

Basic Integration

- 1 Find $\int (6x^2 - 4x + 5) \, dx$. [2]

- 2 Find $\int (3\sqrt{x} - \frac{4}{x^2}) \, dx$. [3]

- 3 Find $\int (8x^3 + 5) \, dx$. [2]

- 4 Find $\int (6x^{\frac{1}{2}} - 4x^{-3}) \, dx$. [2]

- 5 Find $\int \frac{x^4 + 3}{x^2} \, dx$. [2]

- 6 Find $\int x^2(3x - 2) \, dx$. [2]

- 7 A curve has gradient $\frac{dy}{dx} = 4x^3 - 3x^2 + 2$ and passes through the point (1, 5).
Find the equation of the curve. [4]

- 8 Find $\int \frac{(x+4)(x-2)}{\sqrt{x}} \, dx$, giving each term in its simplest form. [5]

- 9 Find $\int (2x+1)(x-3) \, dx$. [3]

- 10 Find $\int \frac{6x^3 - 2x^2 + 5}{x^2} \, dx$. [3]

- 11 Given that $f'(x) = 6x^2 - 4x + 1$ and $f(2) = 10$, find $f(x)$. [3]

- 12 A curve passes through the point (4, 20) and has gradient $\frac{dy}{dx} = 3\sqrt{x}$ for $x > 0$.
Find the equation of the curve. [4]

- 13 A particle moves in a straight line with acceleration $a = 6t - 4 \, \text{m s}^{-2}$ at time t seconds. When $t = 0$ the velocity is $3 \, \text{m s}^{-1}$.
Find the velocity when $t = 2$. [4]

- 14 The curve C has gradient $\frac{dy}{dx} = 3x^2 + kx - 4$, where k is a constant.
 C passes through the point $A(2, 1)$ and the gradient of C at A is 12.
(a) Find the value of k . [2]
(b) Find the equation of C . [5]

- 15 A student writes: " $\int (2x+1)^2 \, dx = \frac{(2x+1)^3}{3} + c$ ".
(a) Show, by differentiating the student's answer, that the working is wrong. [3]
(b) Find the correct expression for $\int (2x+1)^2 \, dx$, by first expanding the bracket. [3]

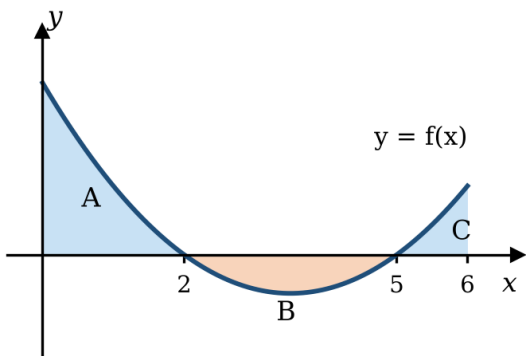
- 16 Find $\int \frac{(x^{\frac{3}{2}} - 2)^2}{\sqrt{x}} \, dx$. [5]

- 17** A curve C passes through the origin and satisfies $\frac{dy}{dx} = 3x^2 - 12x + 9$.
Find the equation of C, and show that C touches the x-axis at the point (3, 0). [5]
- 18** The curve C has gradient $\frac{dy}{dx} = 3x^2 - 2x + k$, where k is a constant. C passes through the points (0, 1) and (2, 9).
Find the value of k. [5]
- 19** A student writes: " $\int \sqrt{x} \, dx = \frac{1}{2}x^{-\frac{1}{2}} + c$ ".
(a) Explain the error in the student's working. [2]
(b) Write down the correct answer. [2]
- 20** A particle moves on a straight line. Its velocity at time t seconds is $v = 3t^2 - 8t + 4 \text{ m s}^{-1}$.
When $t = 0$ its displacement from the origin O is 5 m.
Find the displacement of the particle from O when $t = 3$. [5]
- 21** The function f satisfies $f'(x) = 12x - 6$, $f(1) = 5$ and $f(2) = 20$.
Find $f(x)$, and hence find $f(0)$. [9]
- 22** Water is pumped in and out of a storage tank. At time t minutes after pumping starts, the volume of water is V litres.
The tank initially contains 200 litres and $\frac{dV}{dt} = 6\sqrt{t} - t$ for $t \geq 0$.
(a) Find V in terms of t. [4]
(b) Find the greatest volume of water in the tank. [3]
(c) Find the time at which the tank again contains 200 litres. [2]
- 23** The depth of water in a harbour channel is h metres at time t hours after midnight. Initially $h = 0$ and, for $0 \leq t \leq 9$, $\frac{dh}{dt} = 6 - 3\sqrt{t}$.
(a) Find h in terms of t. [3]
(b) Find the maximum depth. [3]
(c) Find the next time at which the depth is zero. [2]
- 24** The function f satisfies $f'(x) = 6x - 4$. The graph of $y = f(x)$ has a stationary point at (1, 2).
Find $f(x)$, and hence determine whether the stationary point is a maximum or a minimum. [8]
- 25** A curve C has gradient $\frac{dy}{dx} = \frac{x^3 - 8}{x^2}$ for $x > 0$. The point P(2, 5) lies on C.
(a) Show that $\frac{dy}{dx} = x - 8x^{-2}$. [2]
(b) Find the equation of C. [4]
(c) Find the equation of the tangent to C at P, giving your answer in the form $y = mx + c$. [2]
- 26** Fuel is pumped into a storage tank. The volume of fuel in the tank is $V \text{ m}^3$ at time t minutes after pumping starts. The tank initially holds 12 m^3 and $\frac{dV}{dt} = \frac{3}{2}t^2 - 12t + 40$.
(a) Find V in terms of t. [3]
(b) Find the volume of fuel in the tank after 4 minutes. [2]
(c) Show that the tank is always filling. [3]

- 27** A curve C has gradient $\frac{dy}{dx} = ax^2 + bx$, where a and b are constants.
C has a stationary point at $x = 3$ and passes through the points $(0, 2)$ and $(3, -7)$.
(a) Show that $b = -3a$. [2]
(b) Find the values of a and b and the equation of C. [5]
(c) Find the coordinates of the other stationary point and determine its nature. [4]
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- 28** For $x > 0$, the function f satisfies $f'(x) = (x^{\frac{3}{2}} + \frac{2}{x})^2$. The curve $y = f(x)$ passes through the point $(1, 0)$.
(a) Find $f(x)$. [7]
(b) Find the exact value of $f(4)$. [2]
(c) Explain why the curve $y = f(x)$ crosses the x-axis only once for $x > 0$. [2]
-
- 29** A curve C has gradient $\frac{dy}{dx} = 3x^2 - 6x + k$, where k is a constant.
(a) Show that C has no stationary points when $k > 3$. [4]
(b) Given that $k = 3$ and C passes through $(1, -1)$, find the equation of C. [4]
(c) Hence show that C meets the x-axis exactly once. [4]
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- 30** For $x > 0$ the function f satisfies $f''(x) = 6x + \frac{4}{x^3}$, $f'(1) = 4$ and $f(1) = 4$.
(a) Find $f'(x)$. [3]
(b) Find $f(x)$ and hence the exact value of $f(2)$. [4]
(c) Show that f has exactly one stationary point for $x > 0$, and find its x-coordinate in exact form. [5]
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Definite Integration

- 1** Evaluate $\int_1^3 (2x + 1) dx$. [2]
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- 2** Evaluate $\int_0^2 (3x^2 - 2x) dx$. [2]
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- 3** Evaluate $\int_0^3 (4x + 1) dx$. [2]
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- 4** Evaluate $\int_{-1}^2 (x^2 - 2x) dx$. [2]
-
- 5** Evaluate $\int_1^4 \frac{3}{\sqrt{x}} dx$. [2]
-
- 6** Evaluate $\int_0^2 (x^3 - 4x) dx$. [2]
-
- 7** Find the exact value of $\int_1^4 (3\sqrt{x} + \frac{2}{x^2}) dx$. [4]
-
- 8** Given that $a > 1$ and $\int_1^a (2x + 3) dx = 14$, find the value of a. [4]
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- 9** Find the exact value of $\int_1^3 \frac{x^3 + 2}{x^2} dx$. [4]
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- 10** Evaluate $\int_1^9 \frac{x + 1}{\sqrt{x}} dx$. [4]
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- 11** Given that $k > 0$ and $\int_0^k (3x^2 + 2) \, dx = 12$, find the value of k . [4]
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- 12** Given that $\int_1^5 f(x) \, dx = 7$ and $\int_1^3 f(x) \, dx = 4$, find
- (a) $\int_3^5 f(x) \, dx$, [2]
- (b) $\int_1^5 (2f(x) + 3) \, dx$. [2]
-
- 13** Show that $\int_0^2 x(x - 2)^2 \, dx = \frac{4}{3}$. [4]
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- 14** Show that $\int_1^2 \frac{(x^2 - 1)^2}{x^2} \, dx = \frac{5}{6}$. [6]
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- 15** The diagram shows the graph of $y = f(x)$ for $0 \leq x \leq 6$. The curve crosses the x -axis at $x = 2$ and $x = 5$. The regions A, B and C have areas 7, 4 and 2 respectively. The diagram is not drawn to scale.
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- (a) Write down the value of $\int_0^5 f(x) \, dx$. [1]
- (b) Find $\int_0^6 (2f(x) + 3) \, dx$. [3]
- (c) Find $\int_5^2 f(x) \, dx$, and hence the total area enclosed by the curve, the x -axis and the lines $x = 2$ and $x = 6$. [3]
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- 16** Find the exact value of $\int_1^8 (\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}) \, dx$. [5]
-
- 17** Evaluate $\int_{-3}^3 (x^3 + 2x^2 + x) \, dx$. [4]
-
- 18** The number of bacteria in a culture, P thousand, grows at the rate $\frac{dP}{dt} = 12\sqrt{t}$, where t is the time in hours. Find the increase in the number of bacteria between $t = 4$ and $t = 9$. [4]
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- 19** Given that $a > 1$ and $\int_1^a \frac{1}{x^2} \, dx = \frac{1}{3}$, find the value of a . [4]
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- 20** Given that $\int_0^1 (px + q) \, dx = 5$ and $\int_1^3 (px + q) \, dx = 16$, find the values of the constants p and q . [6]
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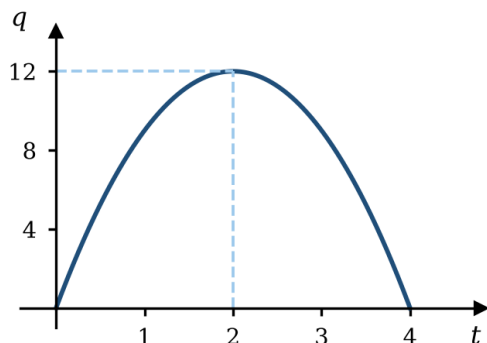
- 21 A student calculates $\int_{-1}^2 \frac{1}{x^2} dx$ as follows:

$$\left[-\frac{1}{x}\right]_{-1}^2 = \left(-\frac{1}{2}\right) - (1) = -\frac{3}{2}.$$

- (a) Explain why the student's answer cannot be correct. [2]
- (b) Find $\int_1^2 \frac{1}{x^2} dx$. [2]
- (c) Hence, or otherwise, find $\int_{-2}^{-1} \frac{1}{x^2} dx$, and explain your answer in terms of symmetry. [4]

- 22 Water flows into a tank at a rate q litres per minute, where $q = 12t - 3t^2$ for $0 \leq t \leq 4$ and t is the time in minutes.

The graph of q against t is shown.



- (a) Find the maximum rate of flow. [3]
- (b) Find the total volume of water that flows into the tank during the first 4 minutes. [3]
- (c) Show that exactly 27 litres have flowed in by $t = 3$, and show that this is the only time in the interval $0 < t < 4$ at which 27 litres have flowed in. [3]

- 23 The mean value of a function f over the interval $a \leq x \leq b$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

- (a) Find the mean value of $f(x) = x^2$ over $1 \leq x \leq 4$. [4]
- (b) Find the mean value of $g(x) = 3x^2 - 4x$ over $0 \leq x \leq k$, giving your answer in terms of k , and find k if the mean value is 8. [4]

- 24 A particle moves on a straight line with velocity $v = t^2 - 4t + 3 \text{ m s}^{-1}$ at time t seconds, for $0 \leq t \leq 4$.

- (a) Find the times at which the particle is instantaneously at rest. [2]
- (b) Find the change in displacement from $t = 0$ to $t = 4$. [3]
- (c) Find the total distance travelled from $t = 0$ to $t = 4$. [4]

- 25 For $a > 0$, let $I(a) = \int_{-a}^a (12 - 3x^2) dx$.

- (a) Show that $I(a) = 24a - 2a^3$. [3]
- (b) Find the value of a for which $I(a)$ is greatest, and the greatest value. [4]

- 26 For $x > 0$ let $F(x) = \int_1^x (3t^2 + 1) dt$.

- (a) Show that $F(x) = x^3 + x - 2$. [3]
- (b) Find $F'(x)$ and explain why the equation $F(x) = 8$ has only one solution for $x > 0$. Find it. [4]

- 27 Given that $k > 1$ and $\int_1^k (4x + \frac{6}{x^2}) dx = 12$:

- (a) show that $k^3 - 4k - 3 = 0$. [4]
- (b) Hence find the exact value of k . [4]

28 Consider integrals of cubic polynomials between limits that are symmetrical about zero.

- (a) Prove that, for $a > 0$, $\int_{-a}^a (px^3 + qx^2 + rx + s) \, dx = 2a(\frac{qa^2}{3} + s)$, where p , q , r and s are constants. [5]
- (b) Hence evaluate $\int_{-2}^2 (5x^3 - 3x^2 + 4x + 7) \, dx$. [2]
- (c) Find the positive value of a such that $\int_{-a}^a (6x^2 + 7x^3 - 4) \, dx = 0$. [3]

29 The function f is given by $f(x) = x^2 - 4x + 3$. For $k > 0$ let $I(k) = \int_0^k f(x) \, dx$.

- (a) Show that $I(k) = \frac{k(k-3)^2}{3}$. [4]
- (b) Hence find all values of $k > 0$ for which $I(k) = 0$, and explain why $I(k) \geq 0$ for all $k > 0$. [3]
- (c) Interpret this result in terms of the graph of $y = f(x)$. [3]

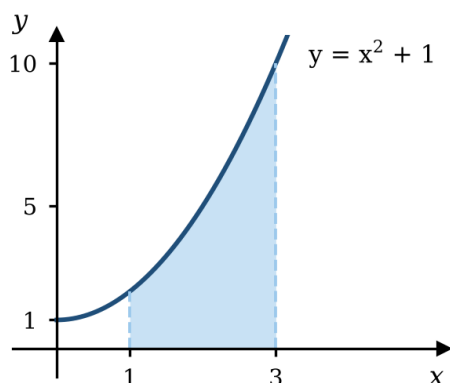
30 For a positive integer n , let $J(n) = \int_0^1 (x^n + x^{2n}) \, dx$.

- (a) Show that $J(n) = \frac{1}{n+1} + \frac{1}{2n+1}$. [3]
- (b) Find the value of n for which $J(n) = \frac{11}{28}$. [5]

Area Under and Between Curves

1 Find the area of the region bounded by the curve $y = 3x^2$, the x -axis and the lines $x = 0$ and $x = 2$. [2]

2 The diagram shows the curve $y = x^2 + 1$. The shaded region is bounded by the curve, the x -axis and the lines $x = 1$ and $x = 3$. Find the area of the shaded region.

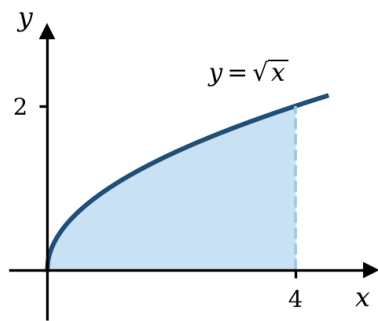


[3]

3 Find the area of the region bounded by the line $y = 4x$, the x -axis and the lines $x = 1$ and $x = 3$. [2]

4 Find the area of the region bounded by the curve $y = x^2 + 2$, the x -axis, the y -axis and the line $x = 3$. [2]

- 5 The diagram shows part of the curve $y = \sqrt{x}$. Find the exact area of the shaded region.

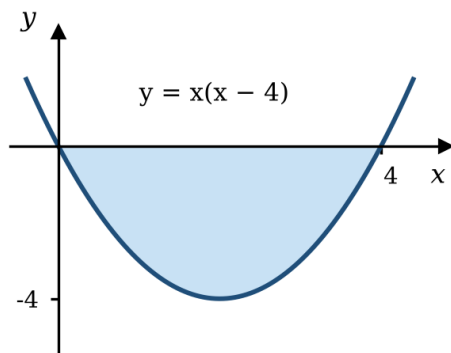


[2]

- 6 Find the area bounded by the curve $y = \frac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 2$.

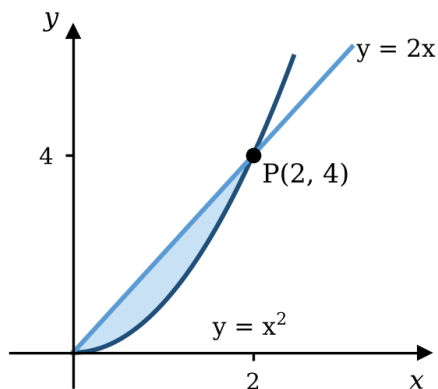
[2]

- 7 The diagram shows the curve $y = x(x - 4)$. The shaded region R lies between the curve and the x-axis. Find the area of R.



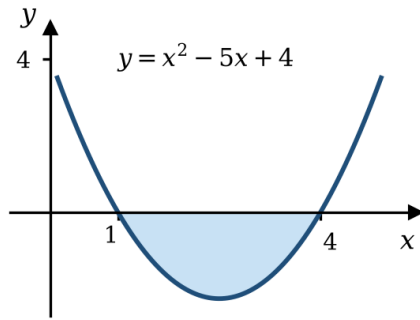
[4]

- 8 The diagram shows the curve $y = x^2$ and the line $y = 2x$, which intersect at the origin and at the point P. Find the coordinates of P and the area of the shaded region between the curve and the line.



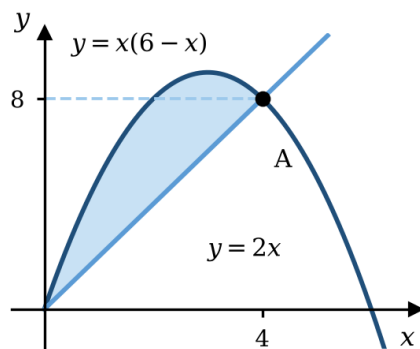
[4]

- 9** The diagram shows the curve $y = x^2 - 5x + 4$, which crosses the x-axis at $x = 1$ and $x = 4$. Find the area of the shaded region.



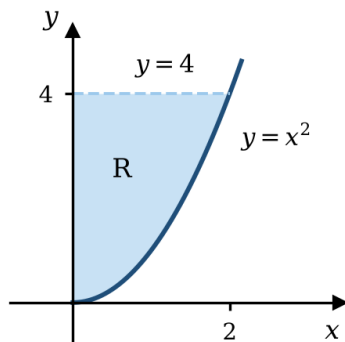
[4]

- 10** The diagram shows the curve $y = x(6 - x)$ and the line $y = 2x$, which meet at the origin and at the point A. Find the area of the shaded region between the curve and the line.



[4]

- 11** The diagram shows the region R bounded by the curve $y = x^2$, the y-axis and the line $y = 4$. By integrating with respect to y , find the area of R.

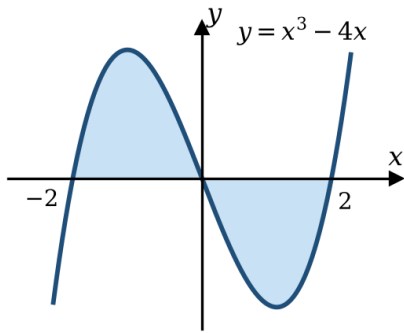


[4]

- 12** The area of the region bounded by the curve $y = 6x^2$, the x-axis and the line $x = k$, where $k > 0$, is 16. Find the value of k .

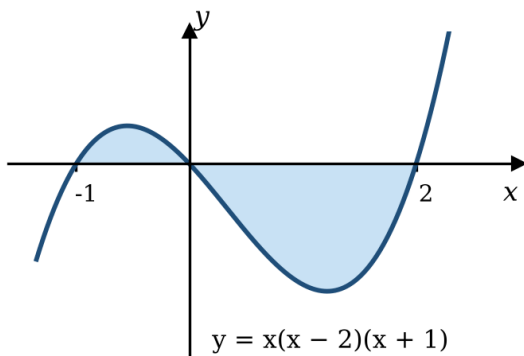
[4]

- 13** The diagram shows the curve $y = x^3 - 4x$. Find the total area of the two shaded regions.



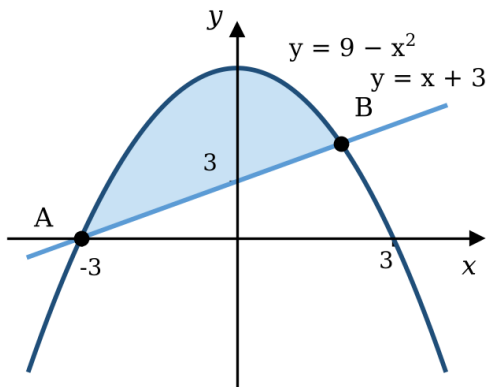
[4]

- 14** The diagram shows the curve $y = x(x - 2)(x + 1)$. Find the total area of the two shaded regions enclosed by the curve and the x-axis.



[7]

- 15** The diagram shows the curve $y = 9 - x^2$ and the line $y = x + 3$, which intersect at A and B.

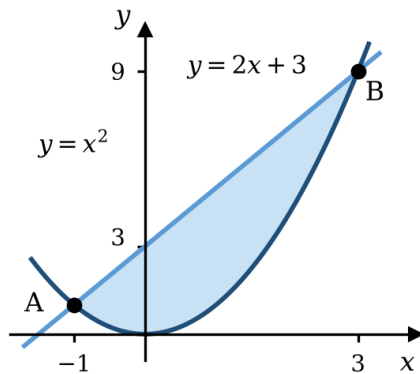


- (a) Find the coordinates of A and B.
(b) Find the area of the shaded region between the curve and the line.

[3]

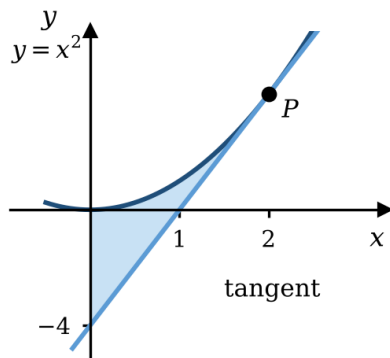
[4]

- 16** The diagram shows the curve $y = x^2$ and the line $y = 2x + 3$, which intersect at the points A and B. Find the coordinates of A and B, and the area of the shaded region.



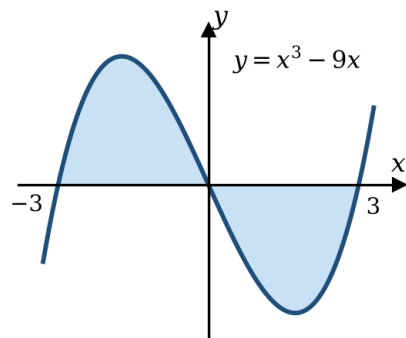
[6]

- 17** The diagram shows the curve $y = x^2$ and the tangent to the curve at the point P(2, 4). Find the equation of the tangent, and the area of the region bounded by the curve, the tangent and the y-axis.



[6]

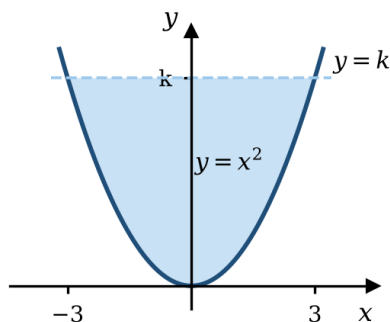
- 18** The diagram shows the curve $y = x^3 - 9x$. Find the total area enclosed between the curve and the x-axis.



[6]

- 19** The diagram shows the curve $y = x^2$ and the line $y = k$, where $k > 0$. The shaded region between them has area 36.

Show that the area is $\frac{4}{3}k^{\frac{3}{2}}$ and hence find the value of k .

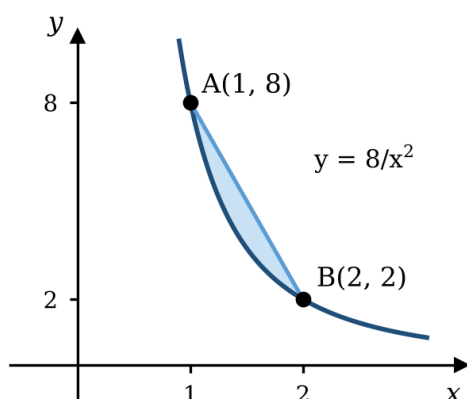


[6]

- 20** The cross-section of a tunnel is bounded by the horizontal ground and the curve $y = 4 - \frac{x^2}{4}$ for $-4 \leq x \leq 4$, where x and y are in metres. Find the area of the cross-section. The tunnel is 50 m long: find the volume of earth removed, to 2 significant figures.

[6]

- 21** The curve C has equation $y = \frac{8}{x^2}$. The points $A(1, 8)$ and $B(2, 2)$ lie on C . The region R is bounded by C and the straight line AB , as shown.



- (a) Find the equation of the line AB .
(b) Find the exact area of R .

[3]

[5]

- 22** For $a > 0$, the curve $y = x(a - x)$ meets the x -axis at the origin and at the point $(a, 0)$. The region R is bounded by the curve and the x -axis.

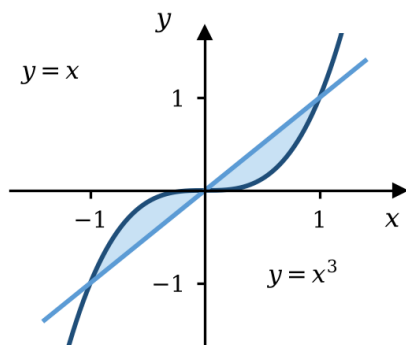
- (a) Show that the area of R is $\frac{a^3}{6}$.
(b) Given that the area of R is 36, find the value of a .
(c) For this value of a , the line $y = 2x$ divides R into two parts. Find the ratio of their areas, in its simplest form.

[3]

[2]

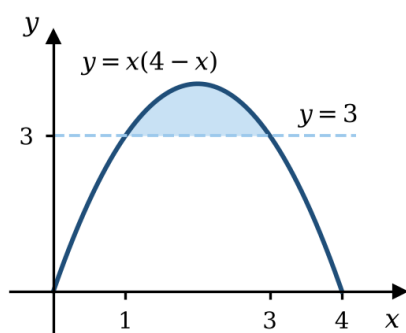
[4]

- 23** The diagram shows the curves $y = x^3$ and the line $y = x$, which meet at three points. Find the coordinates of the three points and the total area of the shaded regions.



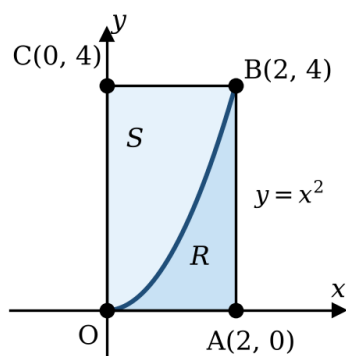
[7]

- 24** The diagram shows the curve $y = x(4 - x)$ and the line $y = 3$. Find the area of the shaded region, which lies above the line and below the curve.



[7]

- 25** The diagram shows the curve $y = x^2$ for $0 \leq x \leq 2$ and the rectangle OABC, where O is the origin, A is (2, 0), B is (2, 4) and C is (0, 4). The curve divides the rectangle into two regions, R and S.



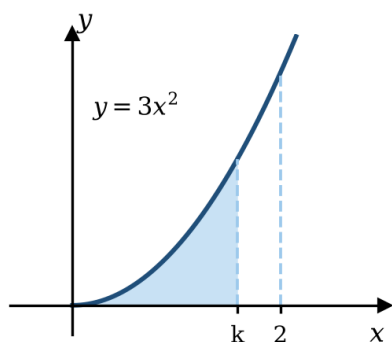
- Find the area of R by integrating with respect to x .
- By writing x in terms of y , find the area of S by integrating with respect to y .
- Hence verify that the areas of R and S add up to the area of the rectangle, and write down the ratio of the area of R to the area of S.

[3]

[4]

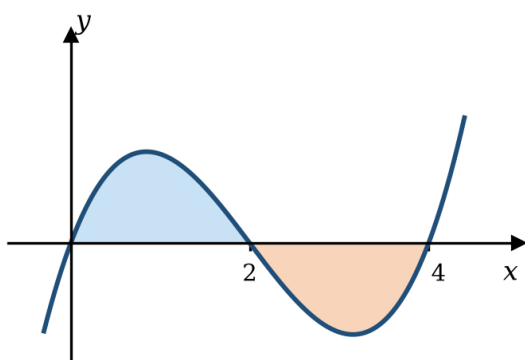
[2]

- 26** The diagram shows the region under the curve $y = 3x^2$ between $x = 0$ and $x = 2$. The line $x = k$ divides the region into two parts of equal area. Find the exact value of k .



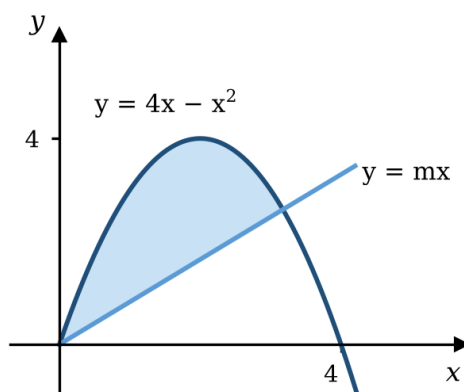
[5]

- 27** The diagram shows the curve C with equation $y = x^3 - 6x^2 + 8x$. The curve crosses the x -axis at the origin and at the points P and Q .



- (a) Find the x -coordinates of P and Q . [2]
 (b) Show that the two shaded regions have equal areas, and state the exact value of the total shaded area. [5]
 (c) Show that $f(2 + h) = -f(2 - h)$ for $f(x) = x^3 - 6x^2 + 8x$, and explain how this relates to your answer to (b). [3]

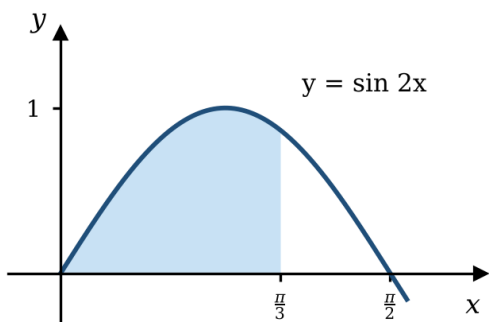
- 28** The diagram shows the curve $y = 4x - x^2$ and the line $y = mx$, where $0 < m < 4$. The shaded region R lies between the curve and the line.

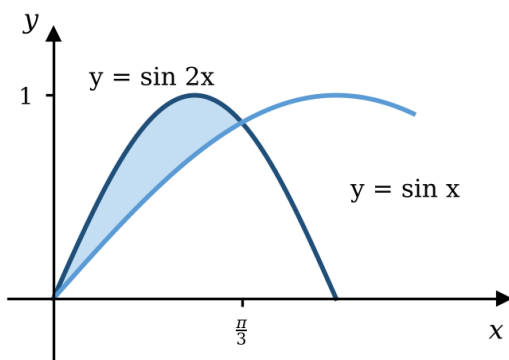


- (a) Find the area of the region enclosed by the curve and the x -axis. [3]
 (b) Show that the line meets the curve again at $x = 4 - m$. [2]
 (c) Show that the area of R is $\frac{(4 - m)^3}{6}$. [3]
 (d) The line divides the area found in (a) into two equal parts. Find the exact value of m , and give it to 3 significant figures. [4]

- 29** The points A and B lie on the curve $y = x^2$ and have x-coordinates a and b respectively, where $a < b$. The shaded region is bounded by the curve and the chord AB.
- (a) Show that the equation of the chord AB is $y = (a + b)x - ab$. [3]
- (b) Show that the area of the shaded region is $\frac{(b - a)^3}{6}$. [5]
- (c) Hence find the area enclosed by the curve $y = x^2$ and the line $y = x + 2$. [4]
-
- 30** The diagram shows the region R bounded by the x-axis, the curve $y = x^2$ for $0 \leq x \leq 2$, the curve $y = \frac{16}{x^2}$ for $2 \leq x \leq 4$ and the line $x = 4$.
- (a) Show that the two curves meet at the point (2, 4). [2]
- (b) Find the area of R. [5]
- (c) The line $x = k$, where $2 < k < 4$, divides R into two parts of equal area. Find the value of k. [5]

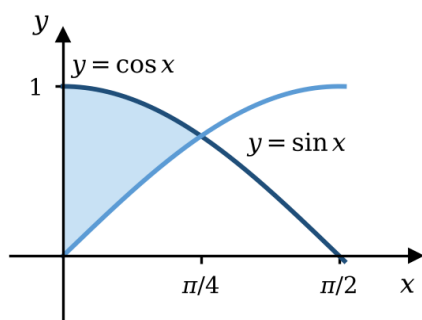
Trigonometric Integration

- 1** Find $\int (4\cos x - 3\sin x) \, dx$. [2]
-
- 2** Find $\int (\sin 3x - 6\cos 2x) \, dx$. [3]
-
- 3** Find $\int (\sec x \tan x + 4\operatorname{cosec}^2 x) \, dx$. [2]
-
- 4** Find $\int 8\sin\left(\frac{x}{2}\right) \, dx$. [2]
-
- 5** Evaluate $\int_0^{\frac{\pi}{2}} (\sin x + \cos x) \, dx$. [2]
-
- 6** Find $\int (5 + 2\sec^2 x) \, dx$. [2]
-
- 7** The diagram shows part of the curve $y = \sin 2x$. The shaded region is bounded by the curve, the x-axis and the line $x = \frac{\pi}{3}$. Find the exact area of the shaded region. Angles are in radians.
- 
- [4]
-
- 8** Find the exact value of $\int_0^{\frac{\pi}{6}} (\cos x + \sec^2 2x) \, dx$. [4]
-
- 9** Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\cot^2 x \, dx$. [5]

- 10** Use $\cos 2x = 1 - 2\sin^2 x$ to find $\int \sin^2 x \, dx$. [4]
-
- 11** Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\cos(2x - \frac{\pi}{3}) \, dx$. [4]
-
- 12** Use $\tan^2 x = \sec^2 x - 1$ to find $\int \tan^2 x \, dx$. [2]
-
- 13** Find the exact value of $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$. [4]
-
- 14** Use a double angle identity in this question.
- (a) Show that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. [2]
- (b) Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sin^2 x \, dx$. [4]
-
- 15** The diagram shows the curves $y = \sin 2x$ and $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$. The curves meet at the origin and at the point where $x = a$. The shaded region lies between the curves.
- 
- (a) Show that $a = \frac{\pi}{3}$. [3]
- (b) Find the exact area of the shaded region. [4]
-
- 16** Find the exact value of $\int_0^{\frac{\pi}{2}} (1 + \cos x)^2 \, dx$. [6]
-
- 17** Show that $\int_0^{\frac{\pi}{6}} \tan^2 2x \, dx = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$. [5]
-
- 18** Given that $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$, find the exact value of the following integral.
- $$\int_0^{\frac{\pi}{2}} \sin 3x \cos x \, dx$$
- [6]
-

- 19** The diagram shows the curves $y = \cos x$ and $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

Find the exact area of the shaded region between the curves and the y-axis.



[6]

- 20** Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec}^2 x \, dx = \frac{2\sqrt{3}}{3}$.

[5]

- 21** A student writes: " $\int \cos^2 x \, dx = \frac{\sin^3 x}{3} + c$ ".

(a) Show that the student's answer is wrong.

[2]

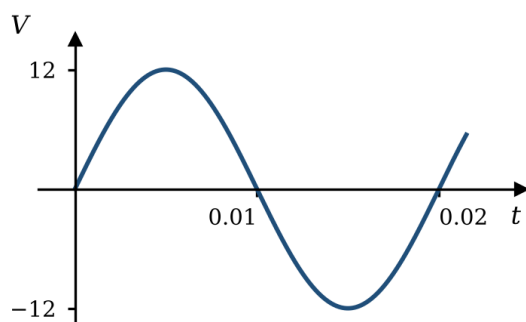
(b) Use $\cos 2x = 2\cos^2 x - 1$ to find $\int \cos^2 x \, dx$.

[3]

(c) Hence find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\cos^2 x \, dx$.

[3]

- 22** The voltage V volts in an alternating supply at time t seconds is modelled by $V = 12\sin(100\pi t)$, as shown in the diagram. One complete cycle takes 0.02 seconds.



(a) Show that $V^2 = 72(1 - \cos 200\pi t)$.

[2]

(b) Find the mean value of V^2 over one complete cycle, given that the mean value of a function over an interval is the integral divided by the length of the interval.

[4]

(c) The root mean square (RMS) voltage is the square root of the mean value of V^2 . Find the RMS voltage, giving your answer to 3 significant figures.

[2]

- 23** Find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$.

[7]

- 24** Consider the function $f(x) = (\sin x + \cos x)^2$.

(a) Show that $f(x) = 1 + \sin 2x$.

[3]

(b) Hence find the exact value of $\int_0^{\frac{\pi}{2}} f(x) \, dx$.

[4]

(c) Hence find the mean value of $f(x)$ over $0 \leq x \leq \frac{\pi}{2}$.

[2]

25 Consider the function $f(x) = \frac{1}{1 + \cos x}$ for $0 \leq x \leq \frac{\pi}{2}$.

(a) Use $\cos 2\theta = 2\cos^2 \theta - 1$ to show that $1 + \cos x = 2\cos^2(\frac{x}{2})$. [2]

(b) Hence find the exact value of $\int_0^{\frac{\pi}{2}} f(x) \, dx$. [4]

(c) Hence find the exact value of $\int_0^{\frac{\pi}{2}} \frac{1 + \sin x}{1 + \cos x} \, dx$. [3]

26 A curve has gradient $\frac{dy}{dx} = 4\sin^2(\frac{x}{2})$ and passes through the point $(\pi, 0)$.

Find the exact value of y when $x = \frac{\pi}{2}$. [7]

27 It is given that $\sin 3x = 3\sin x - 4\sin^3 x$.

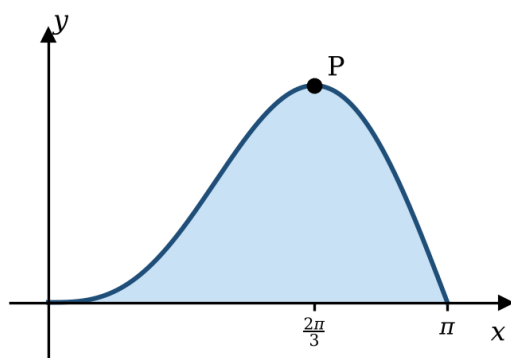
(a) Show that $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$. [2]

(b) Hence find $\int \sin^3 x \, dx$. [4]

(c) Find the exact value of $\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$. [3]

(d) Show that $\int_0^{\pi} \sin^3 x \, dx = \frac{4}{3}$. [2]

28 The diagram shows the curve C with equation $y = 2\sin x - \sin 2x$ for $0 \leq x \leq \pi$. Angles are in radians.



(a) Show that $y = 2\sin x(1 - \cos x)$, and hence explain why $y \geq 0$ for $0 \leq x \leq \pi$. [3]

(b) Find the exact area of the region bounded by C and the x -axis. [4]

(c) Show that the maximum point P on C has x -coordinate $\frac{2\pi}{3}$, and find the exact y -coordinate of P . [5]

29 It is given that $\cos 3x = 4\cos^3 x - 3\cos x$.

(a) Prove this identity, starting from $\cos 3x = \cos(2x + x)$. [4]

(b) Hence show that $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.

Hence find the exact value of $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$. [4]

(c) Find the exact area of the region between the curves $y = \cos x$ and $y = \cos^3 x$ for $0 \leq x \leq \frac{\pi}{2}$, and explain which curve is above. [4]

30 Use the identity $2\sin P \sin Q = \cos(P - Q) - \cos(P + Q)$ in this question.

(a) Show that $\int_0^{\pi} \sin 2x \sin 3x \, dx = 0$. [4]

(b) Let m and n be different positive integers. Prove that $\int_0^{\pi} \sin mx \sin nx \, dx = 0$. [5]

(c) Find $\int_0^{\pi} \sin^2 nx \, dx$ for a positive integer n . [3]

Exponential and Logarithmic Integration

1 Find $\int (e^{4x} + \frac{5}{x}) dx$. [2]

2 Find $\int \frac{6}{3x-1} dx$. [2]

3 Find $\int (3e^x + \frac{2}{x}) dx$. [2]

4 Find $\int e^{-2x} dx$. [2]

5 Find the exact value of $\int_0^1 e^{3x} dx$. [2]

6 Find $\int 12e^{\frac{x}{3}} dx$. [2]

7 Find the exact value of $\int_0^{\ln 3} 4e^{2x} dx$. [4]

8 Show that $\int_1^3 \frac{5}{2x-1} dx = \frac{5}{2} \ln 5$. [4]

9 Find the exact value of $\int_0^{\ln 2} (e^x + e^{-x}) dx$. [4]

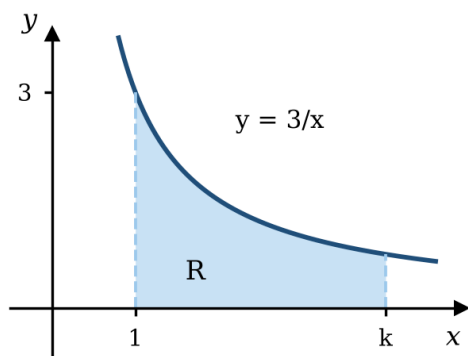
10 Find $\int (e^x + 1)^2 dx$. [3]

11 Find the exact value of $\int_1^2 (\frac{3}{x} - \frac{2}{x^2}) dx$. [4]

12 Show that $\int_0^1 \frac{6x^2}{x^3+2} dx = 2 \ln \frac{3}{2}$. [4]

13 A curve has gradient $\frac{dy}{dx} = 2e^{2x} + \frac{3}{x}$ for $x > 0$ and passes through (1, 5).
Find the equation of the curve. [4]

14 The diagram shows the region R bounded by the curve $y = \frac{3}{x}$, the x-axis and the lines $x = 1$ and $x = k$, where $k > 1$.



- (a) Show that the area of R is $3 \ln k$. [3]
(b) Given that the area of R is $6 \ln 2$, find the value of k . [3]

15 Show that $\int_1^2 \frac{(x+1)^2}{x} dx = \frac{7}{2} + \ln 2$. [6]

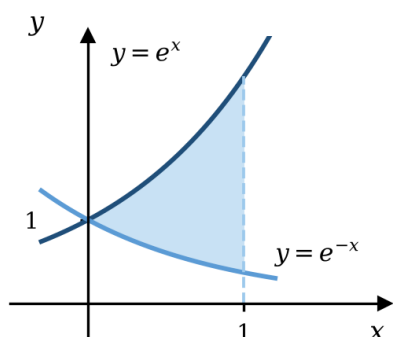
16 Find $\int \frac{e^{2x} + 3}{e^x} dx$. [4]

17 Show that $\int_2^5 \frac{x+1}{x-1} dx = 3 + 4\ln 2$. [5]

18 Given that $a > 0$ and $\int_0^a 2e^{2x} dx = 15$, find the exact value of a . [5]

19 The number of undecayed nuclei in a sample is N thousand at time t hours. It is modelled by $\frac{dN}{dt} = -24e^{-0.3t}$, and $N = 100$ when $t = 0$.
Find N in terms of t , and state what happens to N in the long term. [5]

20 The diagram shows the curves $y = e^x$ and $y = e^{-x}$. Find the exact area of the shaded region bounded by the two curves, the line $x = 1$ and the y -axis. [4]



21 Oil leaks from a tank at a rate of $40e^{-0.1t}$ litres per minute, where t is the time in minutes after the leak starts.

(a) Find the volume of oil lost in the first 10 minutes, giving your answer to 3 significant figures. [3]

(b) Show that, however long the leak continues, the volume of oil lost is always less than 400 litres. [2]

(c) Find the time at which 300 litres have been lost, giving your answer in minutes to 3 significant figures. [3]

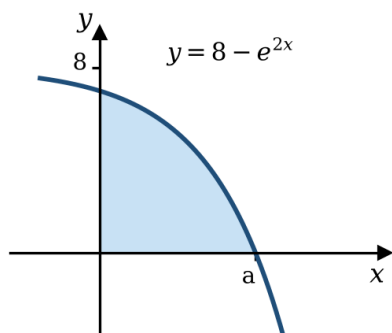
22 A student calculates $\int_{-1}^2 \frac{1}{x} dx = [\ln|x|]_{-1}^2 = \ln 2 - \ln 1 = \ln 2$.

(a) Explain why this working is not valid. [2]

(b) Find the exact value of $\int_{-4}^{-1} \frac{1}{x} dx$. [2]

(c) Explain why your answer to (b) is negative, although the area between the graph and the x -axis from $x = -4$ to $x = -1$ is positive. State the area, in the form $p \ln 2$. [3]

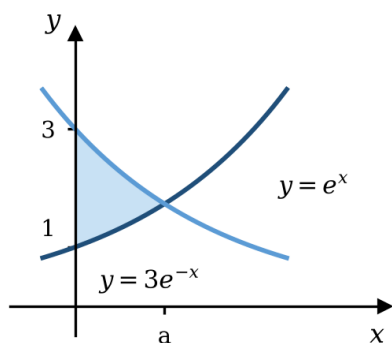
- 23** The diagram shows the curve $y = 8 - e^{2x}$, which meets the x-axis at the point $(a, 0)$.
Find the exact value of a , and show that the area of the shaded region is $12\ln 2 - \frac{7}{2}$.



[7]

- 24** A well produces oil at the rate $r = 50 + 30e^{-0.5t}$ thousand litres per hour, where t is the time in hours after production starts.
- (a) Find the exact total production in the first 4 hours. [4]
- (b) Find the time at which the rate of production first falls to 55 thousand litres per hour. [3]

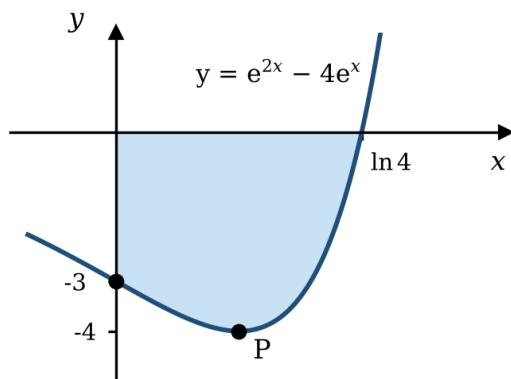
- 25** The diagram shows the curves $y = e^x$ and $y = 3e^{-x}$, which meet at the point where $x = a$.
Show that $a = \frac{1}{2}\ln 3$, and find the exact area of the shaded region bounded by the curves and the y-axis.



[7]

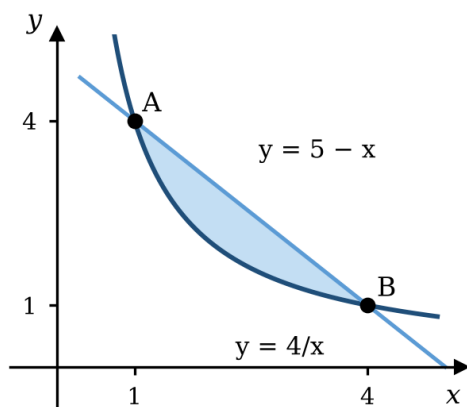
- 26** A student writes: " $\int \frac{1}{1-x} dx = \ln|1-x| + c$ ".
- (a) Show, by differentiating, that this is incorrect. [3]
- (b) Find the correct value of $\int_{-1}^0 \frac{1}{1-x} dx$. [4]

- 27 The diagram shows the curve C with equation $y = e^{2x} - 4e^x$. The curve has a minimum point at P .



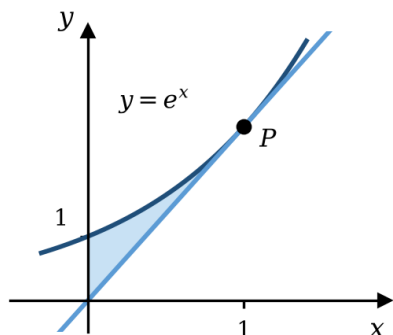
- (a) Find the exact coordinates of the point where C crosses the x -axis. [2]
 (b) Find the exact coordinates of P . [4]
 (c) Find the exact area of the shaded region bounded by C , the x -axis and the y -axis. [5]

- 28 The diagram shows the curve $y = \frac{4}{x}$ and the line $y = 5 - x$, which intersect at the points A and B . The shaded region R lies between the curve and the line.



- (a) Show that the x -coordinates of A and B are 1 and 4. [3]
 (b) Show that the area of R is $\frac{15}{2} - 8\ln 2$. [5]
 (c) Use a trapezium and your answer to (b) to find $\int_1^4 \frac{4}{x} dx$, and verify your answer by integrating directly. [4]

- 29 The curve C has equation $y = e^x$. The tangent to C at the point $P(a, e^a)$ is the line l .

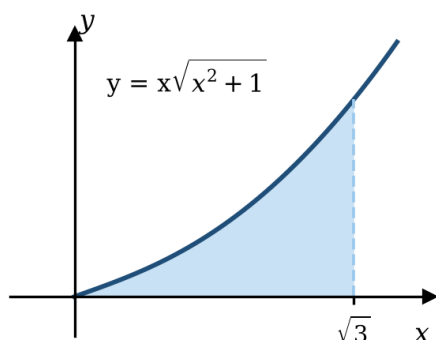


- (a) Show that the equation of l is $y = e^a(x - a + 1)$. [4]
 (b) Show that l passes through the origin only when $a = 1$. [3]
 (c) The diagram shows C , the line l when $a = 1$ and the y -axis. Find the exact area of the shaded region. [5]

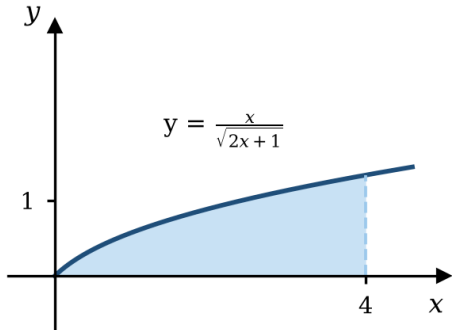
- 30** The diagram shows the curve $y = \frac{1}{x}$ for $x > 0$ and four rectangles. The r th rectangle has width 1 and height $\frac{1}{r}$, for $r = 1, 2, 3, 4$, and its top-left corner lies on the curve.
- (a) Find $\int_1^n \frac{1}{x} dx$, where $n > 1$. [2]
- (b) Explain why $\ln 5 < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, and generalise to show that $\ln n < \sum_{r=1}^{n-1} \frac{1}{r}$ for $n \geq 2$. [4]
- (c) Hence show that $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{99} > 4.6$. [3]

Integration by Substitution

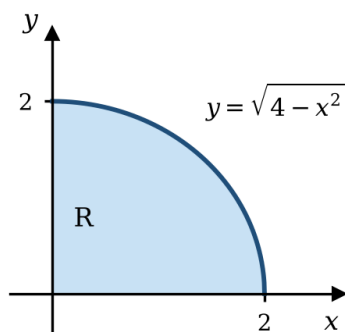
- 1** Use the substitution $u = 3x - 2$ to find $\int (3x - 2)^4 dx$. [3]
- 2** Use the substitution $u = x^2 + 1$ to find $\int 2x(x^2 + 1)^3 dx$. [3]
- 3** Use the substitution $u = 2x - 1$ to find $\int \frac{6}{(2x - 1)^3} dx$. [2]
- 4** Use the substitution $u = x^3 + 1$ to find $\int 3x^2(x^3 + 1)^4 dx$. [2]
- 5** Use the substitution $u = x^2$ to find $\int xe^{x^2} dx$. [2]
- 6** Use the substitution $u = x^2 + 1$ to evaluate $\int_0^1 2x(x^2 + 1)^2 dx$. [2]
- 7** The diagram shows the curve $y = x\sqrt{x^2 + 1}$. The shaded region is bounded by the curve, the x -axis and the line $x = \sqrt{3}$. Use the substitution $u = x^2 + 1$ to find the exact area of the shaded region.



- 8** Use the substitution $u = 1 + \sin x$ to find the exact value of $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)^2} dx$. [4]
- 9** Use the substitution $u = 2x - 1$ to find $\int \frac{4x}{2x - 1} dx$. [4]
- 10** Use the substitution $u = \sin x$ to find the exact value of $\int_0^{\frac{\pi}{2}} \cos x \sin^3 x dx$. [4]
- 11** Use the substitution $u = x^2 + 1$ to find $\int \frac{x}{x^2 + 1} dx$. [4]

- 12** Use the substitution $u = x + 1$ to show that $\int_0^3 \frac{x}{\sqrt{x+1}} dx = \frac{8}{3}$. [4]
- 13** Use the substitution $u = 1 + e^x$ to find the exact value of $\int_0^{\ln 2} \frac{e^x}{1 + e^x} dx$. [4]
- 14** Use the substitution $u = x^2 + 4$ to show that $\int_0^2 \frac{6x}{x^2 + 4} dx = 3 \ln 2$. [5]
- 15** Use the substitution $u = e^x + 1$ to show that $\int_0^{\ln 3} \frac{e^{2x}}{e^x + 1} dx = 2 - \ln 2$. [6]
- 16** Use the substitution $u = 1 - x$ to show that $\int_0^1 x(1 - x)^5 dx = \frac{1}{42}$. [5]
- 17** Show that $\int_0^{\frac{\pi}{3}} \tan x dx = \ln 2$, using the substitution $u = \cos x$. [5]
- 18** Use the substitution $u = 1 + \sqrt{x}$ to find $\int_1^4 \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$. [5]
- 19** Use the substitution $u = 1 + \sin^2 x$ to find the exact value of $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1 + \sin^2 x} dx$. [5]
- 20** Use the substitution $u = \sqrt{x}$ to show that $\int_0^4 \frac{1}{1 + \sqrt{x}} dx = 4 - 2 \ln 3$. [6]
- 21** The diagram shows the curve $y = \frac{x}{\sqrt{2x+1}}$ for $x \geq 0$. The shaded region R is bounded by the curve, the x-axis and the line $x = 4$.
- 
- (a) Use $u = 2x + 1$ to show that the area of R is $\frac{1}{4} \int_1^9 (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) du$. [4]
- (b) Hence find the exact area of R. [4]
- 22** This question is about integrals of odd powers of $\cos x$. Use the substitution $u = \sin x$ throughout.
- (a) Show that $\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^1 (1 - u^2) du$. [4]
- (b) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x dx$. [2]
- (c) Find the exact value of $\int_0^{\frac{\pi}{2}} \cos^5 x dx$. [3]
- 23** Use the substitution $u = 1 - x^2$ to find the exact value of $\int_0^1 x^3 \sqrt{1 - x^2} dx$. [8]

- 24 The diagram shows the curve $y = \sqrt{4 - x^2}$ for $0 \leq x \leq 2$ and the region R under it. Use the substitution $x = 2\sin \theta$ to show that the area of R is π . Explain how you could check this geometrically.

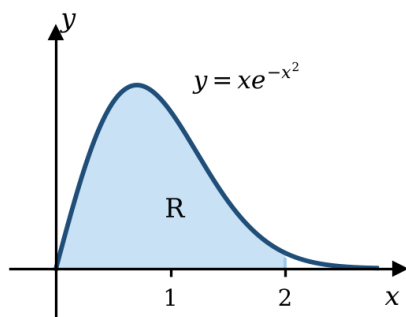


[7]

- 25 Use the substitution $u = e^x$ to find the exact value of $\int_0^{\ln 2} \frac{1}{e^x + 1} dx$.

[6]

- 26 The diagram shows the curve C with equation $y = xe^{-x^2}$ for $x \geq 0$ and the region R bounded by C, the x-axis and the line $x = 2$.

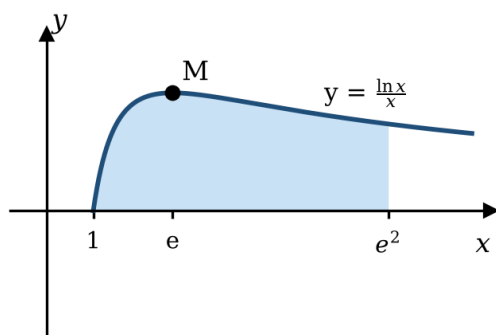


- (a) Find the exact coordinates of the maximum point of C.
(b) Use the substitution $u = x^2$ to find the exact area of R.

[4]

[4]

- 27 The diagram shows the curve C with equation $y = \frac{\ln x}{x}$ for $x > 0$. C has a maximum point M.



- (a) Find the exact coordinates of M.
(b) Use the substitution $u = \ln x$ to find the exact area of the shaded region bounded by C, the x-axis and the line $x = e^2$.
(c) Show that the area between C, the x-axis and the line $x = e^k$ ($k > 0$) is $\frac{k^2}{2}$, and find the value of k for which this area is 8.

[4]

[5]

[3]

- 28** Let f be a function that is continuous on the interval $0 \leq x \leq a$, where $a > 0$.
- (a) By using the substitution $u = a - x$, prove that $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$. [5]
- (b) Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} \, dx$. Use the result of part (a) to show that $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} \, dx$. [3]
- (c) Hence find the exact value of I . [4]
-
- 29** Let $I = \int_0^1 \frac{1}{(1 + x^2)^2} \, dx$.
- (a) Use the substitution $x = \tan \theta$ to show that $I = \int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta$. [4]
- (b) Hence find the exact value of I . [4]
- (c) Hence find the exact value of $\int_0^1 \frac{x^2}{(1 + x^2)^2} \, dx$. [3]
-
- 30** For integers $n \geq 0$, let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$.
- (a) Show that $I_n + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$. [5]
- (b) Find I_0 and I_2 , and hence the exact value of I_4 . [5]
- (c) By considering the sign of I_4 , show that $\pi > \frac{8}{3}$. [2]
-

Integration by Parts

- 1** Use integration by parts to find $\int x e^x \, dx$. [3]
-
- 2** Use integration by parts to find $\int x \cos x \, dx$. [3]
-
- 3** Find $\int x e^{2x} \, dx$. [2]
-
- 4** Find $\int x \sin x \, dx$. [2]
-
- 5** Find $\int 3x e^{-x} \, dx$. [2]
-
- 6** Find $\int x e^{\frac{x}{2}} \, dx$. [2]
-
- 7** Find $\int x \sin 2x \, dx$. [4]
-
- 8** Find $\int x \ln x \, dx$. [4]
-
- 9** Find the exact value of $\int_0^{\frac{\pi}{2}} x \cos x \, dx$. [4]
-
- 10** Find the exact value of $\int_1^e x^2 \ln x \, dx$. [4]
-
- 11** Find $\int x \cos 3x \, dx$. [4]
-

12 Show that $\int_0^2 x e^{-x} dx = 1 - 3e^{-2}$. [4]

13 Find $\int \frac{\ln x}{x^2} dx$. [4]

14 Show that $\int_0^1 (2x + 1)e^x dx = e + 1$. [5]

15 Integration by parts can be used to integrate $\ln x$.

(a) By writing $\ln x$ as $1 \times \ln x$, find $\int \ln x dx$. [3]

(b) Hence show that $\int_1^e (\ln x)^2 dx = e - 2$. [4]

16 Find $\int x^2 e^x dx$. [5]

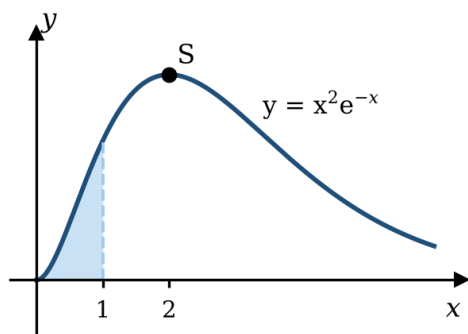
17 Show that $\int_0^{\frac{\pi}{2}} x^2 \cos x dx = \frac{\pi^2}{4} - 2$. [5]

18 Use integration by parts to find the exact value of $\int_0^3 x\sqrt{x+1} dx$. [6]

19 Show that $\int_1^3 \frac{\ln x}{\sqrt{x}} dx = 2\sqrt{3}\ln 3 - 4\sqrt{3} + 4$. [6]

20 Given that $a > 1$ and $\int_1^a \ln x dx = 1$, find the exact value of a . [5]

21 The diagram shows the curve $y = x^2 e^{-x}$ for $x \geq 0$. The point S is the maximum point.

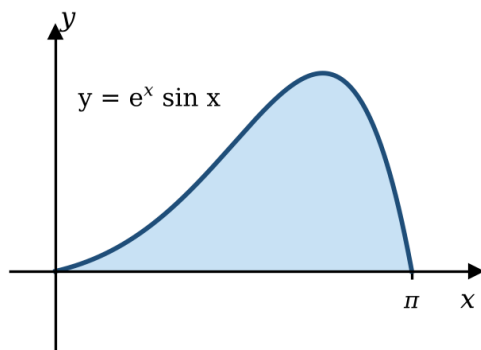


(a) Find $\int x^2 e^{-x} dx$. [4]

(b) Find the exact area of the region bounded by the curve, the x-axis and the line $x = 1$. [2]

(c) Find the exact coordinates of S. [3]

- 22 The diagram shows the curve $y = e^x \sin x$ for $0 \leq x \leq \pi$. Let $I = \int e^x \sin x \, dx$.

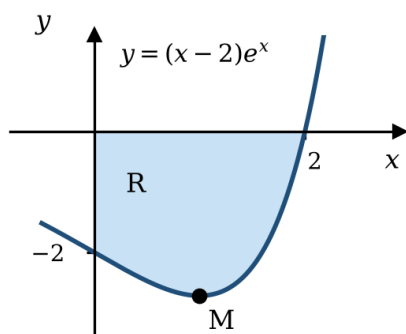


- (a) Use integration by parts twice to show that $I = \frac{1}{2}e^x(\sin x - \cos x) + c$. [6]
 (b) Hence find the exact area of the shaded region, and give it to 3 significant figures. [2]

- 23 Show that $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$. [7]

- 24 The diagram shows the curve $y = (x - 2)e^x$, which crosses the axes at $(2, 0)$ and $(0, -2)$. The curve has a minimum point M.

Find the exact coordinates of M, and the exact area of the shaded region R.



- (a) Find the coordinates of M. [3]
 (b) Find the area of R. [5]

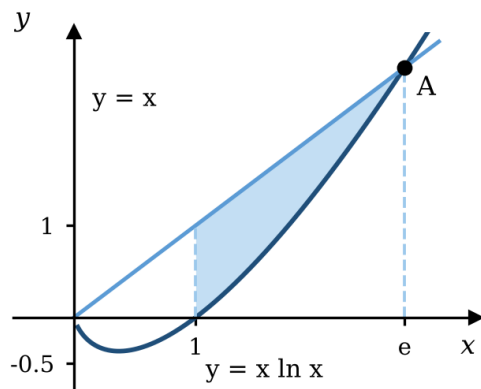
- 25 Find the exact value of $\int_0^{\frac{\pi}{2}} x \cos^2 x \, dx$. [7]

- 26 Use the substitution $u = \sqrt{x}$, followed by integration by parts, to show that $\int_0^1 e^{\sqrt{x}} \, dx = 2$. [6]

- 27 For integers $n \geq 0$, let $I_n = \int_0^1 x^n e^x \, dx$.

- (a) Show that $I_n = e - nI_{n-1}$ for $n \geq 1$. [4]
 (b) Find the exact value of I_0 and use the result in (a) to find the exact value of I_3 . [4]
 (c) Explain why $I_3 > 0$, and hence show that $e < 3$. [2]

- 28** The diagram shows the curve C with equation $y = x \ln x$ for $x > 0$, and the line $y = x$. The curve and the line meet at the point A. The shaded region R is bounded by C, the line $y = x$ and the line $x = 1$.



- (a) Find the exact coordinates of the stationary point of C, and determine its nature. [4]
 (b) Show that $\int_1^e x \ln x \, dx = \frac{1}{4}(e^2 + 1)$. [4]
 (c) Hence find the exact area of R. [3]

- 29** For integers $n \geq 0$, let $J_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$.

- (a) Show that $J_n = \frac{n-1}{n} J_{n-2}$ for $n \geq 2$. [6]
 (b) Hence find the exact values of J_5 and J_6 . [5]

- 30** Let $K = \int e^{2x} \cos 3x \, dx$.

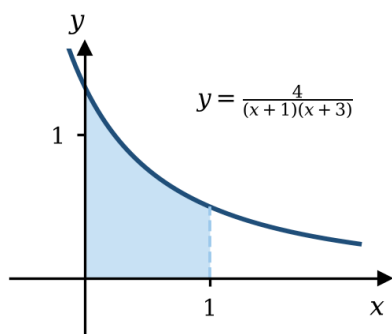
- (a) By using integration by parts twice, show that $K = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + c$. [6]
 (b) The curve $y = e^{2x} \cos 3x$ crosses the x-axis at $x = \frac{\pi}{6}$. Find the exact area of the region between the curve, the y-axis and the x-axis for $0 \leq x \leq \frac{\pi}{6}$, and give it to 3 significant figures. [4]

Integration Using Partial Fractions

- 1** Express $\frac{x+5}{(x+1)(x+3)}$ in partial fractions. [3]
2 Find $\int \frac{2}{(x-1)(x+1)} \, dx$. [5]
3 Find $\int \frac{5x+13}{(x+2)(x+3)} \, dx$. [5]
4 Express $\frac{9x+1}{(2x+1)(x-3)}$ in partial fractions. [3]
5 Express $\frac{5x+8}{x(x+4)}$ in partial fractions. [3]
6 Find $\int \frac{6}{x^2-9} \, dx$. [5]
7 Find $\int \frac{7x+4}{(2x-1)(x+2)} \, dx$. [5]

- 8** Show that $\int_0^1 \frac{3x+5}{(x+1)(x+2)} dx = \ln 6$. [7]
-
- 9** Find the exact value of $\int_2^4 \frac{5x+3}{(x-1)(x+3)} dx$. [7]
-
- 10** Express $\frac{-7x-2}{x(x+2)(x-1)}$ in partial fractions. [4]
-
- 11** Let $f(x) = \frac{5x+10}{(x-1)(x+4)}$.
 (a) Express $f(x)$ in partial fractions. [3]
 (b) Hence find $\int f(x) dx$. [2]
-
- 12** Find $\int \frac{6}{x(x-3)} dx$. [5]
-
- 13** Express $\frac{4x^2+8x}{(x+1)^2(x-1)}$ in partial fractions. [4]
-
- 14** Find $\int \frac{6x^2+19x+11}{(x+2)^2(x-1)} dx$. [6]
-
- 15** Find the exact value of $\int_2^3 \frac{4x^2+8x}{(x+1)^2(x-1)} dx$. [8]
-
- 16** Find $\int \frac{x^2+8x+9}{(x+1)(x+2)} dx$. [7]
-
- 17** Find $\int \frac{4x^2-13x-12}{x(x+2)(x-3)} dx$. [6]
-
- 18** Show that $\int_1^2 \frac{2}{x(x+1)(x+2)} dx = \ln \frac{32}{27}$. [8]
-
- 19** Find the exact value of $\int_2^3 \frac{x^2}{x^2-1} dx$, giving your answer in the form $a + b \ln c$. [8]
-
- 20** Find the exact value of $\int_0^1 \frac{2x^2+11x+14}{(x+1)(x+3)} dx$. [9]
-
- 21** Find $\int \frac{x^3+4x^2+6x-5}{(x-1)(x+2)} dx$. [7]
-
- 22** It is given that $\frac{4x^2+3x-4}{x(x+1)(x-2)} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$.
 (a) Find the values of A, B and C. [4]
 (b) Hence find the exact value of $\int_3^4 \frac{4x^2+3x-4}{x(x+1)(x-2)} dx$, giving your answer in the form $a \ln 2 + b \ln 3 + c \ln 5$ where a, b and c are rational. [5]
-

- 23 The diagram shows the curve with equation $y = \frac{4}{(x+1)(x+3)}$ for $x \geq -0.3$. The shaded region R is bounded by the curve, the coordinate axes and the line $x = 1$.



- (a) Express $\frac{4}{(x+1)(x+3)}$ in partial fractions. [3]
- (b) Hence show that the area of R is $2\ln\frac{3}{2}$. [4]

- 24 Given that $k > 0$ and $\int_0^k \frac{3}{(x+1)(x+4)} dx = \ln 2$, find the value of k . [6]

- 25 Find the exact value of $\int_0^1 \frac{2x^3 + 12x^2 + 21x + 13}{(x+1)^2(x+2)} dx$. [9]

- 26 A student attempts to find $\int \frac{1}{(x+1)(x+2)} dx$ as follows:
 “ $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{1}{x+2}$, so the integral is $\ln|x+1| + \ln|x+2| + c$.”

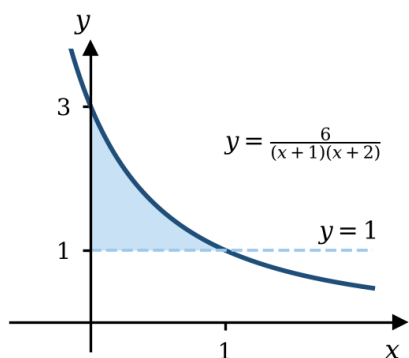
- (a) Show that the student's partial fractions are wrong. [3]
- (b) Find the correct value of $\int \frac{1}{(x+1)(x+2)} dx$. [5]

- 27 For positive integers n let $S_n = \sum_{r=1}^n \frac{1}{r(r+2)}$.

- (a) Express $\frac{1}{x(x+2)}$ in partial fractions. [3]
- (b) Hence show that $S_n = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$. [5]
- (c) Find the exact value of $\int_1^n \frac{1}{x(x+2)} dx$ and show that it is less than $\frac{1}{2}\ln 3$ for every $n > 1$. [4]

- 28 The function f is defined by $f(x) = \frac{px+q}{(x+1)(x+2)}$, where p and q are constants. Given that $f(0) = \frac{7}{2}$ and $\int_0^2 f(x) dx = \ln 72$, find the values of p and q . [9]

- 29** The diagram shows the curve with equation $y = \frac{6}{(x+1)(x+2)}$ and the line $y = 1$. The curve and the line meet at the point P. The shaded region is bounded by the curve, the line and the y-axis.



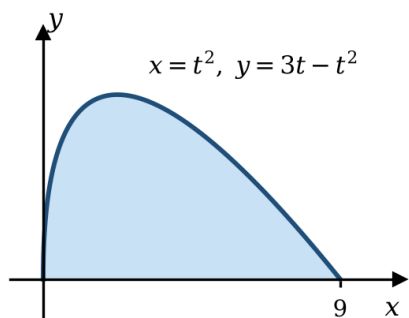
- (a) Show that the x-coordinate of P is 1. [2]
- (b) Express $\frac{6}{(x+1)(x+2)}$ in partial fractions. [3]
- (c) Hence show that the area of the shaded region is $6\ln\frac{4}{3} - 1$. [4]

- 30** Show that $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} dx = \ln\frac{4}{3}$. [7]

Parametric Integration

- 1** The curve C has parametric equations $x = 2t$, $y = t^2$, for $0 \leq t \leq 3$. Find the area of the region bounded by C, the x-axis and the line $x = 6$. [3]
- 2** A curve has parametric equations $x = t^2$, $y = 4t$, $t > 0$. Find the area of the region bounded by the curve, the x-axis and the lines $t = 1$ and $t = 3$ (that is, $x = 1$ and $x = 9$). [3]
- 3** A curve has parametric equations $x = t + 1$, $y = t^2$. Find the area under the curve between $x = 2$ and $x = 4$. [3]
- 4** A curve has parametric equations $x = e^t$, $y = e^{-t}$. Find the area under the curve between $x = 1$ and $x = e$. [4]
- 5** The curve C has parametric equations $x = t^3$, $y = t$, $t \geq 0$. Find the area under C between $t = 1$ and $t = 2$. [3]
- 6** A curve has parametric equations $x = 3t$, $y = 2 + t^2$. Write the area under the curve between $x = 0$ and $x = 6$ as an integral in terms of t , and evaluate it. [3]
- 7** A curve has parametric equations $x = \cos t$, $y = \sin t$, for $0 \leq t \leq \frac{\pi}{2}$. Show that the area under the curve between $x = 0$ and $x = 1$ is $\frac{\pi}{4}$. [5]
- 8** A curve has parametric equations $x = t^3 + 1$, $y = 2t$. Find the area under the curve between $x = 2$ and $x = 9$. [6]

- 9** The diagram shows the curve with parametric equations $x = t^2$, $y = 3t - t^2$, for $0 \leq t \leq 3$. Find the area of the shaded region bounded by the curve and the x-axis.



[5]

- 10** A curve has parametric equations $x = e^{2t}$, $y = e^t$. Find the exact area under the curve between $t = 0$ and $t = \ln 2$. [5]

- 11** The curve C has parametric equations $x = 2\sin t$, $y = 4\cos t$, $0 \leq t \leq \frac{\pi}{2}$. Find the area under C between $x = 0$ and $x = 2$. [6]

- 12** A curve has parametric equations $x = 2t + 1$, $y = \frac{3}{t}$, for $t > 0$. Find the area under the curve between $t = 1$ and $t = e$. [5]

- 13** A curve has parametric equations $x = t^2$, $y = \ln t$, for $t \geq 1$. Find the exact area under the curve between $t = 1$ and $t = e$. [6]

- 14** A curve has parametric equations $x = \tan t$, $y = \cos^2 t$, for $0 \leq t < \frac{\pi}{2}$. Show that the area under the curve between $x = 0$ and $x = 1$ is $\frac{\pi}{4}$. [5]

- 15** A curve has parametric equations $x = 4\sin t$, $y = \sin 2t$, for $0 \leq t \leq \frac{\pi}{2}$. Find the area under the curve between $x = 0$ and $x = 4$. [6]

- 16** The curve C has parametric equations $x = t^2$, $y = 2t + 1$, for $t \geq 0$. The region R is bounded by C, the y-axis and the lines $y = 1$ and $y = 5$. Find the area of R by integrating with respect to y . [5]

- 17** A curve has parametric equations $x = \frac{1}{t}$, $y = t^2 + 1$, for $t > 0$. A student finds the area under the curve between $x = \frac{1}{2}$ and $x = 1$ as follows:

$$\text{“} \frac{dx}{dt} = -\frac{1}{t^2}, \text{ so area} = \int_1^2 (t^2 + 1) \left(-\frac{1}{t^2}\right) dt = -\frac{3}{2} \text{.”}$$

- (a) Explain why this answer cannot be correct and identify the error. [2]
 (b) Find the correct area. [3]

- 18** The ellipse E has parametric equations $x = 2\cos t$, $y = 3\sin t$, for $0 \leq t < 2\pi$. Show that the area enclosed between the upper half of E and the x-axis is 3π . [5]

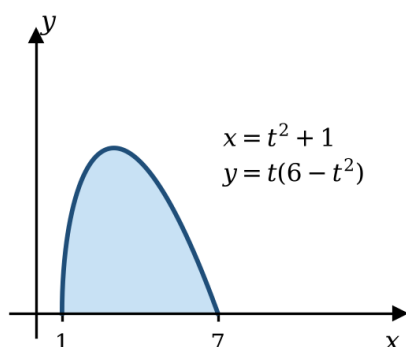
- 19** A curve C has parametric equations $x = 3t + 1$, $y = \frac{18}{t^2}$, for $t > 0$.
The region R is bounded by C, the x-axis and the lines $x = 4$ and $x = 10$.
- (a) Show that the area of R is given by $\int_1^3 \frac{54}{t^2} dt$. [3]
- (b) Hence find the area of R. [2]
- (c) Show that the Cartesian equation of C is $y = \frac{162}{(x-1)^2}$, and verify your answer to part (b) by integrating with respect to x . [4]

- 20** A curve C has parametric equations $x = 9 - t^2$, $y = t(t + 1)$, for $0 \leq t \leq 3$.
C meets the x-axis at the point P and the y-axis at the point Q.
- (a) Find the coordinates of P and Q. [2]
- (b) Find the area of the region bounded by C, the x-axis and the y-axis. [6]

- 21** A curve has parametric equations $x = t^2$, $y = te^{-t}$, for $t \geq 0$.
Find the exact area under the curve between $t = 0$ and $t = 1$. [7]

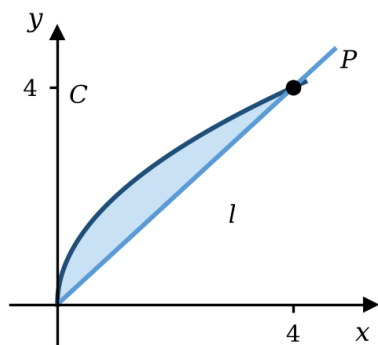
- 22** A curve has parametric equations $x = 3\cos t$, $y = 2\sin 2t$, for $0 \leq t \leq \frac{\pi}{2}$.
Find the area of the region bounded by the curve, the x-axis and the y-axis. [7]

- 23** The diagram shows the curve with parametric equations $x = t^2 + 1$, $y = t(6 - t^2)$, for $t \geq 0$.
The curve meets the x-axis at the points A and B.
Find the coordinates of A and B, and the exact area of the shaded region.



[7]

- 24** The curve C has parametric equations $x = t^2$, $y = 2t$, for $t \geq 0$. The point P on C has parameter $t = 2$, and the line l passes through the origin O and P.
Find the area of the region bounded by C and l.



[7]

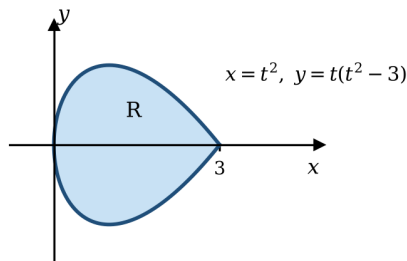
- 25** A curve has parametric equations $x = e^t$, $y = \frac{1}{1 + e^t}$.
Show that the area under the curve between $x = 1$ and $x = 3$ is $\ln 2$. [6]

- 26** A curve C has parametric equations $x = 6\cos \theta$, $y = 2\sin \theta$, for $0 \leq \theta \leq \frac{\pi}{2}$.

The region R is bounded by C, the line $y = 1$ and the y-axis.

- (a) Find the value of θ , and the x-coordinate, at the point where C meets the line $y = 1$. [2]
 (b) Show that the area of R is given by $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12\cos^2 \theta \, d\theta$. [3]
 (c) Hence find the exact area of R. [5]

- 27** The diagram shows the closed loop formed by the curve with parametric equations $x = t^2$, $y = t(t^2 - 3)$, for $-\sqrt{3} \leq t \leq \sqrt{3}$. The loop meets the x-axis at the origin O and at the point A.



- (a) Find the coordinates of A. [2]
 (b) Show that the area of the region R enclosed by the loop is $\frac{24\sqrt{3}}{5}$. [7]

- 28** The ellipse E has parametric equations $x = a \cos t$, $y = b \sin t$, for $0 \leq t < 2\pi$, where a and b are positive constants.

- (a) Show that the area of the region in the first quadrant bounded by E and the axes is $\frac{\pi a b}{4}$. [6]
 (b) Hence write down the area enclosed by E, and state the area enclosed by the ellipse with $a = 5$ and $b = 3$. [3]
 (c) Show that a circle is a special case, and state the corresponding area formula. [2]

- 29** A curve has parametric equations $x = \cos^3 t$, $y = \sin^3 t$, for $0 \leq t \leq \frac{\pi}{2}$.

- (a) Show that the area of the region bounded by the curve and the axes is $3 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t \, dt$. [4]
 (b) Use $\sin 2t = 2\sin t \cos t$ and $\cos 2t = 1 - 2\sin^2 t$ to show that $\sin^4 t \cos^2 t = \frac{1}{8} \sin^2 2t (1 - \cos 2t)$. [3]
 (c) Hence find the exact area. [5]

- 30** A circle of radius a rolls along the x-axis. A point on its circumference traces the curve with parametric equations $x = a(t - \sin t)$, $y = a(1 - \cos t)$ for $0 \leq t \leq 2\pi$ (one arch of a cycloid).

- (a) Find $\frac{dx}{dt}$ and explain why the area under one arch is $\int_0^{2\pi} y \frac{dx}{dt} \, dt$ between suitable limits. [3]
 (b) Show that the area under one arch is $3\pi a^2$. [6]
 (c) Hence state the ratio of the area under one arch to the area of the rolling circle. [2]

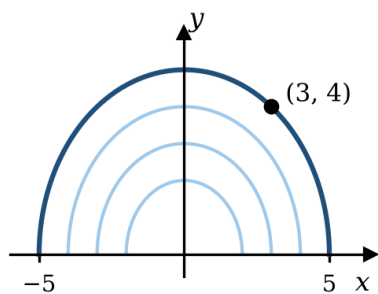
Differential Equations

- 1** Show that $y = 3e^{2x}$ is a solution of the differential equation $\frac{dy}{dx} = 2y$. [2]

- 2** Find the general solution of the differential equation $\frac{dy}{dx} = 4x$. [2]

- 3 Solve $\frac{dy}{dx} = 3y$, given that $y = 2$ when $x = 0$. [2]
- 4 Find the general solution of $\frac{dy}{dx} = \frac{x}{y}$. [2]
- 5 The rate of increase of the population P of a colony is proportional to P . Write down a differential equation for P in terms of the time t , using k for the constant of proportionality. [2]
- 6 Find the general solution of $\frac{dy}{dx} = \frac{y}{x}$, for $x > 0$, $y > 0$. [2]
- 7 Find the general solution of $\frac{dy}{dx} = \frac{x}{y^2}$, giving your answer in the form $y^3 = f(x)$. [4]
- 8 Solve $\frac{dy}{dx} = 3y^2$, given that $y = 1$ when $x = 0$. Give y in terms of x . [4]
- 9 Solve the differential equation $\frac{dy}{dx} = y \cos x$, given that $y = 2$ when $x = 0$. [4]
- 10 Solve $\frac{dy}{dx} = xe^{-y}$, given that $y = 0$ when $x = 0$. Give y in terms of x . [4]
- 11 Find the general solution of the differential equation $\frac{dy}{dx} = 2e^{x-y}$, giving y in terms of x . [3]
- 12 The population P of a town t years after 2020 satisfies $\frac{dP}{dt} = 0.04P$, and $P = 500\,000$ when $t = 0$.
Find P in terms of t , and the time for the population to double, to 3 significant figures. [4]
- 13 Find the general solution of $x\frac{dy}{dx} = y - 1$, for $x > 0$ and $y > 1$, giving y in terms of x . [3]
- 14 Solve the differential equation $\frac{dy}{dx} = \frac{x}{y(1+x^2)}$, given that $y = 2$ when $x = 0$, and $y > 0$. Give y in terms of x . [5]
- 15 A mug of tea at 90°C is left in a room at 20°C . The temperature $\theta^\circ\text{C}$ at time t minutes satisfies $\frac{d\theta}{dt} = -k(\theta - 20)$, where k is a positive constant. After 10 minutes the tea is at 60°C .
Find the value of k and the time taken for the tea to cool to 30°C , to 3 significant figures. [7]
- 16 Water drains from a tank so that the volume V litres at time t minutes satisfies $\frac{dV}{dt} = -k\sqrt{V}$, where k is a positive constant. Initially $V = 100$ and after 5 minutes $V = 64$.
Find the time at which the tank is empty. [6]
- 17 Solve the differential equation $\frac{dy}{dx} = y \tan x$, given that $y = 3$ when $x = 0$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [5]
- 18 A radioactive isotope decays so that the mass m satisfies $\frac{dm}{dt} = -km$, where $k > 0$ and t is in years. The half-life of the isotope is 5 years.
(a) Show that $k = \frac{\ln 2}{5}$. [3]
(b) Find the percentage of the isotope remaining after 12 years, and the time for 90% to decay. [4]

- 19** The differential equation $\frac{dy}{dx} = -\frac{x}{y}$ has solution curves that are semicircles, some of which are shown in the diagram. The solution curve C passes through the point (3, 4). Solve the differential equation and find the equation of C. State the gradient of C at (3, 4).



[5]

- 20** Given that $\sin y \cos y \frac{dy}{dx} = x e^x$ and that $y = \frac{\pi}{2}$ when $x = 0$, show that $\sin^2 y = 2e^x(x - 1) + 3$.

[7]

- 21** A fish population P (in hundreds) in a lake satisfies $\frac{dP}{dt} = 0.1P - 20$, where t is the time in years, and $P = 400$ when $t = 0$. The 20 represents fish caught each year.

- (a) Solve the differential equation to show that $P = 200 + 200e^{0.1t}$. [5]
 (b) Find the time at which the population first reaches 1000 (hundred), to 3 significant figures. [2]

- 22** A drug is infused into a patient at a constant rate. The concentration C mg per litre at time t hours satisfies $\frac{dC}{dt} = 8 - 0.5C$, and $C = 0$ when $t = 0$.

- (a) Show that $C = 16(1 - e^{-0.5t})$. [5]
 (b) Find the long-term concentration, and the time for the concentration to reach 12 mg per litre. [3]

- 23** A curve has the property that, at every point (x, y) with $x > 0$, its gradient is twice the gradient of the line joining (x, y) to the origin. The curve passes through the point (2, 12). Form a differential equation, solve it, and find the equation of the curve. [7]

- 24** A raindrop is a sphere of radius r mm. It evaporates at a rate proportional to its surface area, so that $\frac{dV}{dt} = -4\pi k r^2$, where $V = \frac{4}{3}\pi r^3$ is its volume, t is in seconds and $k > 0$.

- (a) Show that $\frac{dr}{dt} = -k$. [3]
 (b) Initially $r = 2$ and after 5 seconds $r = 1$. Find the time at which the raindrop disappears completely. [3]

- 25** A population P of insects satisfies the differential equation $\frac{dP}{dt} = kP^2$, where t is the time in weeks and k is a positive constant. Initially $P = 1000$ and after 4 weeks $P = 2000$.

- (a) Solve the differential equation to show that $P = \frac{8000}{8 - t}$. [6]
 (b) Find the value of t at which the model predicts that P becomes infinite, and explain why the model is unrealistic for large t . [2]

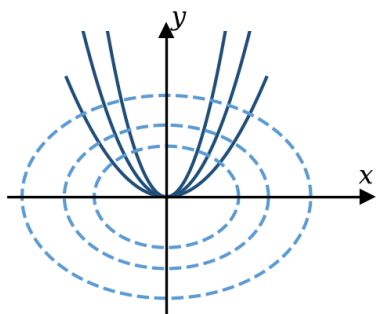
- 26** A tank contains 200 litres of pure water. Brine containing 0.5 kg of salt per litre flows in at 2 litres per minute, and the well-mixed solution flows out at 2 litres per minute. Let x kg be the mass of salt in the tank after t minutes.

- (a) Show that $\frac{dx}{dt} = 1 - \frac{x}{100}$. [3]
 (b) Solve to find x in terms of t , and find the time at which the mass of salt is 50 kg. [5]

- 27** A rumour spreads through a school of 50 pupils. The number x who have heard it at time t days satisfies $\frac{dx}{dt} = \frac{x(50-x)}{50}$, with $x = 5$ when $t = 0$.

- (a) Express $\frac{1}{x(50-x)}$ in partial fractions. [2]
 (b) Solve the differential equation to show that $x = \frac{50}{1 + 9e^{-t}}$. [5]
 (c) Find the time at which half the school has heard the rumour, and the long-term value of x . [3]

- 28** The diagram shows the curves $y = kx^2$ (solid) for three values of the constant k , together with three further curves (dashed) that cross each of them at right angles.



- (a) Show that every curve $y = kx^2$ satisfies $\frac{dy}{dx} = \frac{2y}{x}$. [2]
 (b) A curve that crosses every curve $y = kx^2$ at right angles satisfies $\frac{dy}{dx} = -\frac{x}{2y}$. Solve this equation and show that its solutions are $x^2 + 2y^2 = C$. [4]
 (c) Find the equation of the curve of this second family that passes through (2, 1), and the points where it meets $y = x^2$. [4]

- 29** The function $y(x)$ satisfies $\frac{dy}{dx} = y^2 e^x$ with $y = 1$ when $x = 0$.

- (a) Show that $y = \frac{1}{2 - e^x}$. [4]
 (b) Find the largest value of x for which this solution is defined, and describe what happens to y as x approaches that value from below. [4]
 (c) Explain why a solution curve cannot be continued past $x = \ln 2$ as a smooth function. [2]

- 30** A parachutist falls with speed v m s⁻¹ at time t seconds, where $\frac{dv}{dt} = \frac{100 - v^2}{10}$ and $v = 0$ when $t = 0$.

- (a) Show that $\int \frac{1}{100 - v^2} dv = \int \frac{1}{10} dt$, and express $\frac{1}{100 - v^2}$ in partial fractions. [4]
 (b) Hence show that $v = \frac{10(e^{2t} - 1)}{e^{2t} + 1}$. [5]
 (c) State the terminal speed and find the time to reach half of it. [3]

Volumes of Revolution

- 1 The region bounded by the line $y = x$, the x -axis and the line $x = 3$ is rotated through 360° about the x -axis.
Find the exact volume of the solid formed. [4]

- 2 The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is rotated through 2π radians about the x -axis.
Find the exact volume of the solid formed. [4]

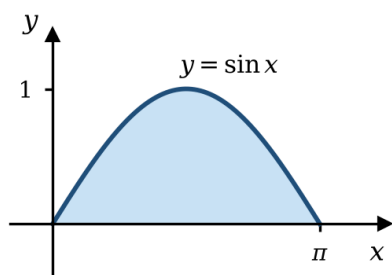
- 3 The region bounded by the curve $y = x^2$, the x -axis and the line $x = 2$ is rotated through 360° about the x -axis.
Find the exact volume of the solid generated. [4]

- 4 The region bounded by $y = e^x$, the axes and the line $x = 1$ is rotated through 360° about the x -axis.
Find the exact volume of the solid formed. [4]

- 5 The region bounded by the curve $y = x^2$ ($x \geq 0$), the y -axis and the line $y = 4$ is rotated through 360° about the y -axis.
Find the exact volume of the solid formed. [3]

- 6 The region bounded by $y = \frac{1}{x}$, the x -axis and the lines $x = 1$ and $x = 3$ is rotated through 360° about the x -axis.
Find the exact volume of the solid formed. [4]

- 7 The diagram shows the curve $y = \sin x$ for $0 \leq x \leq \pi$. The shaded region is rotated through 360° about the x -axis.
Find the exact volume of the solid formed.

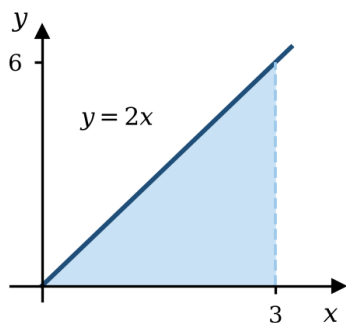


- 8 The region enclosed by the curve $y = 3x - x^2$ and the x -axis is rotated through 360° about the x -axis.
Find the exact volume of the solid formed. [5]

- 9 The semicircle $y = \sqrt{4 - x^2}$ is rotated through 360° about the x -axis to form a sphere of radius 2.
Use integration to show that the volume of the sphere is $\frac{32\pi}{3}$. [5]

- 10** The diagram shows the line $y = 2x$. The shaded triangle, bounded by the line, the x-axis and the line $x = 3$, is rotated through 360° about the x-axis to form a cone.

Find the exact volume of the cone, and check your answer using the formula $\frac{1}{3}\pi r^2 h$.



[5]

- 11** The region bounded by the curve $y = x^3$, the y-axis and the line $y = 8$ is rotated through 360° about the y-axis.

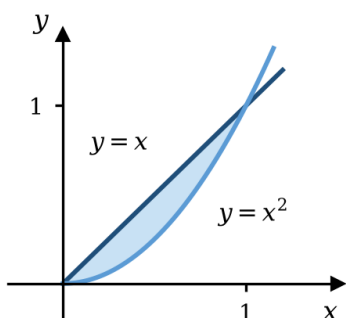
Find the exact volume of the solid formed.

[4]

- 12** The region bounded by the curve $y = 3 + \frac{2}{x}$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed.

[6]

- 13** The diagram shows the line $y = x$ and the curve $y = x^2$. The shaded region between them is rotated through 360° about the x-axis. Find the exact volume of the solid formed.



[6]

- 14** The region R is bounded by the curve $y = \ln x$, the y-axis and the lines $y = 0$ and $y = 2$. R is rotated through 360° about the y-axis.

Find the exact volume of the solid formed.

[5]

- 15** The region bounded by $y = \cos x$, the x-axis and the y-axis (for $0 \leq x \leq \frac{\pi}{2}$) is rotated through 360° about the x-axis.

Find the exact volume of the solid formed.

[5]

- 16** The line $y = 2 + \frac{x}{2}$ between $x = 0$ and $x = 4$ is rotated through 360° about the x-axis to form a frustum of a cone.

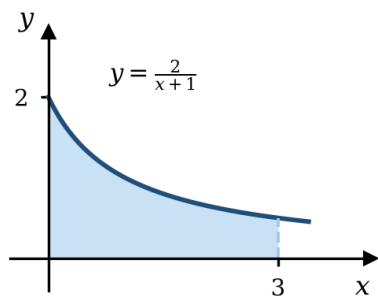
Find the exact volume of the frustum, and verify it using the formula $\frac{\pi h}{3}(R^2 + Rr + r^2)$, where R and r are the end radii and h is the height.

[5]

- 17** The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = k$, where $k > 0$, is rotated through 360° about the x-axis. The volume of the solid is 18π . Find the value of k.

[4]

- 18** The diagram shows the curve $y = \frac{2}{x+1}$. The shaded region is bounded by the curve, the axes and the line $x = 3$, and is rotated through 360° about the x-axis. Find the exact volume of the solid formed.



[5]

- 19** The region bounded by $y = \sec x$, the x-axis, the y-axis and the line $x = \frac{\pi}{4}$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed.

[4]

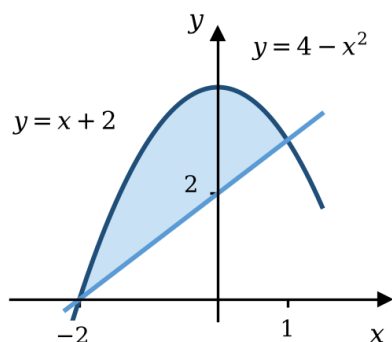
- 20** The region bounded by the curve $y = \frac{x}{(x+2)}$, the x-axis and the line $x = 2$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed.

[7]

- 21** The region bounded by the curve $y = xe^x$, the x-axis and the line $x = 1$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed.

[7]

- 22** The diagram shows the curve $y = 4 - x^2$ and the line $y = x + 2$. The shaded region between them is rotated through 360° about the x-axis. Find the exact volume of the solid generated.



[8]

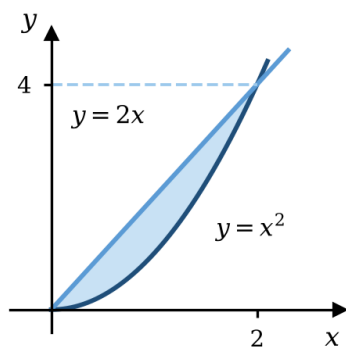
- 23** A curve has parametric equations $x = t^2$, $y = 2t$, for $0 \leq t \leq 3$. The region bounded by the curve, the x-axis and the line $x = 9$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed (a) using the parametric equations, (b) by first finding the Cartesian equation of the curve.

[6]

- 24** A vase is formed by rotating the curve $x^2 = y$, for $0 \leq y \leq 12$, through 360° about the y-axis (x and y in cm). Water is poured into the vase at a constant rate of $20 \text{ cm}^3 \text{ s}^{-1}$. The depth of water is h cm.

- (a) Show that the volume of water V when the depth is h is $V = \frac{\pi h^2}{2}$. [3]
- (b) Find the rate at which the depth is increasing when $h = 5$, to 3 significant figures. [4]

- 25** The diagram shows the curve $y = x^2$ and the line $y = 2x$. The shaded region between them is rotated through 360° about the y -axis. Find the exact volume of the solid formed.



[7]

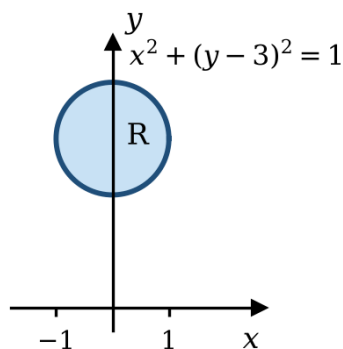
- 26** The region bounded by the curve $y = \ln x$, the x -axis and the line $x = e$ is rotated through 360° about the x -axis. Show that the volume of the solid formed is $\pi(e - 2)$.

[6]

- 27** The circle $x^2 + y^2 = r^2$ is the boundary of a disc. The upper semicircle $y = \sqrt{r^2 - x^2}$ is rotated through 360° about the x -axis.

- (a) Prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. [4]
- (b) A spherical cap of height h ($0 < h < 2r$) is the part of the sphere with $x \geq r - h$. Show that its volume is $\pi(r h^2 - \frac{h^3}{3})$. [6]
- (c) Check that taking $h = 2r$ gives the volume of the whole sphere. [2]

- 28** The diagram shows the circle $x^2 + (y - 3)^2 = 1$. The disc R enclosed by the circle is rotated through 360° about the x -axis to form a ring-shaped solid (a torus).



- (a) Show that the upper and lower halves of the circle have equations $y = 3 + \sqrt{1 - x^2}$ and $y = 3 - \sqrt{1 - x^2}$. [2]
- (b) Show that the volume of the torus is $\pi \int_{-1}^1 12\sqrt{1 - x^2} \, dx$. [4]
- (c) Given that $\int_{-1}^1 \sqrt{1 - x^2} \, dx$ is the area of a semicircle of radius 1, find the exact volume of the torus. [3]

- 29** The region R is bounded by the curve $y = \frac{1}{x}$, the x-axis, the line $x = 1$ and the line $x = k$, where $k > 1$. R is rotated through 360° about the x-axis to form a solid of volume V_k .
- (a) Show that $V_k = \pi(1 - \frac{1}{k})$. [3]
 - (b) Find the limit of V_k as $k \rightarrow \infty$, and explain what this means for the infinitely long solid. [3]
 - (c) Find the smallest integer value of k for which V_k exceeds 99% of this limit. [2]
-
- 30** The ellipse with parametric equations $x = a \cos t$, $y = b \sin t$, $0 \leq t \leq \pi$, is the upper half of an ellipse. It is rotated through 360° about the x-axis to form an ellipsoid (a rugby-ball shape).
- (a) Show that the volume is $V = \pi a b^2 \int_0^\pi \sin^3 t \, dt$. [4]
 - (b) Using $\sin^3 t = \frac{3 \sin t - \sin 3t}{4}$, show that $V = \frac{4}{3} \pi a b^2$. [5]
 - (c) Check that the formula gives the volume of a sphere when $a = b = r$. [1]
-