

Graph Theory

1 (2 marks)

A: 2, B: 4, C: 3, D: 2, E: 3

B1

Sum = 14 and there are 7 edges, $2 \times 7 = 14$

B1

Answer: A 2, B 4, C 3, D 2, E 3; sum 14 = 2×7

2 marks

2 (4 marks)

(a) A connected graph on 4 vertices needs at least 3 edges (it is then a tree); the most is when every pair is joined, $4 \times 3 \div 2 = 6$

B1 B1

Answer: 3 and 6

2 marks

(b) K_4 with one edge removed, for example all edges except AD: AB, AC, BC, BD, CD, with degrees A 2, B 3, C 3, D 2

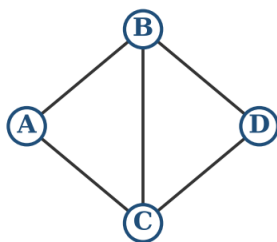
B1

Exactly two vertices (B and C) have odd degree, so the graph is semi-Eulerian

B1

Answer: Semi-Eulerian

2 marks



3 (2 marks)

(a) Each vertex is joined to the other 5

B1

Answer: 5

1 mark

(b) $\frac{6 \times 5}{2} = 15$

B1

Answer: 15

1 mark

4 (2 marks)

It is connected, and it has no cycles

B1

6 vertices and 5 edges ($n - 1$)

B1

Answer: Yes, a tree

2 marks

5 (2 marks)

Entry 1 for each edge, in both positions (symmetric)

M1

Matrix: $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

2 marks

6 (2 marks)

All the degrees are even and the graph is connected

B1

So it has an Eulerian cycle (a closed trail using every edge once)

B1

Answer: Yes

2 marks

7 (3 marks)

Degrees: A 3, B 3, C 2, D 2, E 4

M1

Exactly two vertices (A and B) have odd degree, so there is an Eulerian trail but no Eulerian cycle

A1

A trail must start at one odd vertex and end at the other, for example ABDEACEB

A1

Answer: Trail ABDEACEB; it cannot be closed

3 marks

8 (3 marks)

For example $A - B - D - E - F - C - A$

B1 B1

It is a cycle that visits every vertex exactly once and returns to the start

E1

Answer: e.g. A-B-D-E-F-C-A

3 marks

9 (3 marks)

The sum of all the degrees equals twice the number of edges, so it is even

M1

The even-degree vertices contribute an even sum, so the 5 odd-degree vertices must contribute an even sum

A1

But the sum of five odd numbers is odd, a contradiction

E1

Answer: Impossible: sum of degrees is even

3 marks

10 (4 marks)

(a) 3×4

M1

= 12

A1

Answer: 12

2 marks

(b) Vertices in the set of 3 have degree 4; in the set of 4 have degree 3

B1

There are 4 vertices of odd degree, so there is no Eulerian trail

B1

Answer: Degrees 4 and 3; no Eulerian trail

2 marks

11 (2 marks)

Move D inside triangle ABC and join it to A, B and C

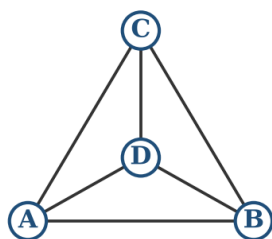
M1

Edges AB, BC, CA form the outer triangle and DA, DB, DC are inside; all 6 edges are present and none cross

A1

Answer: Planar: one vertex inside the triangle of the other three

2 marks

**12 (4 marks)**

(a) Row sums: A 2, B 3, C 3, D 2, E 2

M1

Edges = $12 \div 2 = 6$

A1

Answer: Degrees 2, 3, 3, 2, 2; 6 edges

2 marks

(b) Edges: AB, AC, BC, BD, CE, DE

B1

All vertices are reachable from A, so it is connected

B1

Answer: AB, AC, BC, BD, CE, DE; connected

2 marks

13 (3 marks)

(a) In-degrees: A 1, B 2, C 1, D 2

B1

Out-degrees: A 2, B 1, C 2, D 1

B1

Answer: in-degrees A 1, B 2, C 1, D 2; out-degrees A 2, B 1, C 2, D 1

2 marks

(b) $A \rightarrow B \rightarrow C \rightarrow A$ follows the arrows

B1

Answer: $A \rightarrow B \rightarrow C \rightarrow A$

1 mark

14 (5 marks)

(a) G_1 : A 2, B 4, C 3, D 3, E 4, F 2

B1

G_2 : P 2, Q 4, R 3, S 4, T 3, U 2 (both 4, 4, 3, 3, 2, 2)

B1

Answer: 4, 4, 3, 3, 2, 2 for both

2 marks

(b) A mapping: $A \rightarrow P, B \rightarrow Q, C \rightarrow R, D \rightarrow T, E \rightarrow S, F \rightarrow U$

M1 A1

Check: $AB \rightarrow PQ, BD \rightarrow QT, EF \rightarrow SU$ are edges of G_2

B1

Answer: e.g. $A \rightarrow P, B \rightarrow Q, C \rightarrow R, D \rightarrow T, E \rightarrow S, F \rightarrow U$

3 marks

15 (4 marks)

$V = 8$ and $E = 14$

B1

$F = 2 - V + E = 2 - 8 + 14 = 8$

M1 A1

The diagram has 8 regions including the outer one

B1

Answer: 8 faces

4 marks

16 (6 marks)

(a) The ends of AE (A and E) separate the ends of CF (C and F) around the cycle, so inside the cycle AE and CF must cross

E1

B and D both lie between A and E on the same side, so AE and BD do not cross

E1

Answer: The ends of AE and CF alternate around the cycle; those of AE and BD do not

2 marks

(b) Pairs of chords that cross if drawn on the same side: AE and CF, AE and DF, BD and CF, CF and EG, DF and EG

M1 A1

Put AE inside; then CF and DF must go outside, and then EG (which clashes with CF and DF) goes inside; BD clashes only with CF, so BD goes inside

M1 A1

Answer: Planar: inside AE, BD, EG; outside CF, DF

4 marks

17 (3 marks)

A bipartite graph can have no cycle of odd length (its vertices alternate between two sets)

E1

ABC is a cycle of length 3 (edges AB, BC, CA), so the graph is not bipartite

M1 A1

Answer: Not bipartite: contains the odd cycle ABC

3 marks

18 (5 marks)

(a) A tree with n vertices has $n - 1$ edges: 7

B1

Sum of degrees = $2 \times 7 = 14$

B1

Answer: 7 edges; degree sum 14

2 marks

(b) Vertices of degree 1: $8 - 3 = 5$

M1

Sum of degrees = $4 + 3 + 2 + 5 \times 1 = 14$

M1

This equals the sum 14 found in (a), so the degrees are consistent

A1

Answer: 5 vertices of degree 1

3 marks

19 (4 marks)

(a) K_5 has 10 edges and G has 6, so the complement has 4 edges

M1

Edges: AD, AE, BE, CD

A1

Answer: AD, AE, BE, CD

2 marks

(b) Complement: A 2, B 1, C 1, D 2, E 2

B1

Degree in $G = 4 -$ degree in complement: A 2, B 3, C 3, D 2, E 2

B1

Answer: A 2, B 3, C 3, D 2, E 2

2 marks

20 (5 marks)

(a) Model people as vertices and handshakes as edges: the sum of the degrees would be $9 \times 3 = 27$

M1

The sum of the degrees is twice the number of edges, so it must be even; 27 is odd, so it is impossible

A1 E1

Answer: Impossible: degree sum 27 is odd

3 marks

(b) Place the 9 people round a circle; each shakes hands with the 2 nearest people on each side

M1

Each person has 4 handshakes, giving $9 \times 4 \div 2 = 18$ handshakes in all

A1

Answer: Possible: 9 people in a circle, each shaking hands with the two nearest on each side

2 marks

21 (6 marks)

(a) Degrees: A 5, B 3, C 3, D 3, so all four vertices have odd degree

B1

A trail using every edge once needs at most two odd vertices

M1

There are four, so no such walk exists

A1

Answer: Impossible: four odd vertices

3 marks

(b) The new bridge changes the degrees of its two end vertices by 1, so it must join two odd vertices; any pair works, for example A and B

M1

A and B become even, leaving C and D odd: the walk starts at C and ends at D (or the reverse)

A1

In general the walk starts and ends at the two districts that are not joined by the new bridge; there are 6 possible pairs

A1

Answer: Any of the 6 pairs of districts; the walk starts and finishes at the two districts not joined by the new bridge (e

22 (7 marks)

(a) $K_{3,3}$ has $V = 6$ and $E = 9$

B1

$2V - 4 = 8$ and $9 > 8$, so the inequality fails, so $K_{3,3}$ is not planar

M1 A1

Answer: Not planar: $9 > 8$

3 marks

(b) $V = 6$, $E = 8$ and $2V - 4 = 8$, so $E \leq 2V - 4$ holds (with equality)

B1

Place the 2 vertices at the top and bottom and the 4 vertices in a row between them; join each of the 4 to both. No edges cross

M1 A1

Answer: Planar

3 marks

(c) A graph can satisfy the inequality and still be non-planar (for example a graph containing a subdivision of $K_{3,3}$ plus extra vertices)

E1

Answer: The inequality is necessary, not sufficient

1 mark

23 (6 marks)

(a) Sum = 18, so 9 edges

B1

Example: edges AB, AC, BC, BD, BE, CE, DE, DF, EF has degrees A 2, B 4, C 3, D 3, E 4, F 2

M1 A1

Answer: 9 edges; e.g. the graph above

3 marks

(b) Two vertices of degree 5 are each joined to all the other 5 vertices

M1

So every vertex has degree at least 2, which contradicts the vertex of degree 1

A1 E1

Answer: Not possible

3 marks

24 (6 marks)

(a) Take a longest path in the tree; its end vertex has no neighbour off the path (it would give a longer path) and no neighbour on the path other than the previous vertex (that would give a cycle)

E1

So the end vertex has degree 1

E1

Answer: Take a longest path: its end has degree 1

2 marks

(b) $n = 1$: no edges, $0 = 1 - 1$, true

B1

Assume true for k vertices. A tree with $k + 1$ vertices has a vertex of degree 1; remove it and its edge, leaving a tree with k vertices and $k - 1$ edges

M1 A1

So the tree with $k + 1$ vertices has $(k - 1) + 1 = k$ edges, and the result holds for all n by induction

A1

Answer: $n - 1$ edges

4 marks

25 (4 marks)

(a) In-degrees and out-degrees: A: in 2, out 2, B: in 4, out 4, C: in 2, out 2, D: in 2, out 2

B1 B1

Answer: Equal at every junction

2 marks

(b) Because the in- and out-degrees are equal and the graph is connected, a closed trail exists

M1

One route is $A \rightarrow B \rightarrow D \rightarrow B \rightarrow C \rightarrow A$

A1

Answer: $A \rightarrow B \rightarrow D \rightarrow B \rightarrow C \rightarrow A$

2 marks

26 (6 marks)

$$(a) A^2 = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix} \quad \text{M1 A1}$$

$$\text{Answer: } \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix} \quad \text{2 marks}$$

(b) Entry (A, D) of A^2 is 2 B1

Walks: A-B-D and A-C-D B1

Answer: 2 walks: A-B-D, A-C-D 2 marks

(c) Entry (A, A) of $A^3 = (\text{row A of } A^2) \times (\text{column A of } A) = (2, 1, 1, 2) \cdot (0, 1, 1, 0)$ M1

$$= 2 \times 0 + 1 \times 1 + 1 \times 1 + 2 \times 0 = 2 \quad \text{A1}$$

Answer: 2 2 marks

27 (8 marks)

(a) A cycle alternates between the two sets, so it has equally many vertices from each set M1

A Hamiltonian cycle of $K_{3,4}$ would need $3 = 4$, impossible A1

Answer: Impossible: $3 \neq 4$ 2 marks

(b) Fix the start at a vertex A of the first set; the cycle alternates between sets: A, then one of 3 vertices of the second set, then one of 2 of the first set, then one of 2 of the second, then 1 of the first, then 1 of the second M1

Count = $3 \times 2 \times 2 \times 1 \times 1 = 12$ directed cycles from A, so $12 \div 2 = 6$ distinct cycles A1 A1

Answer: 6 3 marks

(c) Starting from a fixed vertex: $4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 144$ directed cycles M1 A1

$$144 \div 2 = 72 \quad \text{A1}$$

Answer: 72 3 marks

28 (8 marks)

(a) Counting edge-face incidences: each face has at least 3 edges and each edge is counted at most twice, so $3F \leq 2E$ M1 A1

$$F = 2 - V + E, \text{ so } 3(2 - V + E) \leq 2E \quad \text{M1}$$

$$6 - 3V + 3E \leq 2E \text{ gives } E \leq 3V - 6 \quad \text{A1}$$

Answer: $E \leq 3V - 6$ 4 marks

(b) K_5 : $V = 5, E = 10$; $3V - 6 = 9$ B1

$10 > 9$, so K_5 is not planar A1

Answer: $10 > 9$ 2 marks

(c) $K_{3,3}$: $V = 6, E = 9$ and $3V - 6 = 12$, so $9 \leq 12$ and the inequality is satisfied B1

In a bipartite graph every face has at least 4 edges, which gives $E \leq 2V - 4 = 8 < 9$ E1

Answer: $E \leq 2V - 4$ 2 marks

29 (7 marks)

(a) This is true for connected graphs: all degrees even is both necessary and sufficient for an Eulerian cycle B1

(It would be false for a disconnected graph such as two separate triangles, which is why connectedness matters) E1

Answer: True for connected graphs 2 marks

(b) True: a vertex on the cycle has two different cycle neighbours B1

Answer: True 1 mark

(c) False: two triangles sharing one vertex (a bowtie) has all degrees at least 2 B1

The shared vertex would have to be visited twice, so there is no Hamiltonian cycle E1

Answer: False: the bowtie graph 2 marks

(d) A vertex of degree 9 is joined to all other vertices, so every other vertex is a leaf of degree 1 M1

So no vertex has degree 8: the claim is false A1

Answer: False**2 marks****30 (8 marks)****(a)** Degrees: A 2, B 4, C 2, D 2, E 4, F 2, G 2, H 2**M1**

All degrees are even and the graph is connected, so there is an Eulerian cycle

A1**Answer: All degrees even, connected****2 marks****(b)** Split into closed trails and join them, e.g. triangle ABC, then the cycles through B and E**M1**

One cycle: A-B-D-E-F-H-G-E-B-C-A

A1 A1**Answer: e.g. A-B-D-E-F-H-G-E-B-C-A****3 marks****(c)** Starting at any vertex and following unused edges, a walk can only get stuck at its start (every other vertex entered by an edge has an unused exit because its degree is even), so it closes into a cycle**E1 E1**

Remove the cycle: degrees stay even; repeat. A connected graph's cycles share vertices, so they can be joined into one closed trail

E1**Answer: Even degrees allow every walk to return to its start****3 marks**

Minimum Spanning Trees

1 (2 marks)

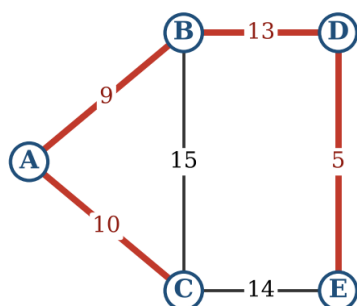
A spanning tree is a subgraph that includes every vertex, is connected and contains no cycles

B1A tree on 8 vertices has $8 - 1 = 7$ edges**B1****Answer: Connects all vertices with no cycles; 7 edges****2 marks****2 (3 marks)**

Edges in ascending order of weight: DE 5, AB 9, AC 10, BD 13, CE 14, BC 15

M1

Accept in turn: DE (5), AB (9), AC (10), BD (13)

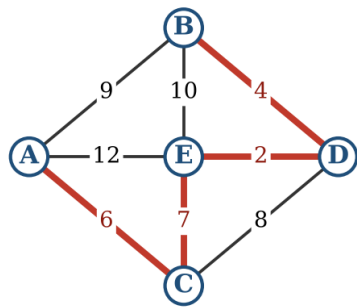
A1Total weight = $5 + 9 + 10 + 13 = 37$ **A1****Answer: DE, AB, AC, BD; weight 37****3 marks****3 (3 marks)**

Start at A. Vertices are added in the order: A, C, E, D, B

M1

Edges chosen in turn: AC (6), CE (7), ED (2), DB (4)

A1Total weight = $6 + 7 + 2 + 4 = 19$ **A1****Answer: Vertices A, C, E, D, B; weight 19****3 marks**

**4 (3 marks)**

(a) AB and BE are already accepted, so A, B and E are already connected; adding AE makes the cycle A-B-E, so it is rejected

B1

Answer: Rejected

1 mark

(b) D is in the component C, D and E is in A, B, E: different components, so DE is accepted

B1

Answer: Accepted

1 mark

(c) F is not yet connected to anything, so CF joins a new vertex without a cycle and is accepted

B1

Answer: Accepted

1 mark**5 (3 marks)**

Shortest distances in order: AC 4, BC 5, CD 6; these do not form a cycle

M1 A1

Total = $4 + 5 + 6 = 15$ km

A1

Answer: 15 km

3 marks**6 (2 marks)**

Order: BC 6, DE 7, AC 8, CD 9, BE 10, AB 12, CE 14, AD 15

M1

BC, DE, AC and CD are accepted; BE is rejected because it would form a cycle (B and E are already joined through C and D). The tree is complete with 4 edges for 5 vertices, so AB, CE and AD are not considered

A1

Answer: Accept BC, DE, AC, CD (weight 30); reject BE

2 marks**7 (4 marks)**

Edges in ascending order of weight: AB 8, CE 9, EF 11, BD 12, AC 16, BE 18, DE 21, BC 24, DF 25

M1

Accept in turn: AB (8), CE (9), EF (11), BD (12), AC (16)

A1

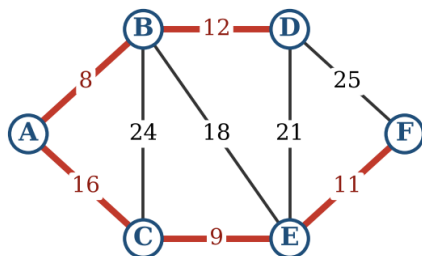
No edge needs to be rejected

A1

Total weight = $8 + 9 + 11 + 12 + 16 = 56$

A1

Answer: AB, CE, EF, BD, AC; weight 56

4 marks**8 (4 marks)**

Start at A. Vertices are added in the order: A, B, C, E, D, F

M1

Edges chosen in turn: AB (7), BC (3), CE (8), ED (6), DF (14)

A1

Total weight = $7 + 3 + 8 + 6 + 14 = 38$

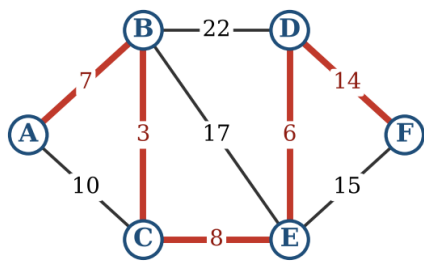
A1

Check: 5 edges for 6 vertices

B1

Answer: Vertices A, B, C, E, D, F; edges AB, BC, CE, ED, DF; weight 38

4 marks

**9 (4 marks)**

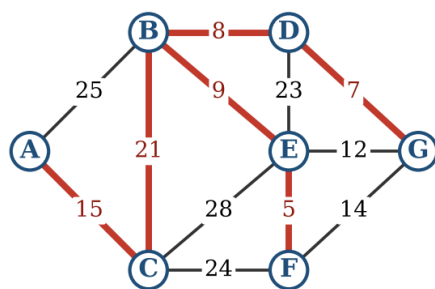
Start at A. Vertices are added in the order: A, C, B, F, E, D
 Edges chosen in turn: AC (6), CB (4), CF (12), FE (15), ED (9)
 Total weight = $6 + 4 + 12 + 15 + 9 = 46$
 Check: 5 edges for 6 vertices
Answer: 46 km

M1
 A1
 A1
 B1
 4 marks

10 (4 marks)

Edges in ascending order of weight: EF 5, DG 7, BD 8, BE 9, EG 12, FG 14, AC 15, BC 21, DE 23, CF 24, AB 25, CE 28
 Accept in turn: EF (5), DG (7), BD (8), BE (9), AC (15), BC (21)
 Reject (would form a cycle): EG (12), FG (14)
 Total weight = $5 + 7 + 8 + 9 + 15 + 21 = 65$
Answer: EF, DG, BD, BE, AC, BC; weight 65

M1
 A1
 A1
 A1
 4 marks

**11 (4 marks)**

(a) Order: CD 3.4, AC 3.8, EF 3.9, BC 4.1, DE 4.6 (accepted; 5 edges for 6 vertices)
 CD, AC, EF, BC, DE: total = $3.4 + 3.8 + 3.9 + 4.1 + 4.6 = 19.8$ km
Answer: 19.8 km
 (b) $19.8 \times 2400 = £47520$
Answer: £47520

M1 A1
 A1
 3 marks
 B1
 1 mark

12 (4 marks)

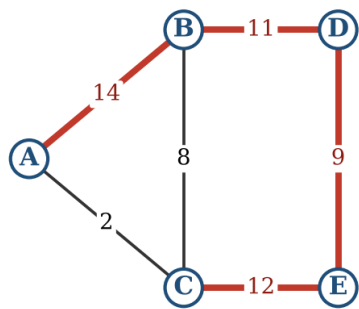
(a) A spanning tree of a 5-cycle has 4 of the 5 edges, so it is found by deleting exactly one edge
 Deleting any one of the 5 edges leaves a tree (a path), giving 5 spanning trees
Answer: 5
 (b) Total of all edges = $6 + 4 + 7 + 5 + 8 = 30$
 Delete the heaviest edge EA (8): weight = $30 - 8 = 22$
Answer: Delete EA; weight 22

M1
 A1
 2 marks
 M1
 A1
 2 marks

13 (3 marks)

Order from the largest weight: AB 14, CE 12, BD 11, DE 9, BC 8, AC 2
 Accept in turn, rejecting any edge that forms a cycle: AB (14), CE (12), BD (11), DE (9)
 Total weight = $14 + 12 + 11 + 9 = 46$
Answer: 46

M1
 A1
 A1
 3 marks

**14 (5 marks)**

Start at D. Vertices are added in the order: D, E, C, B, A, G, F

Edges chosen in turn: DE (5), EC (8), CB (7), BA (4), EG (21), GF (11)

Total weight = $5 + 8 + 7 + 4 + 21 + 11 = 56$

Check: 6 edges for 7 vertices, tree connects all sites

Draw: edges DE, EC, CB, BA, EG, GF

Answer: Vertices D, E, C, B, A, G, F; edges DE, EC, CB, BA, EG, GF; weight 56

M1

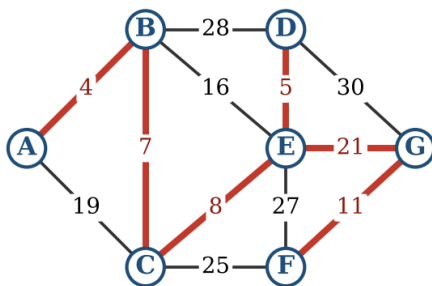
A1

A1

B1

A1

5 marks

**15 (4 marks)**

Edges in ascending order of weight: EF 3, DG 4, EG 6, AB 7, BC 9, CF 10, BE 12, EH 13, DE 18, FH 19, GH 23, CE 25, AC 32, BD 34

Accept in turn: EF (3), DG (4), EG (6), AB (7), BC (9), CF (10), EH (13)

Reject (would form a cycle): BE (12)

Total weight = $3 + 4 + 6 + 7 + 9 + 10 + 13 = 52$

Answer: EF, DG, EG, AB, BC, CF, EH; rejected BE; weight 52

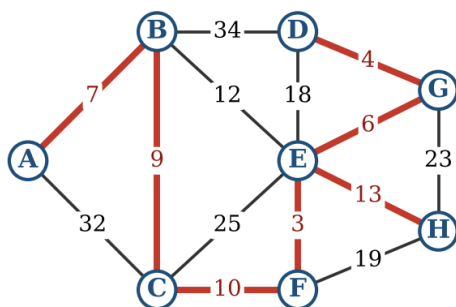
M1

A1

A1

A1

4 marks

**16 (6 marks)**

(a) Edges in ascending order of weight: BD 4, AB 5, AC 5, BC 5, CE 6, EF 6, DE 7, DF 8, BE 9

Accept in turn: BD (4), AB (5), AC (5), CE (6), EF (6)

Reject (would form a cycle): BC (5)

Total weight = $4 + 5 + 5 + 6 + 6 = 26$

Answer: weight 26

(b) When edges of equal weight form a choice, a different choice gives another tree of the same

M1

A1

A1

A1

4 marks

M1

weight	
A second minimum spanning tree: AB, BC, BD, CE, EF (weight 26)	A1
Answer: Another tree of weight 26	2 marks
17 (7 marks)	
(a) Edges in ascending order of weight: AC 8, BD 10, BE 11, EF 13, CE 18, DE 19, AB 21, BC 23, DF 24	M1
Accept in turn: AC (8), BD (10), BE (11), EF (13), CE (18)	A1
No edge needs to be rejected	A1
Total weight = $8 + 10 + 11 + 13 + 18 = 60$	A1
Answer: weight 60	4 marks
(b) Start with DF (24) already accepted, then continue Kruskal's algorithm	M1
Edges: DF (24), AC (8), BD (10), BE (11), CE (18)	A1
Weight = 71	A1
Answer: weight 71	3 marks
18 (5 marks)	
(a) Kruskal on the remaining edges gives: BD (3), EF (4), BE (10), AB (19), BC (21)	M1 A1
Path from A to C: A-B-C	A1
Answer: Path A-B-C	3 marks
(b) Edge AC replaces the heaviest edge on that path if it is lighter than it; the heaviest edge on the path is 21	M1
So $x \leq 21$ (at $x = 21$ there is a second minimum spanning tree of equal weight)	A1
Answer: $x = 21$	2 marks
19 (8 marks)	
(a) Edges in ascending order of weight: BC 7, DF 11, CE 12, AC 13, DE 16, AB 20, BE 23, BD 24, EF 25	M1
Accept in turn: BC (7), DF (11), CE (12), AC (13), DE (16)	A1
No edge needs to be rejected	A1
Total weight = $7 + 11 + 12 + 13 + 16 = 59$	A1
Answer: weight 59	4 marks
(b) Kruskal's algorithm on all edges gives the tree: BC (7), GD (7), GB (9), DF (11), CE (12), GA (12)	M1 A1
Weight = 58	A1
The new weight is 58, which is 1 less than the answer to (a): the links from G replace the heavier edges AC (13) and DE (16) of the old tree, so adding a town can reduce the total	E1
Answer: weight 58, which is 1 less than in (a)	4 marks
20 (5 marks)	
(a) After AB (8) and BE (5), the arc ED (7) is chosen, then EF (9); EF is chosen before CE, so $x > 9$	M1 A1
CE is the last arc added, so it is cheaper than the other arcs available to C, AC (13) and BC (15), giving $x < 13$	A1
Answer: $9 < x < 13$	3 marks
(b) Total weight = $8 + 5 + 7 + 9 + x = 29 + x$	M1
$29 + x = 40$, so $x = 11$, which satisfies $9 < x < 13$	A1
Answer: $x = 11$	2 marks
21 (9 marks)	
(a) Edges in ascending order of weight: EG 3, BD 8, BC 10, EF 11, CF 12, AC 13, AB 15, DE 20, CE 21, DG 26, FG 28, BE 30	M1
Accept in turn: EG (3), BD (8), BC (10), EF (11), CF (12), AC (13)	A1
No edge needs to be rejected	A1
Total weight = $3 + 8 + 10 + 11 + 12 + 13 = 57$	A1
Answer: weight 57	4 marks
(b) Start at A. Vertices are added in the order: A, C, B, D, F, E, G	M1

Edges chosen in turn: AC (13), CB (10), BD (8), CF (12), FE (11), EG (3)

Total weight = $13 + 10 + 8 + 12 + 11 + 3 = 57$

Answer: weight 57

A1

A1

3 marks

(c) All the edge weights are different so the minimum spanning tree is unique and both algorithms must find it

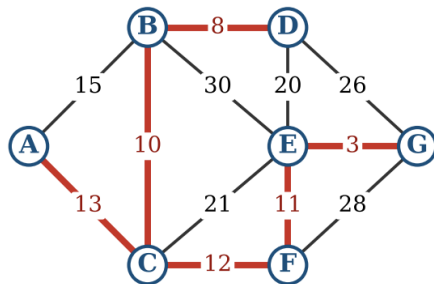
E1

Prim's algorithm works directly from a distance table (matrix) and needs no sorting of all the edges

E1

Answer: The tree is unique

2 marks



22 (8 marks)

(a) Edges in ascending order of weight: AC 3, EH 5, EF 6, CF 7, AB 10, BD 11, GH 16, EG 20, FH 21, DG 24, BC 26, CE 30, DE 32, BE 34

M1

Accept in turn: AC (3), EH (5), EF (6), CF (7), AB (10), BD (11), GH (16)

A1

No edge needs to be rejected

A1

Total weight = $3 + 5 + 6 + 7 + 10 + 11 + 16 = 58$

A1

Answer: weight 58

4 marks

(b) Continue Kruskal's algorithm without CF: tree AC (3), EH (5), EF (6), AB (10), BD (11), GH (16), DG (24)

M1 A1

Weight = 75

A1

CF is replaced by DG (24)

E1

Answer: weight 75

4 marks

23 (9 marks)

(a) Edges in ascending order of weight: EG 3, AB 4, DE 6, BE 7, AC 12, CF 14, CE 17, DG 18, FG 19, BC 25, BD 27, EF 28

M1

Accept in turn: EG (3), AB (4), DE (6), BE (7), AC (12), CF (14)

A1

No edge needs to be rejected

A1

Total weight = $3 + 4 + 6 + 7 + 12 + 14 = 46$

A1

Answer: weight 46

4 marks

(b) Removing CF from the tree splits it into two parts: A, B, C, D, E, G and F

M1

Other edges joining the two parts: EF (28), FG (19)

A1

CF stays in a minimum spanning tree while $y \leq 19$, the smallest of these

A1

Answer: $y = 19$

3 marks

(c) CF is replaced by the cheapest alternative of weight 19

M1

Weight = $46 - 14 + 19 = 51$

A1

Answer: 51

2 marks

24 (5 marks)

(a) Kruskal's algorithm rejects an edge if both its ends are already in the same component

E1

Adding an edge joining two vertices of the same component would create a cycle, so it is rejected, and edges joining different components cannot create a cycle

E1

Answer: Edges within a component are rejected

2 marks

(b) Suppose there were two different minimum spanning trees T and T'. Let e be the lightest edge in one but not the other, say e in T

M1

Adding e to T' creates a cycle containing an edge f not in T, and f is heavier than e (since e is the lightest edge in the difference and the weights are distinct)

A1

Replacing f by e gives a lighter spanning tree than T', contradicting minimality

E1

Answer: Distinct weights give a unique tree**3 marks****25 (7 marks)****(a)** Start at A. Vertices are added in the order: A, C, B, F, E, D, G, H**M1**

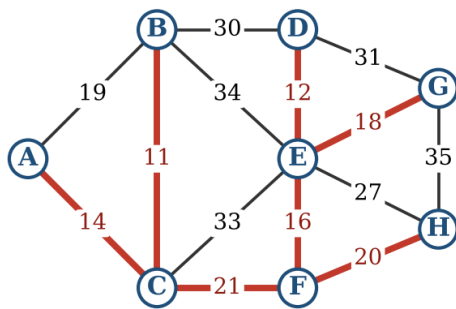
Edges chosen in turn: AC (14), CB (11), CF (21), FE (16), ED (12), EG (18), FH (20)

A1Total weight = $14 + 11 + 21 + 16 + 12 + 18 + 20 = 112$ **A1**

Check: 7 lines for 8 villages

A1**Answer: 112 thousand pounds****4 marks****(b)** Adding HC (9) to the tree creates a cycle with the path H-F-C in the tree (HF 20, FC 21)**M1**

The heaviest edge on the cycle is FC (21), which is heavier than HC (9), so FC is replaced by HC

A1New minimum cost = $112 - 21 + 9 = 100$ thousand pounds**A1****Answer: 100 thousand pounds****3 marks****26 (7 marks)****(a)** Edges in ascending order of weight: DE 3, FG 5, AC 8, BD 9, CE 12, BE 13, DG 14, AB 18, BC 20, EG 21, CF 26, EF 30**M1**

Accept in turn: DE (3), FG (5), AC (8), BD (9), CE (12), DG (14)

A1

Reject (would form a cycle): BE (13)

A1

Path: A-C-E-D-G

A1**Answer: weight 51****4 marks****(b)** Adding AG to T creates a cycle: the path A to G together with AG**M1**

The heaviest edge on the path is DG (14), which is heavier than AG (12)

A1So DG is replaced by AG and the new weight is $51 - 14 + 12 = 49$ **A1****Answer: 49****3 marks****27 (10 marks)****(a)** A spanning tree has 3 edges and K_4 has 6 edges: $C(6, 3) = 20$ sets of 3 edges**M1**The 4 sets that form a triangle contain a cycle (4 triangles), leaving $20 - 4 = 16$ **M1 A1**This equals $4^2 = 16$ **A1****Answer: 16****4 marks****(b)** $6^4 = 1296$ **B1**

A star is decided by its central vertex, so there are 6 stars

B1**Answer: 1296; 6****2 marks****(c)** $K_{2,3}$ has 6 edges and a spanning tree has 4, so there are $C(6, 4) = 15$ edge sets**M1**A set of 4 edges containing a cycle must be a 4-cycle, which uses both vertices of the first set and 2 of the 3 vertices of the second set: $C(3, 2) = 3$ such sets**M1 A1** $15 - 3 = 12$ **A1****Answer: 12****4 marks****28 (8 marks)****(a)** Edges in ascending order of weight: BD 3, FG 4, EF 5, CE 7, CF 9, BE 11, AB 13, AC 19, BC 22, DG 24, DE 26, EG 29**M1**

Accept in turn: BD (3), FG (4), EF (5), CE (7), BE (11), AB (13)

A1

Route in T: A-B-E-F-G with bottleneck 13

A1 A1**Answer: bottleneck 13****4 marks****(b)** Suppose a route had every edge lighter than 13**M1**

Then in Kruskal's algorithm all its edges are considered before the edge of weight 13, so A and G would already be connected, and the bottleneck edge of weight 13 in T would have been rejected (it would form a cycle), a contradiction

A1 E1

Answer: Bottleneck 13 is the minimum

3 marks

(c) For any two vertices, the route between them in the minimum spanning tree has the smallest possible bottleneck over all routes in the network

E1

Answer: Minimax property of the minimum spanning tree

1 mark

29 (6 marks)

(a) $x = 5$: the tree is AE (5), BE (6), AC (9), CD (11), weight 31

M1 A1

$x = 15$: the tree is CE (5), BE (6), AB (8), CD (11), weight 30

A1

Answer: 31 and 30

3 marks

(b) Evaluate the weight for $x = 1, 2, \dots, 19$: 1:27, 2:28, 3:29, 4:30, 5:31, 6:32, 7:33, 8:34, 9:34, 10:34, 11:34, 12:33, 13:32, 14:31, 15:30, 16:29, 17:28, 18:27, 19:26

M1 A1

The least weight is 26, at $x = 19$

A1

Answer: $x = 19$, weight 26

3 marks

30 (10 marks)

(a) Edges in ascending order of weight: AB 3, AC 4, BE 5, DE 8, CF 10, DG 13, BD 16, CE 18, BC 25, EF 27, EG 28, FG 29

M1

Accept in turn: AB (3), AC (4), BE (5), DE (8), CF (10), DG (13)

A1

No edge needs to be rejected

A1

Total weight = $3 + 4 + 5 + 8 + 10 + 13 = 43$

A1

Edges at A in this tree: 2

B1

Answer: weight 43, using 2 edges at A

5 marks

(b) A is a leaf, so the other six vertices are joined by a spanning tree of the network without A, plus one edge at A

E1

Kruskal on the network without A: BE (5), DE (8), CF (10), DG (13), CE (18) = 54

M1 A1

Cheapest edge at A: AB (3), total $54 + 3 = 57$

A1 A1

Answer: weight 57

5 marks

Shortest Path Algorithms

1 (2 marks)

Possible routes: A-C-E (10), A-B-C-E (9), A-B-D-E (18)

M1

Shortest route A-B-C-E, length 9

A1

Answer: A-B-C-E, length 9

2 marks

2 (3 marks)

A-B-D: 9; A-B-C-D: 12; A-C-D: 9; A-C-B-D: 8

M1 A1

Shortest: A-C-B-D, length 8

A1

Answer: A-C-B-D, length 8

3 marks

3 (2 marks)

A final label is the length of the shortest route from the start vertex to that vertex

B1

C has the smaller working value (3), so C is labelled next with final label 3

B1

Answer: Shortest distance from the start; C with 3

2 marks

4 (2 marks)

Direct A-D is 9; A-C-B-D = $2 + 3 + 5 = 10$; A-C-D = $2 + 8 = 10$; A-B-D = $6 + 5 = 11$

M1

Shortest route A-D, length 9

A1

Answer: A-D, length 9

2 marks

5 (2 marks)

Routes: A-B-D = 8, A-C-D = 10, A-C-B-D = $3 + 2 + 2 = 7$

M1

Shortest: A-C-B-D, length 7

A1

Answer: A-C-B-D, length 7

2 marks

6 (2 marks)

The smallest working value among unlabelled vertices is 9

C is labelled next, with final label 9

Answer: C (9)

M1

A1

2 marks

7 (5 marks)

Order of labelling and final labels: A (0), C (8), B (20), E (20), D (30), F (31)

Working values: A: 0; C: 8; B: 20; E: 20; D: 36, 30; F: 31

Route (trace back): A-C-E-F

Shortest distance = 31

Answer: A-C-E-F, length 31

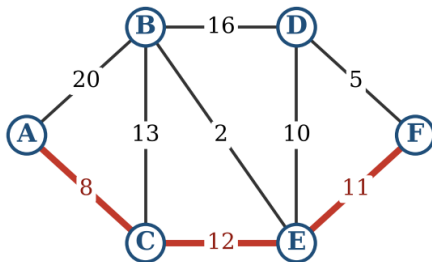
M1

A1 A1

A1

A1

5 marks

**8 (4 marks)**

Order of labelling and final labels: A (0), B (3), D (7), C (8), E (10)

Working values: A: 0; B: 3; D: 7; C: 8; E: 13, 10

Route (trace back): A-B-D

Shortest distance = 7

Answer: A-B-D, length 7

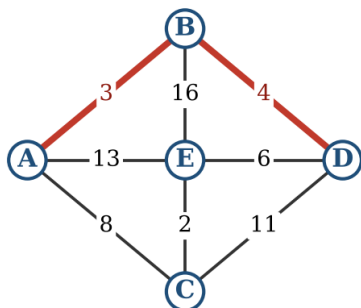
M1

A1

A1

A1

4 marks

**9 (5 marks)**

Order of labelling and final labels: A (0), C (13), B (21), D (23), G (27), E (30), F (31)

Working values: A: 0; C: 13; B: 21; D: 23; G: 27; E: 36, 30; F: 31

Route (trace back): A-B-D-G

Shortest distance = 27

Answer: A-B-D-G, length 27

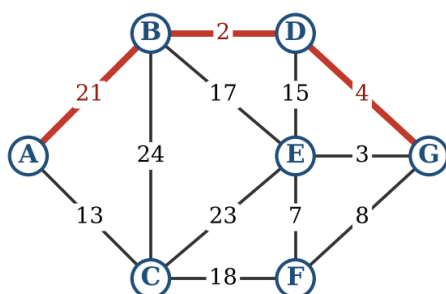
M1

A1 A1

A1

A1

5 marks



10 (4 marks)

Order of labelling and final labels: A (0), B (5), C (8), E (18), F (20), D (22)

M1

Working values: A: 0; B: 5; C: 15, 8; E: 18; F: 20; D: 24, 22

A1

Route (trace back): A-B-C-F

A1

Shortest distance = 20

A1

Answer: A-B-C-F, length 20 km

4 marks

11 (4 marks)

Order of labelling and final labels: A (0), B (16), C (18), D (26), E (28), F (33)

M1

Working values: A: 0; B: 16; C: 18; D: 26; E: 28; F: 34, 33

A1

Route (trace back): A-B-E-F

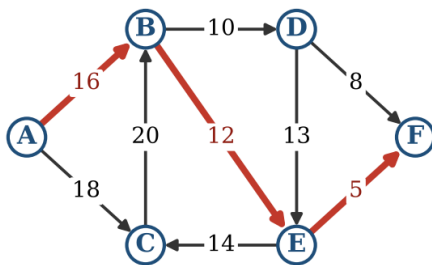
A1

Shortest distance = 33

A1

Answer: A-B-E-F, 33 minutes

4 marks

**12 (5 marks)**

(a) Order of labelling and final labels: A (0), B (10), C (13), D (16), E (16), G (18), F (21)

M1

Route A-C-E, distance 16

A1

Answer: A-C-E, 16 km

2 marks

(b) Shortest E to G: E-G, distance 2

M1 A1

Total = 16 + 2 = 18 km via E

A1

Answer: 18 km

3 marks

13 (6 marks)

(a) Order of labelling and final labels: A (0), B (6), C (15), D (26), E (38)

M1

Working values: A: 0; B: 6; C: 15; D: 26; E: 40, 38

A1

Route (trace back): A-B-D-E

A1

Shortest distance = 38

A1

Answer: A-B-D-E, 38 km

4 marks

(b) Time = $38 \div 20 = 1.90$ hours

M1

= 114 minutes

A1

Answer: 114 minutes

2 marks

14 (6 marks)

Order of labelling and final labels: A (0), C (9), B (13), E (16), D (23), F (28), H (30), G (39)

M1 A1

Working values: A: 0; C: 9; B: 17, 13; E: 25, 16; D: 23; F: 28; H: 30; G: 39

A1 A1

Route (trace back): A-C-B-E-H

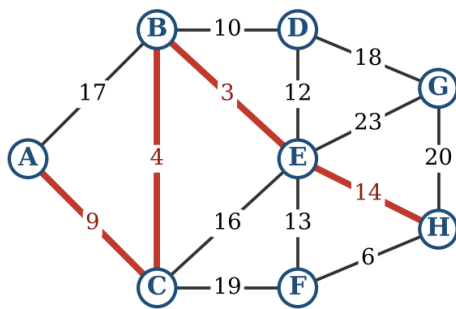
A1

Shortest distance = 30

A1

Answer: A-C-B-E-H, length 30

6 marks

**15 (6 marks)**

Order of labelling and final labels: A (0), D (3), G (9), E (11), H (14), B (15), F (20), C (24), I (24)

M1 A1

Working values: A: 0; D: 3; G: 9; E: 13, 11; H: 14; B: 15; F: 20; C: 26, 24; I: 29, 24

A1 A1

Route (trace back): A-D-G-H-I

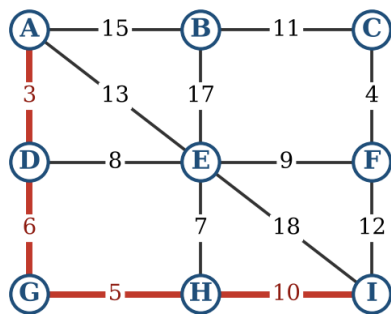
A1

Shortest distance = 24

A1

Answer: A-D-G-H-I, length 24

6 marks

**16 (8 marks)**

(a) Order of labelling and final labels: A (0), B (3), C (7), E (14), D (22), G (26), F (28)

M1 A1

Working values: A: 0; B: 3; C: 8, 7; E: 17, 14; D: 22; G: 26; F: 28

A1

Route (trace back): A-B-C-E-G

A1

Shortest distance = 26

A1

Answer: A-B-C-E-G, length 26

5 marks

(b) Remove C and all roads at C and run Dijkstra again

M1

Order of labelling and final labels: A (0), B (3), E (17), D (22), G (29), F (34)

A1

Route A-B-E-G, length 29 (an increase of 3)

A1

Answer: A-B-E-G, length 29

3 marks

17 (6 marks)

(a) Order of labelling and final labels: A (0), B (2), C (4), E (4), D (6), F (7)

M1

Working values: A: 0; B: 2; C: 5, 4; E: 4; D: 7, 6; F: 7

A1 A1

Shortest distance = 7

A1

Answer: 7

4 marks

(b) Trace back from F along every edge on which final label difference equals the edge length

M1

Routes: A-B-C-F; A-B-E-F

A1

Answer: 2 routes

2 marks

18 (5 marks)

(a) A-B-C-D-E = 6 + 2 + 3 + 5 = 16

M1

Other routes: A-B-D-E = 18, A-C-D-E = 9 + 3 + 5 = 17, so the shortest is 16

A1

Answer: 16

2 marks

(b) A-B-C-D-E = 13 + x; A-B-D-E = 18

M1

13 + x < 18 gives x < 5 (A-C-D-E = 14 + x is always longer than 13 + x); when x = 5 the route

A1

A-B-D-E is equally short

Answer: $0 < x < 5$

2 marks

(c) A-B-C-D-E = 23, A-C-D-E = 24, A-B-D-E = 18, so 18

B1

Answer: 18

1 mark

19 (5 marks)

(a) The algorithm is applied n times (once from each vertex), each application taking order n^2

M1

So the total is $n \times n^2 = n^3$, which is of order n^3

A1

Answer: $n \times n^2 = n^3$

2 marks

(b) Time $\propto n^3$, so the time is multiplied by $(\frac{240}{60})^3 = 4^3 = 64$

M1

$0.15 \times 64 = 9.6$ seconds

A1

Answer: 9.6 seconds

2 marks

(c) Order n^3 describes only the dominant term for large n ; smaller terms in the true running time are ignored, so the time is not exactly proportional to n^3

E1

Answer: Only the dominant term is modelled

1 mark

20 (6 marks)

(a) Row D, column A of the route table is E, so the route starts D-E; then row E, column A is C, and row C, column A is A

M1 A1

Route D-E-C-A, and the distance table gives 14 for D to A ($3 + 4 + 7 = 14$)

A1

Answer: D-E-C-A, length 14

3 marks

(b) From A the nearest is C (7); from C the nearest unvisited is E (4); from E, D (3); from D, B (13)

M1 A1

Return from B to A (10): total $7 + 4 + 3 + 13 + 10 = 37$, so an upper bound is 37

A1

Answer: Tour A-C-E-D-B-A, upper bound 37

3 marks

21 (8 marks)

(a) Initial matrix (rows and columns A, B, C, D):

$$\begin{pmatrix} 0 & 7 & 3 & \infty \\ 7 & 0 & 2 & 4 \\ 3 & 2 & 0 & 9 \\ \infty & 4 & 9 & 0 \end{pmatrix}$$

M1 A1

Answer:

$$\begin{pmatrix} 0 & 7 & 3 & \infty \\ 7 & 0 & 2 & 4 \\ 3 & 2 & 0 & 9 \\ \infty & 4 & 9 & 0 \end{pmatrix}$$

2 marks

(b) After A:

$$\begin{pmatrix} 0 & 7 & 3 & \infty \\ 7 & 0 & 2 & 4 \\ 3 & 2 & 0 & 9 \\ \infty & 4 & 9 & 0 \end{pmatrix}$$

M1

After B:

$$\begin{pmatrix} 0 & 7 & 3 & 11 \\ 7 & 0 & 2 & 4 \\ 3 & 2 & 0 & 6 \\ 11 & 4 & 6 & 0 \end{pmatrix}$$

A1

After C:

$$\begin{pmatrix} 0 & 5 & 3 & 9 \\ 5 & 0 & 2 & 4 \\ 3 & 2 & 0 & 6 \\ 9 & 4 & 6 & 0 \end{pmatrix}$$

A1

After D:

$$\begin{pmatrix} 0 & 5 & 3 & 9 \\ 5 & 0 & 2 & 4 \\ 3 & 2 & 0 & 6 \\ 9 & 4 & 6 & 0 \end{pmatrix} \text{ (final)}$$

A1

Answer:
$$\begin{pmatrix} 0 & 5 & 3 & 9 \\ 5 & 0 & 2 & 4 \\ 3 & 2 & 0 & 6 \\ 9 & 4 & 6 & 0 \end{pmatrix}$$

4 marks

(c) Final matrix entry (A, D) = 9

B1

Route A-C-B-D: $3 + 2 + 4 = 9$

B1

Answer: 9 via A-C-B-D

2 marks

22 (8 marks)

(a) Order of labelling and final labels: A (0), B (1), C (3), F (6), E (9), D (10), I (10), H (16), G (19)

M1 A1

Working values: A: 0; B: 1; C: 3; F: 6; E: 19, 9; D: 10; I: 10; H: 16; G: 19

A1 A1

Route (trace back): A-B-C-F-I

A1

Shortest distance = 10

A1

Answer: A-B-C-F-I, 10 minutes

6 marks

(b) Final labels: A 0, B 1, C 3, F 6, E 9, D 10, I 10, H 16, G 19

M1

Vertices with a larger label than I (10): H (16), G (19); the greatest label is 19 at G

A1

Answer: H and G are further than I; G is furthest

2 marks

23 (6 marks)

(a) In this network every route moves forward to T and no vertex can be revisited, so a vertex's largest label is final once all its predecessors are labelled

E1

In a network with cycles a route could be lengthened indefinitely by going round a cycle, so there is no longest route

E1

Answer: The network has no cycles, so no route can be lengthened by repeating vertices; with a cycle there would be no longest route

(b) S 0, A 5, B 4; C = $\max(5 + 6, 4 + 2) = 11$; D = $\max(5 + 3, 4 + 7) = 11$

M1 A1

T = $\max(11 + 4, 11 + 5) = 16$

A1

Longest route S-B-D-T, length 16

A1

Answer: S-B-D-T, length 16

4 marks

24 (8 marks)

(a) Order of labelling and final labels: A (0), C (7), B (8), D (11), E (17), F (22)

M1

Working values: A: 0; C: 7; B: 8; D: 11; E: 22, 20, 17; F: 22

A1

Route (trace back): A-B-D-F

A1

Shortest distance = 22

A1

Answer: A-B-D-F, 22

4 marks

(b) Add the waiting time at the start of each edge that leaves an intermediate junction

M1

Quickest route including waiting: A-B-D-F, total time 25 minutes

A1 A1

This route is the same as the route in (a)

B1

Answer: A-B-D-F, 25 minutes

4 marks

25 (6 marks)

(a) B (10) was labelled before C (3)

B1

At each stage the unlabelled vertex with the smallest working value must be labelled next, and $3 < 10$

E1

Answer: Wrong order: smallest working value must go first

2 marks

(b) Order of labelling and final labels: A (0), C (3), B (10), E (17), F (23), D (27)

M1

Working values: A: 0; C: 3; B: 16, 10; E: 17; F: 23; D: 27

A1

Route (trace back): A-C-E-F

A1

Shortest distance = 23

A1

Answer: A-C-E-F, length 23

4 marks

26 (8 marks)

(a) Final labels from A: A (0), C (3), B (11), E (18), D (19), F (19), G (24)

M1 A1

Answer: A to C: 3, A to E: 18

2 marks

(b) From C: C (0), A (3), B (14), F (16), E (17), D (22), G (27)

M1

From E: E (0), B (7), D (12), G (15), C (17), A (18), F (18)

M1

C to E: 17, C to G: 27, E to G: 15

Answer: 17, 27, 15

A1
3 marks

(c) A-C-E-G: $3 + 17 + 15 = 35$

M1

A-E-C-G: $18 + 17 + 27 = 62$

M1

Shortest: 35 km

A1

Answer: 35 km

3 marks

27 (10 marks)

(a) Initial matrix:

$$\begin{pmatrix} 0 & 4 & 9 & \infty & \infty \\ 4 & 0 & 3 & 6 & \infty \\ 9 & 3 & 0 & \infty & 2 \\ \infty & 6 & \infty & 0 & 5 \\ \infty & \infty & 2 & 5 & 0 \end{pmatrix}$$

M1 A1

Answer:

$$\begin{pmatrix} 0 & 4 & 9 & \infty & \infty \\ 4 & 0 & 3 & 6 & \infty \\ 9 & 3 & 0 & \infty & 2 \\ \infty & 6 & \infty & 0 & 5 \\ \infty & \infty & 2 & 5 & 0 \end{pmatrix}$$

2 marks

(b) After B:

$$\begin{pmatrix} 0 & 4 & 7 & 10 & \infty \\ 4 & 0 & 3 & 6 & \infty \\ 7 & 3 & 0 & 9 & 2 \\ 10 & 6 & 9 & 0 & 5 \\ \infty & \infty & 2 & 5 & 0 \end{pmatrix}$$

M1 A1

After C:

$$\begin{pmatrix} 0 & 4 & 7 & 10 & 9 \\ 4 & 0 & 3 & 6 & 5 \\ 7 & 3 & 0 & 9 & 2 \\ 10 & 6 & 9 & 0 & 5 \\ 9 & 5 & 2 & 5 & 0 \end{pmatrix}$$

A1 A1

Answer: After B:

$$\begin{pmatrix} 0 & 4 & 7 & 10 & \infty \\ 4 & 0 & 3 & 6 & \infty \\ 7 & 3 & 0 & 9 & 2 \\ 10 & 6 & 9 & 0 & 5 \\ \infty & \infty & 2 & 5 & 0 \end{pmatrix}$$

After C:

$$\begin{pmatrix} 0 & 4 & 7 & 10 & 9 \\ 4 & 0 & 3 & 6 & 5 \\ 7 & 3 & 0 & 9 & 2 \\ 10 & 6 & 9 & 0 & 5 \\ 9 & 5 & 2 & 5 & 0 \end{pmatrix}$$

4 marks

(c) Final matrix:

$$\begin{pmatrix} 0 & 4 & 7 & 10 & 9 \\ 4 & 0 & 3 & 6 & 5 \\ 7 & 3 & 0 & 7 & 2 \\ 10 & 6 & 7 & 0 & 5 \\ 9 & 5 & 2 & 5 & 0 \end{pmatrix}$$

A1

Shortest A to E: 9

B1

Route: A-B-C-E

B1

Dijkstra from A: A (0), B (4), C (7), E (9), D (10), agrees: E = 9

A1

Answer: 9 via A-B-C-E

4 marks

28 (8 marks)

(a) A 0; working B 2, C 5; B has the smallest working value, so B is labelled 2 and D gets working value 9

M1 A1

C labelled 5 (the edge CB is ignored since B is already final), then $D = \min(9, 8) = 8$: D labelled 8

A1

Answer: B = 2, D = 8

3 marks

(b) $A-C-B = 5 + (-4) = 1 < 2$

M1

True shortest to B is 1 and to D is $A-C-B-D = 1 + 7 = 8$ or $A-C-D = 8$, so $D = 8$

A1 A1

Answer: B = 1, D = 8

3 marks

(c) The algorithm labels the vertex with the smallest working value as final because any other route to it must pass through unlabelled vertices with larger working values; with negative weights a later edge could reduce the total

E1 E1

Answer: A later negative edge can reduce a final label

2 marks

29 (6 marks)

(a) Each shortest route moves 2 steps across and 2 steps down: 4 sections

M1

Every step goes right or down, so the length is 4

A1

Answer: 4

2 marks

(b) A: 1; B: 1; D: 1; C: 1; E: $B + D = 2$; G: 1

M1 A1

$F = C + E = 3$; $H = E + G = 3$

A1

$I = F + H = 6$

A1

Answer: 6

4 marks

30 (9 marks)

(a) Order of labelling and final labels: A (0), C (10), B (15), E (24), F (29), G (32), D (33)

M1 A1

Working values: A: 0; C: 10; B: 16, 15; E: 24; F: 29; G: 39, 32; D: 33

A1

Route (trace back): A-C-F-G

A1

Shortest distance = 32

A1

Answer: 32, via A-C-F-G

5 marks

(b) A new road ab of length 6 gives a route A to a, then ab, then b to G, of length $d(A, a) + 6 + d(b, G)$ (or the same with a and b exchanged)

M1

Shortest distances from A: B 15, C 10, D 33, E 24, F 29. Shortest distances to G: B 24, C 22, D 6, E 15, F 3

M1

The possible roads are CD, BF, DF; the best routes using them are CD: 22, BF: 24, DF: 41 (a value over 32 means the road does not help)

A1

The best road is CD, with route A to C (10), road (6), D to G (6), a total of 22 km, a saving of 10 km

A1

Answer: CD, giving 22 km

4 marks

Route Inspection and Network Problems

1 (2 marks)

B and C have degree 3 (the others have degree 2)

B1

A closed route uses an even number of edge-ends at every vertex, so odd vertices force a repeated edge

E1

Answer: B and C

2 marks

2 (2 marks)

Two vertices (degrees 3, 3) are odd

B1

Exactly two odd vertices means semi-Eulerian: a trail through every edge exists but it cannot be closed

B1

Answer: Semi-Eulerian, 2 odd vertices

2 marks

3 (2 marks)

Route length = total weight + repeated length

M1

$86 + 14 = 100$

A1

Answer: 100

2 marks

4 (2 marks)

PQ and RS; PR and QS; PS and QR

M1 A1

Answer: PQ+RS, PR+QS, PS+QR

2 marks

5 (2 marks)

$(5 - 1)! \div 2 = 24 \div 2$

M1

12

A1

Answer: 12

2 marks

6 (2 marks)

Classical: every vertex is visited exactly once and the tour returns to start, in a complete graph satisfying the triangle inequality B1

Practical: every vertex is visited at least once, so vertices and edges may be repeated B1

Answer: Classical: exactly once; practical: at least once 2 marks

7 (4 marks)

Odd vertices: A, B, C, D (degrees A 3, B 3, C 3, D 3) B1

Pairings: $AC\ 5 + BD\ 10 = 15$; $AB\ 8 + CD\ 11 = 19$; $AD\ 16 + BC\ 7 = 23$ M1

Least repeated length 15: repeat the shortest paths for AC and BD A1

Route length = $61 + 15 = 76$ A1

Answer: 76 4 marks

8 (4 marks)

Odd vertices: A, B, C, D (degrees A 3, B 3, C 5, D 3) B1

Pairings: $AD\ 9 + BC\ 5 = 14$; $AB\ 6 + CD\ 15 = 21$; $AC\ 8 + BD\ 15 = 23$ M1

Least repeated length 14: repeat the shortest paths for AD and BC A1

Route length = $105 + 14 = 119$ A1

Answer: 119 4 marks

9 (4 marks)

Odd vertices: B, C (degrees B 3, C 3) B1

Pairings: $BC\ 8 = 8$ M1

Least repeated length 8: repeat the shortest paths for BC A1

Route length = $53 + 8 = 61$ A1

Answer: 61 m 4 marks

10 (4 marks)

(a) Odd vertices: C, D (degrees C 3, D 3) B1

Pairings: $CD\ 19 = 19$ M1

Answer: C and D 2 marks

(b) Closed route = $87 + 19 = 106$ B1

Starting at one odd vertex and finishing at the other, no edge needs repeating, so the length is 87 B1

Answer: closed 106; open 87 2 marks

11 (3 marks)

From A the nearest is C (9) M1

Tour: A-C-B-D-A A1

Length = $9 + 7 + 10 + 15 = 41$ A1

Answer: A-C-B-D-A, 41 3 marks

12 (3 marks)

(a) The optimal tour is the shortest of all tours, so it is no longer than any particular tour E1

Answer: Optimal $\leq$ any tour 1 mark

(b) The best (lowest) upper bound is 58 km B1

The optimal tour is at most 58 km B1

Answer: 58 km 2 marks

13 (4 marks)

(a) Using Dijkstra's algorithm: E to F = 9 M1 A1

Answer: 9 2 marks

(b) Total weight + repeated path = $87 + 9$ M1

= 96 A1

Answer: 96 2 marks

14 (4 marks)

Odd vertices: D, F, G, H (degrees D 3, F 3, G 3, H 3)

B1

Pairings: DG 24 + FH 16 = 40; DF 33 + GH 19 = 52; DH 27 + FG 25 = 52

M1

Least repeated length 40: repeat the shortest paths for DG and FH

A1

Route length = 198 + 40 = 238

A1

Answer: 238

4 marks

15 (4 marks)

Odd vertices: C, D, E, G (degrees C 3, D 3, E 5, G 3)

B1

Pairings: CE 10 + DG 15 = 25; CD 13 + EG 18 = 31; CG 28 + DE 3 = 31

M1

Least repeated length 25: repeat the shortest paths for CE and DG

A1

Route length = 142 + 25 = 167

A1

Answer: 167

4 marks

16 (7 marks)

(a) Odd vertices: D, E, F, G (degrees D 3, E 5, F 3, G 3)

B1

Pairings: DG 12 + EF 13 = 25; DE 21 + FG 20 = 41; DF 32 + EG 25 = 57

M1

Least repeated length 25: repeat the shortest paths for DG and EF

A1

Route length = 204 + 25 = 229

A1

Answer: 229

4 marks

(b) The two unpaired odd vertices are the start and finish, so pair up the other two odd vertices with the shortest connecting path

M1

Best: start E, finish F, repeat length 12

A1

Route length = 204 + 12 = 216

A1

Answer: 216 from E to F

3 marks

17 (4 marks)

(a) Using shortest routes: AB 17; AC 15; AD 22; AE 13; BC 11; BD 13; BE 4; CD 10; CE 7; DE 9

M1 A1

Answer: AB 17; AC 15; AD 22; AE 13; BC 11; BD 13; BE 4; CD 10; CE 7; DE 9

2 marks

(b) Nearest neighbour from A: A-E-B-C-D-A

M1

Length = 60 (upper bound)

A1

Answer: 60

2 marks

18 (8 marks)

(a) Tour A-C-B-E-D-F-A

M1 A1

Upper bound = 132

A1

Answer: 132

3 marks

(b) MST of the other five towns has weight 71

M1 A1

Two shortest edges at E: 13 and 17

A1

Lower bound = 71 + 13 + 17 = 101

A1

Answer: 101

4 marks

(c) $101 \leq \text{optimal} \leq 132$

B1

Answer: 101 to 132

1 mark

19 (3 marks)

Remove the deleted vertex from the optimal tour: what remains is a path through all other vertices, which is a spanning tree of them

E1

So its length is at least the weight of the minimum spanning tree of those vertices

E1

The optimal tour also uses two edges at the deleted vertex, each at least as long as the two shortest edges there

E1

Answer: Tour minus a vertex is a spanning tree plus two edges at that vertex

3 marks

20 (5 marks)

Odd vertices: C, D, E, G (degrees C 3, D 3, E 5, G 3)

B1

Pairings: CD 18 + EG 11 = 29; CE 10 + DG 24 = 34; CG 21 + DE 19 = 40

M1

Least repeated length 29: repeat the shortest paths for CD and EG

A1

Route length = 138 + 29 = 167

A1

Time = $167 \div 30 \times 60 = 334$ minutes

B1

Answer: 167 km, 334 minutes

5 marks

21 (6 marks)

(a) $\frac{6!}{2^3 \times 3!} = 15$

B1

Answer: 15

1 mark

(b) Odd vertices: A, B, D, F, H, I

B1

Distances between odd vertices: AB 14, AD 9, AF 28, AH 23, AI 34, BD 15, BF 18, BH 13, BI

M1

24, DF 19, DH 14, DI 25, FH 17, FI 10, HI 15

Best of the 15 pairings: AD + BH + FI = 32

M1 A1

Route length = 175 + 32 = 207

A1

Answer: 207

5 marks

22 (7 marks)

(a) Odd vertices: A, B, C, D (degrees A 3, B 3, C 5, D 3)

B1

Pairings: AB 6 + CD 14 = 20; AC 3 + BD 17 = 20; AD 11 + BC 9 = 20

M1

Least repeated length 20: repeat the shortest paths for AB and CD

A1

Route length = 139 + 20 = 159

A1

Answer: 159

4 marks

(b) The new road joins two odd vertices, leaving two odd vertices C and D

M1

Repeat the shortest path between them: 14

A1

New length = 139 + 1 + 14 = 154

A1

Answer: 154

3 marks

23 (8 marks)

(a) Not every pair of towns is joined directly, so it is not a complete graph

E1

Replace the network by a complete network whose edge weights are the shortest distances between towns

E1

Answer: Use shortest distances to complete the graph

2 marks

(b) From A: A-B-E-C-D-F-A, length 84

M1 A1

From D: D-F-E-B-A-C-D, length 70

A1

Better upper bound: 70

A1

Answer: 70

4 marks

(c) $(84 - 70) \div 70 \times 100$

M1

= 20%

A1

Answer: 20%

2 marks

24 (8 marks)

(a) Odd vertices: D, F, G, H (degrees D 3, F 3, G 3, H 3)

B1

Pairings: DG 16 + FH 17 = 33; DF 15 + GH 22 = 37; DH 30 + FG 7 = 37

M1

Least repeated length 33: repeat the shortest paths for DG and FH

A1

Route length = 193 + 33 = 226

A1

Answer: 226

4 marks

(b) The start and finish are two of the odd vertices; the other two are joined by a shortest path

M1

Smallest repeated length over the choices: 7

A1

Route length = 193 + 7 = 200

A1

Answer: 200

3 marks

(c) $226 - 200 = 26$

B1

Answer: 26

1 mark

25 (7 marks)**(a)** From A: A-C-B-E-F-D-A, length 107

M1 A1

From B: B-C-D-E-F-A-B, length 105

A1

From C: C-B-E-F-A-D-C, length 104

A1

Answer: A: 107, B: 105, C: 104

4 marks

(b) Minimum spanning tree weight: 77 (edges BC, EF, BE, DE, AC)

M1 A1

Removing any one edge from a tour leaves a spanning tree, so the tour is longer than the minimum spanning tree

E1

Answer: Lower bound 77

3 marks

26 (9 marks)**(a)** Tour A-B-E-D-F-C-G-A

M1 A1

Length 106

A1

Answer: 106

3 marks

(b) MST of the other six towns: weight 39

M1 A1

Two shortest edges at A: 12, 17

A1

Lower bound = 68

A1

Answer: 68

4 marks

(c) The optimal is at least 68, so the error is at most $(106 - 68) \div 68 \times 100$

M1

= 55.9%

A1

Answer: 55.9%

2 marks

27 (7 marks)**(a)** All degrees are even so an Eulerian cycle exists, using every edge exactly once

E1

No route can be shorter than W since every edge must be used

E1

Answer: W

2 marks

(b) Repeat the edges of a shortest path from X to Y: this makes X and Y even, so there is an Eulerian cycle of length $W + d(X, Y)$

M1 A1

Any closed route must repeat a set of edges that joins X to Y (to make all degrees even), whose total is at least $d(X, Y)$

E1 E1

Answer: $W + d(X, Y)$

4 marks

(c) The sum of the degrees is even, so the number of odd vertices is even

E1

Answer: Number of odd vertices is even

1 mark

28 (8 marks)**(a)** Each time move to the nearest unvisited town: A to D (8) then D to B (10) then B to E (7) then E to C (20) then C to A (14)

M1 A1

Tour A-D-B-E-C-A, length 59

A1

Answer: A-D-B-E-C-A, 59

3 marks

(b) Try A-C-B-E-D-A: $14 + 16 + 7 + 11 + 8 = 56$

M1 A1

 $56 < 59$, so nearest neighbour was not optimal

E1

Answer: A-C-B-E-D-A, 56

3 marks

(c) The shortest of the 12 tours has length 56 (enumerating them all)

M1 A1

Answer: 56

2 marks

29 (8 marks)**(a)** Routes from B to C: $B-C = 5$, $B-D-C = 7 + x \geq 8$, $B-A-C = 6 + 8 = 14$

M1

The least is 5

A1

Answer: 5

2 marks

(b) Total edge weight = $6 + 8 + 5 + 7 + x = 26 + x$

B1

Repeated length = 5, so length = $26 + x + 5 = 31 + x$

M1 A1

Answer: $31 + x$

3 marks

(c) $31 + x = 40$, so $x = 9$

M1 A1

The edge BC (5 km) is repeated

A1

Answer: $x = 9$; BC repeated

3 marks

30 (10 marks)**(a)** Odd vertices: C, D, E, G (degrees C 3, D 3, E 5, G 3)**B1**

Pairings: CE 10 + DG 14 = 24; CD 16 + EG 17 = 33; CG 27 + DE 13 = 40

M1

Least repeated length 24: repeat the shortest paths for CE and DG

A1

Route length = 139 + 24 = 163

A1**Answer: 163****4 marks****(b)** Pairing totals: 24, 33, 40**M1 A1**

Longest route 179, shortest 163

A1 $(179 - 163) \div 163 \times 100 = 9.82\%$ **A1****Answer: 9.82%****4 marks****(c)** Remove that edge, recompute the shortest distances between odd vertices on the reduced network, choose the pairing again and add the new repeated length to the new total weight**E1 E1****Answer: Recompute on the reduced network****2 marks**