks)

$$\int \sin y \cos y \, dy = \int x e^x \, dx \quad \text{M1}$$

$$\text{LHS} = \frac{1}{2}\sin^2 y \quad \text{A1}$$

$$\text{RHS by parts (} u = x, \frac{dv}{dx} = e^x \text{): } x e^x - \int e^x \, dx = e^x(x - 1) \quad \text{M1 A1}$$

$$\frac{1}{2}\sin^2 y = e^x(x - 1) + c; x = 0, y = \frac{\pi}{2}: \frac{1}{2} = -1 + c, \text{ so } c = \frac{3}{2} \quad \text{M1 A1}$$

$$\sin^2 y = 2e^x(x - 1) + 3 \quad \text{A1}$$

Answer: shown 7 marks

21 (7 marks)

$$(a) \int \frac{1}{0.1P - 20} \, dP = \int 1 \, dt \quad \text{M1}$$

$$10\ln|0.1P - 20| = t + c \quad \text{M1 A1}$$

$$t = 0, P = 400: 0.1P - 20 = 20, \text{ so } c = 10\ln 20 \quad \text{M1}$$

$$0.1P - 20 = 20e^{0.1t}, \text{ so } P = 200 + 200e^{0.1t} \quad \text{A1}$$

Answer: shown 5 marks

$$(b) 200 + 200e^{0.1t} = 1000, \text{ so } e^{0.1t} = 4 \quad \text{M1}$$

$$t = 10\ln 4 = 13.9 \text{ years} \quad \text{A1}$$

Answer: 13.9 years 2 marks

22 (8 marks)

$$(a) \int \frac{1}{8 - 0.5C} \, dC = \int 1 \, dt \quad \text{M1}$$

$$-2\ln|8 - 0.5C| = t + c \quad \text{M1 A1}$$

$$t = 0, C = 0: c = -2\ln 8, \text{ so } 8 - 0.5C = 8e^{-0.5t} \quad \text{M1}$$

$$C = 16(1 - e^{-0.5t}) \quad \text{A1}$$

Answer: shown 5 marks

$$(b) \text{ As } t \rightarrow \infty, e^{-0.5t} \rightarrow 0 \text{ so } C \rightarrow 16 \quad \text{B1}$$

$$16(1 - e^{-0.5t}) = 12, \text{ so } e^{-0.5t} = \frac{1}{4} \quad \text{M1}$$

$$t = 2\ln 4 = 2.77 \text{ hours} \quad \text{A1}$$

Answer: 16 mg per litre; 2.77 hours 3 marks

23 (7 marks)

The line to the origin has gradient $\frac{y}{x}$, so $\frac{dy}{dx} = \frac{2y}{x}$

M1 A1

$$\int \frac{1}{y} dy = \int \frac{2}{x} dx$$

M1

$\ln y = 2 \ln x + c$, so $y = Ax^2$

M1 A1

(2, 12): $A = 3$, so $y = 3x^2$

M1 A1

Answer: $y = 3x^2$

7 marks

24 (6 marks)

(a) $\frac{dV}{dr} = 4\pi r^2$

M1

$$\frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr} = \frac{-4\pi k r^2}{4\pi r^2} = -k$$

M1 A1

Answer: shown

3 marks

(b) $r = -kt + 2$

M1

$r = 1$ at $t = 5$: $k = 0.2$

A1

$r = 0$: $t = \frac{2}{0.2} = 10$ seconds

A1

Answer: 10 seconds

3 marks

25 (8 marks)

(a) $\int P^{-2} dP = \int k dt$

M1

$$-\frac{1}{P} = kt + c$$

A1

$t = 0, P = 1000$: $c = -\frac{1}{1000}$

M1

$t = 4, P = 2000$: $-\frac{1}{2000} = 4k - \frac{1}{1000}$, so $k = \frac{1}{8000}$

M1 A1

$$\frac{1}{P} = \frac{1}{1000} - \frac{t}{8000} = \frac{8-t}{8000}, \text{ so } P = \frac{8000}{8-t}$$

A1

Answer: shown

6 marks

(b) The denominator is zero when $t = 8$, so P becomes infinite after 8 weeks

B1

A real population cannot be infinite: the model ignores limits on food and space

E1

Answer: $t = 8$; resources are limited

2 marks

26 (8 marks)

(a) Salt in: $0.5 \times 2 = 1$ kg per minute

M1

Salt out: concentration $\frac{x}{200} \times 2 = \frac{x}{100}$ kg per minute

M1

$$\frac{dx}{dt} = 1 - \frac{x}{100}$$

A1

Answer: shown

3 marks

(b) $\int \frac{1}{1-x/100} dx = \int 1 dt$, so $-100 \ln|1 - \frac{x}{100}| = t + c$

M1 A1

$x = 0$ at $t = 0$: $c = 0$, so $x = 100(1 - e^{-t/100})$

M1 A1

$50 = 100(1 - e^{-t/100})$: $t = 100 \ln 2 = 69.3$ minutes

A1

Answer: $x = 100(1 - e^{-t/100})$; 69.3 minutes

5 marks

27 (10 marks)

(a) $1 \equiv A(50 - x) + Bx$: $x = 0$ gives $A = \frac{1}{50}$; $x = 50$ gives $B = \frac{1}{50}$

M1 A1

Answer: $\frac{1}{50}(\frac{1}{x} + \frac{1}{50-x})$

2 marks

(b) $\int \frac{1}{x(50-x)} dx = \int \frac{1}{50} dt$

M1

$$\frac{1}{50} [\ln x - \ln(50-x)] = \frac{t}{50} + c, \text{ so } \ln \frac{x}{50-x} = t + k$$

M1 A1

$$t = 0, x = 5: k = \ln \frac{1}{9}, \text{ so } \frac{x}{50 - x} = \frac{e^t}{9}$$

M1

$$x = \frac{50e^t}{9 + e^t} = \frac{50}{1 + 9e^{-t}}$$

A1

Answer: shown

5 marks

(c) $x = 25: 1 + 9e^{-t} = 2$, so $e^t = 9$ and $t = \ln 9 = 2.20$ days

M1 A1

As $t \rightarrow \infty$, $x \rightarrow 50$

B1

Answer: 2.20 days; 50

3 marks

28 (10 marks)

$$(a) \frac{dy}{dx} = 2kx \text{ and } k = \frac{y}{x^2}$$

M1

$$\frac{dy}{dx} = \frac{2y}{x}$$

A1

Answer: shown

2 marks

$$(b) \int 2y \, dy = \int -x \, dx$$

M1

$$y^2 = -\frac{x^2}{2} + c$$

M1 A1

$$x^2 + 2y^2 = C$$

A1

Answer: shown

4 marks

(c) $C = 4 + 2 = 6$

M1

 $x^2 + 2x^4 = 6$, so $2u^2 + u - 6 = 0$ with $u = x^2$: $(2u - 3)(u + 2) = 0$

M1 A1

$$u = \frac{3}{2}, \text{ so } x = \pm\sqrt{\frac{3}{2}} \text{ and } y = \frac{3}{2}$$

A1

Answer: $x^2 + 2y^2 = 6$

4 marks

29 (10 marks)

$$(a) \int \frac{1}{y^2} \, dy = \int e^x \, dx$$

M1

$$-\frac{1}{y} = e^x + c$$

A1

$$x = 0, y = 1: -1 = 1 + c, \text{ so } c = -2$$

M1

$$\frac{1}{y} = 2 - e^x, \text{ so } y = \frac{1}{2 - e^x}$$

A1

Answer: shown

4 marks

(b) The solution is undefined when $2 - e^x = 0$, i.e. $x = \ln 2$

M1 A1

As $x \rightarrow \ln 2$ from below, $2 - e^x \rightarrow 0^+$, so $y \rightarrow +\infty$

M1 A1

Answer: $x < \ln 2$; $y \rightarrow \infty$

4 marks

(c) y is unbounded at $x = \ln 2$ (a vertical asymptote), so it is not defined there

E1

The part for $x > \ln 2$ would be a separate branch, not connected to the one through $(0, 1)$

E1

Answer: vertical asymptote

2 marks

30 (12 marks)

$$(a) \text{ Separating: } \frac{dv}{100 - v^2} = \frac{dt}{10}$$

M1

$$1 \equiv A(10 + v) + B(10 - v): v = 10 \text{ gives } A = \frac{1}{20}; v = -10 \text{ gives } B = \frac{1}{20}$$

M1 A1

$$\frac{1}{100 - v^2} \equiv \frac{1}{20} \left(\frac{1}{10 - v} + \frac{1}{10 + v} \right)$$

A1

$$\text{Answer: } \frac{1}{20} \left(\frac{1}{10 - v} + \frac{1}{10 + v} \right)$$

4 marks

$$(b) \frac{1}{20} [-\ln(10 - v) + \ln(10 + v)] = \frac{t}{10} + c$$

M1 A1

$$\ln \frac{10 + v}{10 - v} = 2t + k; t = 0, v = 0 \text{ gives } k = 0$$

M1

$$\frac{10 + v}{10 - v} = e^{2t}, \text{ so } v = \frac{10(e^{2t} - 1)}{e^{2t} + 1}$$

M1 A1

Answer: shown**5 marks****(c)** As $t \rightarrow \infty$, $v \rightarrow 10 \text{ m s}^{-1}$ **B1**

$$v = 5: e^{2t} = 3, \text{ so } t = \frac{1}{2} \ln 3 = 0.549 \text{ s}$$

M1 A1**Answer: 10 m s^{-1} ; 0.549 s** **3 marks**

Volumes of Revolution

1 (4 marks)

$$V = \pi \int_0^3 y^2 \, dx = \pi \int_0^3 x^2 \, dx$$

M1 A1

$$= \pi \left[\frac{x^3}{3} \right]_0^3$$

M1

$$= 9\pi$$

A1**Answer: 9π** **4 marks****2 (4 marks)**

$$V = \pi \int_0^4 y^2 \, dx = \pi \int_0^4 x \, dx$$

M1 A1

$$= \pi \left[\frac{x^2}{2} \right]_0^4$$

M1

$$= 8\pi$$

A1**Answer: 8π** **4 marks****3 (4 marks)**

$$V = \pi \int_0^2 y^2 \, dx = \pi \int_0^2 x^4 \, dx$$

M1 A1

$$= \pi \left[\frac{x^5}{5} \right]_0^2$$

M1

$$= \frac{32\pi}{5}$$

A1**Answer: $\frac{32\pi}{5}$** **4 marks****4 (4 marks)**

$$V = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 e^{2x} \, dx$$

M1 A1

$$= \pi \left[\frac{e^{2x}}{2} \right]_0^1$$

M1

$$= \frac{\pi(e^2 - 1)}{2}$$

A1**Answer: $\frac{\pi(e^2 - 1)}{2}$** **4 marks****5 (3 marks)**

$$x^2 = y$$

B1

$$V = \pi \int_0^4 x^2 \, dy = \pi \int_0^4 y \, dy = \pi \left[\frac{y^2}{2} \right]_0^4$$

M1

$$= 8\pi$$

A1**Answer: 8π** **3 marks**

6 (4 marks)

$$\begin{aligned}
 V &= \pi \int_1^3 y^2 \, dx = \pi \int_1^3 x^{-2} \, dx && \text{M1 A1} \\
 &= \pi \left[-\frac{1}{x} \right]_1^3 && \text{M1} \\
 &= \frac{2\pi}{3} && \text{A1}
 \end{aligned}$$

Answer: $\frac{2\pi}{3}$ 4 marks

7 (5 marks)

$$\begin{aligned}
 V &= \pi \int_0^\pi \sin^2 x \, dx && \text{M1} \\
 \sin^2 x &= \frac{1}{2}(1 - \cos 2x) && \text{M1} \\
 &= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^\pi && \text{M1 A1} \\
 &= \frac{\pi^2}{2} && \text{A1}
 \end{aligned}$$

Answer: $\frac{\pi^2}{2}$ 5 marks

8 (5 marks)

The curve meets the x-axis at $x = 0$ and $x = 3$ B1

$$\begin{aligned}
 V &= \pi \int_0^3 y^2 \, dx = \pi \int_0^3 (9x^2 - 6x^3 + x^4) \, dx && \text{M1 A1} \\
 &= \pi \left[3x^3 - \frac{3x^4}{2} + \frac{x^5}{5} \right]_0^3 && \text{M1} \\
 &= \frac{81\pi}{10} && \text{A1}
 \end{aligned}$$

Answer: $\frac{81\pi}{10}$ 5 marks

9 (5 marks)

$$\begin{aligned}
 V &= \pi \int_{-2}^2 (4 - x^2) \, dx && \text{M1 A1} \\
 &= \pi \left[4x - \frac{x^3}{3} \right]_{-2}^2 && \text{M1} \\
 &= \pi \left(\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right) = \pi \left(16 - \frac{16}{3} \right) && \text{M1} \\
 &= \frac{32\pi}{3} && \text{A1}
 \end{aligned}$$

Answer: shown 5 marks

10 (5 marks)

$$\begin{aligned}
 V &= \pi \int_0^3 (2x)^2 \, dx = \pi \int_0^3 4x^2 \, dx && \text{M1 A1} \\
 &= \pi \left[\frac{4x^3}{3} \right]_0^3 = 36\pi && \text{M1 A1}
 \end{aligned}$$

Radius 6, height 3: $\frac{1}{3}\pi(36)(3) = 36\pi$ A1

Answer: 36π 5 marks

11 (4 marks)

$x = y^{\frac{1}{3}}$, so $x^2 = y^{\frac{2}{3}}$ M1

$$V = \pi \int_0^8 y^{\frac{2}{3}} \, dy = \pi \left[\frac{3y^{\frac{5}{3}}}{5} \right]_0^8 && \text{M1 A1}$$

$$= \pi \times \frac{3 \times 32}{5} = \frac{96\pi}{5} && \text{A1}$$

Answer: $\frac{96\pi}{5}$ 4 marks

12 (6 marks)

$$V = \pi \int_1^2 \left(3 + \frac{2}{x}\right)^2 dx = \pi \int_1^2 \left(9 + \frac{12}{x} + \frac{4}{x^2}\right) dx$$

M1 A1

$$= \pi \left[9x + 12\ln x - \frac{4}{x}\right]_1^2$$

M1 A1

$$= \pi[(18 + 12\ln 2 - 2) - (9 - 4)]$$

M1

$$= \pi(11 + 12\ln 2)$$

A1

Answer: $\pi(11 + 12\ln 2)$

6 marks

13 (6 marks)Intersections: $x = x^2$, so $x = 0$ and $x = 1$

M1 A1

$$V = \pi \int_0^1 (x^2 - x^4) dx \text{ (outer radius } x, \text{ inner radius } x^2)$$

M1 A1

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5}\right)$$

M1

$$= \frac{2\pi}{15}$$

A1

Answer: $\frac{2\pi}{15}$

6 marks

14 (5 marks) $y = \ln x$ gives $x = e^y$, so $x^2 = e^{2y}$

M1 A1

$$V = \pi \int_0^2 e^{2y} dy = \pi \left[\frac{e^{2y}}{2}\right]_0^2$$

M1 A1

$$= \frac{\pi}{2}(e^4 - 1)$$

A1

Answer: $\frac{\pi}{2}(e^4 - 1)$

5 marks

15 (5 marks)

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x dx$$

M1

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

M1

$$= \frac{\pi}{2} \left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \times \frac{\pi}{2}$$

M1 A1

$$= \frac{\pi^2}{4}$$

A1

Answer: $\frac{\pi^2}{4}$

5 marks

16 (5 marks)

$$V = \pi \int_0^4 \left(2 + \frac{x}{2}\right)^2 dx = \pi \left[\frac{2(2 + \frac{x}{2})^3}{3}\right]_0^4$$

M1 A1

$$= \frac{2\pi}{3}(4^3 - 2^3)$$

M1

$$= \frac{2\pi}{3} \times 56 = \frac{112\pi}{3}$$

A1

$$\text{Check: radii 2 and 4, } h = 4: \frac{4\pi}{3}(16 + 8 + 4) = \frac{112\pi}{3}$$

B1

Answer: $\frac{112\pi}{3}$

5 marks

17 (4 marks)

$$V = \pi \int_0^k x dx = \pi \left[\frac{x^2}{2}\right]_0^k = \frac{\pi k^2}{2}$$

M1 A1

$$\frac{\pi k^2}{2} = 18\pi, \text{ so } k^2 = 36$$

M1

$$k = 6$$

A1

Answer: $k = 6$

4 marks

18 (5 marks)

$$V = \pi \int_0^3 \frac{4}{(x+1)^2} dx$$

M1 A1

$$= \pi \left[-\frac{4}{x+1} \right]_0^3$$

M1 A1

$$= \pi(-1 + 4) = 3\pi$$

A1

Answer: 3π

5 marks

19 (4 marks)

$$V = \pi \int_0^{\frac{\pi}{4}} \sec^2 x dx$$

M1

$$= \pi [\tan x]_0^{\frac{\pi}{4}}$$

M1 A1

$$= \pi(1 - 0) = \pi$$

A1

Answer: π

4 marks

20 (7 marks)

$$V = \pi \int_0^2 \frac{x^2}{(x+2)^2} dx$$

M1

$$\text{Writing } x = (x+2) - 2: \frac{x^2}{(x+2)^2} = 1 - \frac{4}{x+2} + \frac{4}{(x+2)^2}$$

M1 A1

$$\pi \left[x - 4 \ln(x+2) - \frac{4}{x+2} \right]_0^2$$

M1 A1

$$= \pi[(2 - 4 \ln 4 - 1) - (-4 \ln 2 - 2)]$$

M1

$$= \pi(3 - 4 \ln 2)$$

A1

Answer: $\pi(3 - 4 \ln 2)$

7 marks

21 (7 marks)

$$V = \pi \int_0^1 x^2 e^{2x} dx$$

M1 A1

$$\text{Parts } (u = x^2, \frac{dv}{dx} = e^{2x}): \left[\frac{x^2 e^{2x}}{2} \right]_0^1 - \int_0^1 x e^{2x} dx$$

M1 A1

$$\int_0^1 x e^{2x} dx = \left[\frac{x e^{2x}}{2} \right]_0^1 - \frac{1}{2} \int_0^1 e^{2x} dx = \frac{e^2}{2} - \frac{e^2 - 1}{4}$$

M1

$$\int_0^1 x^2 e^{2x} dx = \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4} \right) = \frac{e^2 - 1}{4}$$

M1

$$V = \frac{\pi}{4}(e^2 - 1)$$

A1

Answer: $\frac{\pi}{4}(e^2 - 1)$

7 marks

22 (8 marks)

$$4 - x^2 = x + 2 \text{ gives } x^2 + x - 2 = 0, \text{ so } x = -2 \text{ and } x = 1$$

M1 A1

$$V = \pi \int_{-2}^1 ((4 - x^2)^2 - (x + 2)^2) dx$$

M1 A1

$$= \pi \int_{-2}^1 (x^4 - 9x^2 - 4x + 12) dx$$

M1

$$= \pi \left[\frac{x^5}{5} - 3x^3 - 2x^2 + 12x \right]_{-2}^1 = \pi \left(\frac{36}{5} + \frac{72}{5} \right)$$

M1 A1

$$= \frac{108\pi}{5}$$

A1

Answer: $\frac{108\pi}{5}$

8 marks

23 (6 marks)

$$(a) \frac{dx}{dt} = 2t; V = \pi \int_0^3 y^2 \frac{dx}{dt} dt = \pi \int_0^3 4t^2 \times 2t dt$$

$$= \pi [2t^4]_0^3 = 162\pi$$

M1 A1

M1 A1

$$(b) y^2 = 4t^2 = 4x, \text{ so } V = \pi \int_0^9 4x dx = \pi [2x^2]_0^9 = 162\pi$$

M1 A1

Answer: 162π

6 marks

24 (7 marks)

$$(a) V = \pi \int_0^h x^2 dy = \pi \int_0^h y dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^h = \frac{\pi h^2}{2}$$

M1 A1

A1

Answer: shown

3 marks

$$(b) \frac{dV}{dh} = \pi h$$

M1

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{20}{\pi h}$$

M1 A1

$$h = 5; \frac{20}{5\pi} = 1.27 \text{ cm s}^{-1}$$

A1

Answer: 1.27 cm s⁻¹

4 marks

25 (7 marks)

$$\text{Curve: } x = \sqrt{y}; \text{ line: } x = \frac{y}{2}; \text{ they meet at } (2, 4)$$

M1 A1

$$\text{At height } y \text{ the line is nearer the } y\text{-axis, so the outer radius is } \sqrt{y} \text{ and the inner radius is } \frac{y}{2}$$

M1

$$V = \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy$$

M1 A1

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4 = \pi \left(8 - \frac{16}{3} \right)$$

M1

$$= \frac{8\pi}{3}$$

A1

Answer: $\frac{8\pi}{3}$

7 marks

26 (6 marks)

$$V = \pi \int_1^e (\ln x)^2 dx$$

M1

$$\text{Parts: } u = (\ln x)^2, \frac{dv}{dx} = 1: [x(\ln x)^2]_1^e - 2 \int_1^e \ln x dx$$

M1 A1

$$\int_1^e \ln x dx = [x \ln x - x]_1^e = 1$$

M1 A1

$$V = \pi(e - 2)$$

A1

Answer: shown

6 marks

27 (12 marks)

$$(a) V = \pi \int_{-r}^r (r^2 - x^2) dx$$

M1 A1

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r$$

M1

$$= \pi \left(2r^3 - \frac{2r^3}{3} \right) = \frac{4}{3} \pi r^3$$

A1

Answer: shown

4 marks

$$(b) V = \pi \int_{r-h}^r (r^2 - x^2) dx$$

M1 A1

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^r$$

M1

$$= \pi \left(r^3 - \frac{r^3}{3} \right) - \pi \left(r^2(r-h) - \frac{(r-h)^3}{3} \right)$$

M1

Expanding and simplifying: $\pi(r h^2 - \frac{h^3}{3})$

M1 A1

Answer: shown

6 marks

$$(c) \pi(r(2r)^2 - \frac{(2r)^3}{3}) = \pi(4r^3 - \frac{8r^3}{3})$$

M1

$$= \frac{4}{3}\pi r^3$$

A1

Answer: $\frac{4}{3}\pi r^3$

2 marks

28 (9 marks)

$$(a) (y - 3)^2 = 1 - x^2$$

M1

$$y = 3 \pm \sqrt{1 - x^2}$$

A1

Answer: shown

2 marks

$$(b) V = \pi \int_{-1}^1 ((3 + s)^2 - (3 - s)^2) dx, \text{ where } s = \sqrt{1 - x^2}$$

M1 A1

$$(3 + s)^2 - (3 - s)^2 = 12s, \text{ so } V = \pi \int_{-1}^1 12\sqrt{1 - x^2} dx$$

M1 A1

Answer: shown

4 marks

$$(c) \text{ The integral equals } \frac{\pi}{2}$$

B1

$$V = 12\pi \times \frac{\pi}{2} = 6\pi^2$$

M1 A1

Answer: $6\pi^2$

3 marks

29 (8 marks)

$$(a) V_k = \pi \int_1^k x^{-2} dx = \pi \left[-\frac{1}{x}\right]_1^k$$

M1 A1

$$= \pi \left(-\frac{1}{k} + 1\right)$$

A1

Answer: shown

3 marks

$$(b) \text{ As } k \rightarrow \infty, \frac{1}{k} \rightarrow 0 \text{ so } V_k \rightarrow \pi$$

M1 A1

The solid extends to infinity but has finite volume π

E1

Answer: π ; finite volume

3 marks

$$(c) 1 - \frac{1}{k} > 0.99, \text{ so } \frac{1}{k} < 0.01$$

M1

$k > 100$, so the smallest integer is $k = 101$

A1

Answer: $k = 101$

2 marks

30 (10 marks)

$$(a) \frac{dx}{dt} = -a \sin t; x \text{ runs from } a \text{ (} t = 0 \text{) to } -a \text{ (} t = \pi \text{)}$$

M1 A1

$$V = \pi \int_{\pi}^0 b^2 \sin^2 t \times (-a \sin t) dt = \pi a b^2 \int_0^{\pi} \sin^3 t dt \text{ (limits reversed, 0 to } \pi \text{)}$$

M1 A1

Answer: shown

4 marks

$$(b) \int_0^{\pi} \sin^3 t dt = \frac{1}{4} \left[-3\cos t + \frac{\cos 3t}{3}\right]_0^{\pi}$$

M1 A1

$$= \frac{1}{4} \left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{1}{4} \times \frac{16}{3} = \frac{4}{3}$$

M1 A1

$$V = \pi a b^2 \times \frac{4}{3} = \frac{4}{3}\pi a b^2$$

A1

Answer: shown

5 marks

$$(c) a = b = r: V = \frac{4}{3}\pi r^3$$

B1

Answer: $\frac{4}{3}\pi r^3$

1 mark