

Function Notation, Domain and Range

1 (2 marks)

$$f(3) = 2(9) - 5(3) + 3 = 6$$

B1

$$f(-1) = 2(1) - 5(-1) + 3 = 10$$

B1

Answer: (a) 6 (b) 10

2 marks

2 (3 marks)

$$(a) g(13) = \sqrt{13 - 4} = \sqrt{9} = 3$$

B1

Answer: 3

1 mark

(b) Square root is never negative and $x \geq 4$ gives every value from 0 upwards: $g(x) \geq 0$

B1

Answer: $g(x) \geq 0$

1 mark

(c) $g(2) = \sqrt{-2}$, which is not a real number

E1

Answer: $\sqrt{-2}$ is not real

1 mark

3 (1 mark)

The denominator cannot be zero: $x + 4 \neq 0$

B1

Answer: all real x except $x = -4$

1 mark

4 (1 mark)

$x^2 \geq 0$ for all real x, so $g(x) \geq 3$

B1

Answer: $g(x) \geq 3$

1 mark

5 (2 marks)

$$h(3) = 2^3 = 8$$

B1

$$h(-2) = 2^{-2} = \frac{1}{4}$$

B1

Answer: $h(3) = 8$, $h(-2) = \frac{1}{4}$

2 marks

6 (3 marks)

$$(a) f(0) = -1, f(1) = 2, f(2) = 5$$

B1

$$f(3) = 8, f(4) = 11$$

B1

Answer: -1, 2, 5, 8, 11

2 marks

(b) Different inputs always give different outputs, so each value in the range comes from exactly one input

E1

Answer: different inputs give different outputs

1 mark

7 (4 marks)

$$(a) f(-1) = \frac{3(-1) + 1}{-1 - 2} = \frac{-2}{-3} = \frac{2}{3}$$

B1

Answer: $\frac{2}{3}$

1 mark

$$(b) 3x + 1 = 4(x - 2)$$

M1

$$3x + 1 = 4x - 8$$

M1

$$x = 9$$

A1

Answer: $x = 9$

3 marks

8 (4 marks)

$$(a) x^2 - 6x + 11 = (x - 3)^2 - 9 + 11$$

M1

$$= (x - 3)^2 + 2$$

A1

Answer: $(x - 3)^2 + 2$

2 marks

$$(b) (x - 3)^2 \geq 0 \text{ so } f(x) \geq 2$$

B1

Answer: $f(x) \geq 2$

1 mark

(c) The graph is symmetrical about $x = 3$, so for example $f(2) = f(4) = 3$: two inputs give the same output

E1

Answer: $f(2) = f(4)$

1 mark

9 (4 marks)

$$f(-2) = 2(-2) + 1 = -3, f(3) = 10 - 3 = 7, f(8) = 10 - 8 = 2$$

B1 B1

$$2x + 1 = 5 \text{ gives } x = 2, \text{ valid since } 2 < 3$$

M1

$$10 - x = 5 \text{ gives } x = 5, \text{ valid since } 5 \geq 3$$

A1

Answer: (a) -3, 7, 2 (b) $x = 2$ or $x = 5$

4 marks

10 (6 marks)

(a) Need $2x - 6 \geq 0$

M1

$$x \geq 3$$

A1

Answer: $x \geq 3$

2 marks

(b) Need $5 - x > 0$

M1

$$x < 5$$

A1

Answer: $x < 5$

2 marks

(c) Denominator is zero when $x = 1$ or $x = -3$

M1

$$x \in \mathbb{R}, x \neq 1, x \neq -3$$

A1

Answer: $x \neq 1, -3$

2 marks

11 (5 marks)

(a) Minimum is $g(2) = 1$

M1

Maximum is $g(5) = 10$ ($g(0) = 5$ is smaller)

M1

$$1 \leq g(x) \leq 10$$

A1

Answer: $1 \leq g(x) \leq 10$

3 marks

(b) f is decreasing, $f(-1) = 5$ (not included) and $f(4) = -5$

M1

$$-5 \leq f(x) < 5$$

A1

Answer: $-5 \leq f(x) < 5$

2 marks

12 (3 marks)

(a) $f(3) = 5 = f(-3)$: two different inputs give the same output

E1

Answer: $f(3) = f(-3)$

1 mark

(b) The vertex is at $x = 0$, so $k = 0$

B1

$$\text{Range: } f(x) \geq -4$$

B1

Answer: $k = 0$; $f(x) \geq -4$

2 marks

13 (3 marks)

$$2a + b = 7 \text{ and } -a + b = -2$$

M1

$$\text{Subtract: } 3a = 9, \text{ so } a = 3$$

M1

$$b = 7 - 6 = 1$$

A1

Answer: $a = 3, b = 1$

3 marks

14 (6 marks)

(a) Domain: $-2 \leq x \leq 5$

B1

$$\text{Lowest value is } -2, \text{ highest is } 4, \text{ so range: } -2 \leq f(x) \leq 4$$

B1

Answer: Domain $-2 \leq x \leq 5$; range $-2 \leq f(x) \leq 4$

2 marks

(b) First segment has gradient $\frac{-6}{3} = -2$, so $y = -2x$ for $-2 \leq x \leq 1$, giving $x = 0$

M1

$$\text{Second segment has gradient } \frac{4}{4} = 1, \text{ so } y = x - 3 \text{ for } 1 \leq x \leq 5, \text{ giving } x = 3$$

A1

Answer: $x = 0$ or $x = 3$

2 marks

(c) $f(-2) = 4$

M1

$$f(4) = 4 - 3 = 1, \text{ so } f(f(-2)) = 1$$

A1

Answer: 1

2 marks

15 (7 marks)

(a) $2x^2 - 8x + 5 = 2(x^2 - 4x) + 5$

M1

$$= 2[(x - 2)^2 - 4] + 5$$

A1

$$= 2(x - 2)^2 - 3$$

A1

Answer: $2(x - 2)^2 - 3$

3 marks

(b) The vertex is at $x = 2$; for $x \geq k$ to be one-to-one need $k \geq 2$, so least k is 2

Answer: $k = 2$

B1

1 mark

(c) Minimum value is -3 at $x = 2$, so $f(x) \geq -3$

Answer: $f(x) \geq -3$

B1

1 mark

(d) $2x^2 - 8x + 5 = 5$, so $2x(x - 4) = 0$

$x = 0$ or 4 , but $x \geq 2$, so $x = 4$

Answer: $x = 4$

M1

A1

2 marks

16 (6 marks)

(a) $g(0) = \frac{-1}{2}$

B1

$3x - 1 = 2(x + 2) = 2x + 4$

M1

$x = 5$

A1

Answer: $g(0) = -\frac{1}{2}$; $x = 5$

3 marks

(b) $3 - \frac{7}{x+2} = \frac{3(x+2) - 7}{x+2}$

M1

$= \frac{3x - 1}{x + 2}$

A1

Answer: shown

2 marks

(c) $\frac{7}{x+2}$ is never zero, so $g(x) = 3 - \frac{7}{x+2}$ is never 3

E1

Answer: $\frac{7}{x+2} \neq 0$

1 mark

17 (4 marks)

(a) $e^{2x} > 0$ for all x , so $h(x) > -3$

B1

Answer: $h(x) > -3$

1 mark

(b) $e^{2x} = 8$

M1

$2x = \ln 8 = 3 \ln 2$

M1

$x = \frac{3}{2} \ln 2$

A1

Answer: $x = \frac{3}{2} \ln 2$

3 marks

18 (5 marks)

(a) $-1 \leq \sin x \leq 1$, so $-2 \leq f(x) \leq 4$

B1 B1

Answer: $-2 \leq f(x) \leq 4$

2 marks

(b) $\sin x = \frac{1}{2}$

M1

$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

A1 A1

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

3 marks

19 (4 marks)

(a) $f(x + h) = x^2 + 2xh + h^2 - 3x - 3h$

M1

$f(x + h) - f(x) = 2xh + h^2 - 3h = h(2x + h - 3)$

A1

Answer: shown

2 marks

(b) Using (a) or directly: $(a + 1)^2 - 3(a + 1) - (a - 1)^2 + 3(a - 1)$

M1

$= 4a - 6$

A1

Answer: $4a - 6$

2 marks

20 (6 marks)

(a) f is decreasing and $f(2) = 2$

M1

As $x \rightarrow \infty$, $f(x) \rightarrow 0$ from above, so $0 < f(x) \leq 2$

A1

Answer: $0 < f(x) \leq 2$

2 marks

(b) g is increasing and $g(4) = 3 + \ln 3$

M1

$g(x) \geq 3 + \ln 3$

A1

Answer: $g(x) \geq 3 + \ln 3$ **2 marks****(c)** h is decreasing and $h(0) = 3$ **M1**As $x \rightarrow \infty$, $h(x) \rightarrow 2$ from above, so $2 < h(x) \leq 3$ **A1****Answer: $2 < h(x) \leq 3$** **2 marks****21 (8 marks)****(a)** Side parallel to the wall = $40 - 2x$ **B1**

$$A = x(40 - 2x) = 40x - 2x^2$$

A1**Answer: shown****2 marks****(b)** Both lengths must be positive: $x > 0$ and $40 - 2x > 0$ **M1**

$$0 < x < 20$$

A1**Answer: $0 < x < 20$** **2 marks****(c)** $A = 200 - 2(x - 10)^2$ **M1**Maximum 200 at $x = 10$, and $A \rightarrow 0$ at both ends: $0 < A \leq 200$ **A1****Answer: $0 < A \leq 200$** **2 marks****(d)** $40x - 2x^2 = 150$, so $x^2 - 20x + 75 = 0$ **M1** $(x - 5)(x - 15) = 0$, so $x = 5$ or $x = 15$ **A1****Answer: $x = 5$ or 15** **2 marks****22 (8 marks)**

(a) $2 + \frac{5}{x-1} = \frac{2(x-1)+5}{x-1}$

M1

$$= \frac{2x+3}{x-1} = f(x), \text{ as required}$$

A1**Answer: shown****2 marks****(b)** $\frac{5}{x-1}$ can take every real value except 0**M1**So $f(x)$ takes every real value except 2**A1**Because $5/(x-1)$ can never equal 0, $f(x)$ can never equal 2 (the horizontal asymptote)**E1****Answer: $f(x)$ takes all real values except 2****3 marks****(c)** $2x + 3 = x(x - 1)$ **M1**

$$x^2 - 3x - 3 = 0$$

A1

$$x = \frac{3 \pm \sqrt{21}}{2}$$

A1

Answer: $x = \frac{3 \pm \sqrt{21}}{2}$

3 marks**23 (8 marks)****(a)** $H = 1.8 - 5(t^2 - 2.4t)$ **M1**

$$= 1.8 - 5[(t - 1.2)^2 - 1.44] = 9 - 5(t - 1.2)^2$$

A1

Maximum height is 9 m

A1**Answer: $H = 9 - 5(t - 1.2)^2$; maximum height 9 m****3 marks****(b)** $9 - 5(t - 1.2)^2 = 0$ so $(t - 1.2)^2 = 1.8$ **M1**

$$t = 1.2 + \sqrt{1.8} = 2.54 \text{ s (the negative root is rejected)}$$

A1**Answer: 2.54 s****2 marks****(c)** Domain $0 \leq t \leq 2.54$ **B1**Range $0 \leq H \leq 9$ **B1****Answer: $0 \leq t \leq 2.54$; $0 \leq H \leq 9$** **2 marks****(d)** After this time H is negative, which would put the ball below ground level, but the ball has landed**E1****Answer: $H < 0$, ball below the ground****1 mark****24 (6 marks)****(a)** Need $x^2 - x - 6 = (x - 3)(x + 2) > 0$ **M1**Positive outside the roots: $x < -2$ or $x > 3$ **A1****Answer: $x < -2$ or $x > 3$** **2 marks**

(b) $x^2 - x - 6 = -6$ at $x = 0$, and the logarithm of a negative number is not defined

E1

Answer: $\ln(-6)$ undefined

1 mark

(c) $x^2 - x - 6 = 1$

M1

$$x^2 - x - 7 = 0, x = \frac{1 \pm \sqrt{29}}{2}$$

A1

Both roots lie in the domain (≈ 3.19 and -2.19)

E1

Answer: $x = \frac{1 \pm \sqrt{29}}{2}$

3 marks

25 (8 marks)

(a) $2 + a = 0$, so $a = -2$

B1

As $x \rightarrow 3$ from below, $f \rightarrow 6 - 2 = 4$

M1

$9 - b = 4$, so $b = 5$

A1

Answer: $a = -2$, $b = 5$

3 marks

(b) For $x < 3$: $f(x) = 2x - 2 < 4$, taking every value below 4

M1

For $x \geq 3$: $f(x) = x^2 - 5 \geq 4$

M1

Together they cover all reals: $f(x) \in \mathbb{R}$

A1

Answer: all real numbers

3 marks

(c) $2x - 2 = 12$ gives $x = 7$, rejected as $7 > 3$

M1

$x^2 - 5 = 12$ gives $x = \sqrt{17}$ ($x \geq 3$)

A1

Answer: $x = \sqrt{17}$

2 marks

26 (11 marks)

(a) $f(5) = f(1) = 1$

B1

$f(-3) = f(-3 + 4) = f(1) = 1$

M1

$f(10) = f(2) = 0$

A1

Answer: $f(5) = 1$, $f(-3) = 1$, $f(10) = 0$

3 marks

(b) For $0 \leq x \leq 2$: an arch with maximum point $(1, 1)$ meeting the x-axis at $(0, 0)$ and $(2, 0)$

B1

f is odd, so for $-2 \leq x \leq 0$ the arch is inverted, with minimum point $(-1, -1)$

B1

Period 4: the pattern repeats, so there is a maximum at $(-3, 1)$ and a minimum at $(3, -1)$

B1

Zeros at $x = -4, -2, 0, 2, 4$

B1

Answer: wave through $(-4, 0)$, $(-3, 1)$, $(-2, 0)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 0)$, $(3, -1)$, $(4, 0)$

4 marks

(c) For $2 \leq x \leq 4$, $f(x) \leq 0$, so only $0 \leq x \leq 2$ gives solutions

B1

$$x(2 - x) = \frac{3}{4}, \text{ so } 4x^2 - 8x + 3 = 0$$

M1

$$(2x - 1)(2x - 3) = 0$$

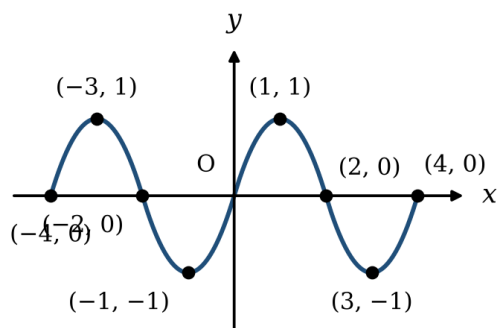
M1

$$x = \frac{1}{2} \text{ or } x = \frac{3}{2}$$

A1

Answer: $x = \frac{1}{2}$ or $x = \frac{3}{2}$

4 marks



27 (10 marks)

(a) $x^2 + 2x + 4 = (x + 1)^2 + 3$

M1

$(x + 1)^2 \geq 0$, so the denominator is at least 3, hence never 0

E1

Answer: shown

2 marks

(b) Denominator takes every value from 3 upwards

M1

Greatest value of f is $\frac{3}{3} = 1$

A1

f is always positive, and gets arbitrarily close to 0, so $0 < f(x) \leq 1$

A1

Answer: $0 < f(x) \leq 1$

3 marks

(c) $x^2 + 2x + 4 = 6$

M1

$x^2 + 2x - 2 = 0$, so $(x + 1)^2 = 3$

M1

$x = -1 \pm \sqrt{3}$

A1

Answer: $x = -1 \pm \sqrt{3}$

3 marks

(d) Range is unchanged: $0 < f(x) \leq 1$

B1

For $x \geq -1$, $(x + 1)^2$ is increasing, so f is decreasing: one-to-one

E1

Answer: $0 < f(x) \leq 1$; one-to-one

2 marks

28 (9 marks)

(a) $f(1) = 5 - 0 = 5$

B1

As x approaches 1 from below, $2x + 3$ approaches 5, the same value, so the two pieces join

E1

Answer: $f(1) = 5$; the pieces join

2 marks

(b) First piece: f increases from $f(-2) = -1$ up to (but not including) 5

M1

Second piece: f decreases from 5 to $f(4) = -4$

M1

Range is $-4 \leq f(x) \leq 5$

A1

Answer: $-4 \leq f(x) \leq 5$

3 marks

(c) First piece gives $x = \frac{k-3}{2}$, valid when $-1 \leq k < 5$

M1

Second piece gives $x = 1 + \sqrt{5-k}$ ($x \geq 1$), valid when $-4 \leq k \leq 5$

M1

Both valid for $-1 \leq k < 5$

A1

At $k = 5$ both pieces give $x = 1$ only (one solution), and for $k < -1$ only the second piece works, so exactly two solutions only for $-1 \leq k < 5$

A1

Answer: $-1 \leq k < 5$

4 marks

29 (10 marks)

(a) $1 - \frac{2}{x^2 + 1} = \frac{x^2 + 1 - 2}{x^2 + 1}$

M1

$= \frac{x^2 - 1}{x^2 + 1}$

A1

Answer: shown

2 marks

(b) $x^2 + 1 \geq 1$, so $0 < \frac{2}{x^2 + 1} \leq 2$

M1 A1

So $-1 \leq f(x) < 1$

A1

Answer: $-1 \leq f(x) < 1$

3 marks

(c) $f(-x) = \frac{(-x)^2 - 1}{(-x)^2 + 1} = f(x)$

M1

The graph is symmetrical about the y -axis

E1

Answer: even; symmetric in the y -axis

2 marks

(d) $2(x^2 - 1) = x^2 + 1$

M1

$x^2 = 3$, $x = \pm\sqrt{3}$

A1

Answer: $x = \pm\sqrt{3}$

2 marks

(e) 1 is not in the range, as $\frac{2}{x^2 + 1}$ is never zero

E1

Answer: 1 is not in the range

1 mark

30 (10 marks)

(a) $g(-x) = \frac{-x}{x^2 + 1} = -g(x)$

M1 A1

Answer: shown

2 marks

(b) $2(x^2 + 1) = 5x$, so $2x^2 - 5x + 2 = 0$

M1

$(2x - 1)(x - 2) = 0$, so $x = \frac{1}{2}$ or $x = 2$

M1 A1

Answer: $x = \frac{1}{2}, 2$

3 marks

(c) $y(x^2 + 1) = x$, so $yx^2 - x + y = 0$

M1

A real x exists only if the discriminant $1 - 4y^2 \geq 0$

M1

$y^2 \leq \frac{1}{4}$, so $-\frac{1}{2} \leq y \leq \frac{1}{2}$

A1

Answer: shown

3 marks

(d) For $y = \frac{1}{2}$: $\frac{1}{2}x^2 - x + \frac{1}{2} = 0$, so $(x - 1)^2 = 0$, $x = 1$

M1

For $y = -\frac{1}{2}$: $x = -1$ by symmetry (odd function)

A1

Answer: greatest at $x = 1$, least at $x = -1$

2 marks

Composite and Inverse Functions

1 (2 marks)

$fg(2) = f(4) = 11$

B1

$gf(2) = g(7) = 49$

B1

Answer: (a) 11 (b) 49

2 marks

2 (2 marks)

$y = x^3 - 1$, so $x^3 = y + 1$

M1

$f^{-1}(x) = \sqrt[3]{x + 1}$

A1

Answer: $f^{-1}(x) = \sqrt[3]{x + 1}$

2 marks

3 (3 marks)

Let $y = 3x - 5$, so $x = \frac{y + 5}{3}$

M1

$f^{-1}(x) = \frac{x + 5}{3}$

A1

$f^{-1}(7) = \frac{12}{3} = 4$

B1

Answer: $f^{-1}(x) = \frac{x + 5}{3}$; $f^{-1}(7) = 4$

3 marks

4 (2 marks)

$y = \frac{x}{2} + 3$, so $x = 2(y - 3)$

M1

$f^{-1}(x) = 2x - 6$

A1

Answer: $f^{-1}(x) = 2x - 6$

2 marks

5 (2 marks)

$ff(x) = 2(2x + 3) + 3$

M1

$= 4x + 9$

A1

Answer: $4x + 9$

2 marks

6 (4 marks)

(a) $gf(x) = g(x^2 + 1)$

M1

$= \frac{1}{(x^2 + 1) - 2} = \frac{1}{x^2 - 1}$

A1

Answer: $gf(x) = \frac{1}{x^2 - 1}$

2 marks

(b) $x^2 - 1 = 3$

M1

$x = \pm 2$

A1

Answer: $x = \pm 2$

2 marks

7 (5 marks)

(a) $y(x - 3) = 2x + 1$, so $xy - 3y = 2x + 1$

M1

$$x(y - 2) = 3y + 1, \text{ so } x = \frac{3y + 1}{y - 2}$$

M1

$$f^{-1}(x) = \frac{3x + 1}{x - 2}$$

A1

Answer: $f^{-1}(x) = \frac{3x + 1}{x - 2}$

3 marks

(b) Domain of f^{-1} is the range of f : $x \neq 2$

B1

Range of f^{-1} is the domain of f : $f^{-1}(x) \neq 3$

B1

Answer: Domain $x \neq 2$; range $f^{-1}(x) \neq 3$

2 marks

8 (3 marks)

$$fg(x) = 3(x - 4) + 2 = 3x - 10$$

M1

$$gf(x) = (3x + 2) - 4 = 3x - 2$$

M1

$$3x - 10 = 3x - 2 \text{ gives } -10 = -2, \text{ which is impossible, so no solutions}$$

E1

Answer: no solutions

3 marks

9 (3 marks)

$$y(x - 2) = 5, \text{ so } x = 2 + \frac{5}{y}$$

M1 A1

$$f^{-1}(x) = 2 + \frac{5}{x}, \text{ domain } x \neq 0$$

A1

Answer: $f^{-1}(x) = 2 + \frac{5}{x}, x \neq 0$

3 marks

10 (3 marks)

$$y + 1 = e^{2x}, \text{ so } 2x = \ln(y + 1)$$

M1

$$f^{-1}(x) = \frac{1}{2} \ln(x + 1)$$

A1

Domain: $x > -1$ (the range of f)

B1

Answer: $f^{-1}(x) = \frac{1}{2} \ln(x + 1), x > -1$

3 marks

11 (3 marks)

$$y - 1 = (x - 3)^2, \text{ so } x = 3 + \sqrt{y - 1} \text{ (} x \geq 3 \text{)}$$

M1 A1

$$f^{-1}(x) = 3 + \sqrt{x - 1}, \text{ domain } x \geq 1$$

A1

Answer: $f^{-1}(x) = 3 + \sqrt{x - 1}, x \geq 1$

3 marks

12 (6 marks)

(a) $x^2 - 4x + 7 = (x - 2)^2 + 3$

M1

$$x \geq 2 \text{ so } f(x) \geq 3$$

A1

Answer: $f(x) \geq 3$

2 marks

(b) $y = (x - 2)^2 + 3$, so $x - 2 = \sqrt{y - 3}$ (positive root since $x \geq 2$)

M1

$$x = 2 + \sqrt{y - 3}$$

M1

$$f^{-1}(x) = 2 + \sqrt{x - 3}$$

A1

Answer: $f^{-1}(x) = 2 + \sqrt{x - 3}$

3 marks

(c) Domain of f^{-1} is $x \geq 3$ (the range of f); the graphs are reflections of each other in the line $y = x$

B1

Answer: $x \geq 3$; reflection in $y = x$

1 mark

13 (7 marks)

(a) $fg(x) = 2(x^2 + 3) + 1 = 2x^2 + 7$

B1

$$gf(x) = (2x + 1)^2 + 3 = 4x^2 + 4x + 4$$

B1

Answer: $fg(x) = 2x^2 + 7$; $gf(x) = 4x^2 + 4x + 4$

2 marks

(b) $2x^2 + 7 = 4x^2 + 4x + 4$

M1

$$2x^2 + 4x - 3 = 0$$

A1

$$x = \frac{-4 \pm \sqrt{16 + 24}}{4} = \frac{-2 \pm \sqrt{10}}{2}$$

A1

Answer: $x = \frac{-2 \pm \sqrt{10}}{2}$

3 marks

(c) $gf(x) = (2x + 1)^2 + 3$

M1

$(2x + 1)^2 \geq 0$, so $gf(x) \geq 3$

A1

Answer: $gf(x) \geq 3$

2 marks

14 (6 marks)

(a) $gf(x) = (\sqrt{x+3})^2 - 5$

M1

$= x - 2$, for $x \geq -3$

A1

Answer: $gf(x) = x - 2$, $x \geq -3$

2 marks

(b) $fg(x) = \sqrt{x^2 - 5} + 3 = \sqrt{x^2 - 2}$

M1

Need $x^2 - 2 \geq 0$: $x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$

A1

Answer: $fg(x) = \sqrt{x^2 - 2}$, $|x| \geq \sqrt{2}$

2 marks

(c) $x^2 - 2 = 25$

M1

$x = \pm\sqrt{27} = \pm 3\sqrt{3}$

A1

Answer: $x = \pm 3\sqrt{3}$

2 marks

15 (6 marks)

(a) $f(x) = (x + 3)^2 - 7$

B1

Range: $f(x) \geq -7$

B1

Answer: $(x + 3)^2 - 7$; $f(x) \geq -7$

2 marks

(b) $(x + 3)^2 = y + 7$, and $x + 3 \geq 0$

M1

$f^{-1}(x) = -3 + \sqrt{x + 7}$

A1

Domain: $x \geq -7$

B1

Answer: $f^{-1}(x) = -3 + \sqrt{x + 7}$, $x \geq -7$

3 marks

(c) $-3 + \sqrt{9} = 0$

B1

Answer: 0

1 mark

16 (6 marks)

(a) $f(2) = 4$ and $f(5) = 6$, so $f^{-1}(4) = 2$ and $f^{-1}(6) = 5$

B1 B1

Answer: 2 and 5

2 marks

(b) $f(0) = 1$, then $f(1) = 1 + \frac{3}{2} = \frac{5}{2}$

M1 A1

Answer: $\frac{5}{2}$

2 marks

(c) Reflect in $y = x$: the points are (1, 0), (4, 2) and (6, 5)

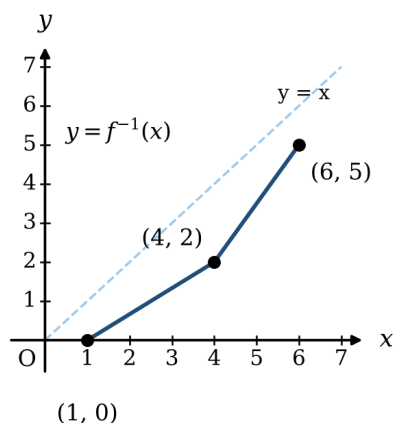
B1

Two straight segments joining them

B1

Answer: two straight segments through (1, 0), (4, 2) and (6, 5)

2 marks



17 (5 marks)

$$(a) \frac{9}{5}C = F - 32 \quad \text{M1}$$

$$C = \frac{5}{9}(F - 32) \quad \text{A1}$$

$$\text{Answer: } C = \frac{5}{9}(F - 32) \quad \text{2 marks}$$

$$(b) K = \frac{5}{9}(F - 32) + 273.15 \quad \text{M1}$$

$$\frac{5}{9}(F - 32) = 26.85, \text{ so } F - 32 = 48.33... \quad \text{M1}$$

$$F = 80.3 \quad \text{A1}$$

$$\text{Answer: } K = \frac{5}{9}(F - 32) + 273.15; F = 80.3 \quad \text{3 marks}$$

18 (11 marks)

$$(a) fgh(x) = f(g(x^2)) = f(3x^2) = 3x^2 + 2 \quad \text{M1 A1}$$

$$hgf(x) = h(g(x + 2)) = h(3x + 6) = (3x + 6)^2 = 9(x + 2)^2 \quad \text{M1 A1}$$

$$\text{Answer: } fgh(x) = 3x^2 + 2; hgf(x) = 9(x + 2)^2 \quad \text{4 marks}$$

$$(b) 3x^2 + 2 = 9(x^2 + 4x + 4) \quad \text{M1}$$

$$6x^2 + 36x + 34 = 0, \text{ so } 3x^2 + 18x + 17 = 0 \quad \text{M1 A1}$$

$$x = \frac{-18 \pm \sqrt{120}}{6} = \frac{-9 \pm \sqrt{30}}{3} \quad \text{M1 A1}$$

$$\text{Answer: } x = \frac{-9 \pm \sqrt{30}}{3} \quad \text{5 marks}$$

$$(c) 9(x + 2)^2 \geq 0 \text{ for all real } x, \text{ with equality when } x = -2 \quad \text{M1}$$

$$\text{Range: } hgf(x) \geq 0 \quad \text{A1}$$

$$\text{Answer: } hgf(x) \geq 0 \quad \text{2 marks}$$

19 (7 marks)

$$(a) y = 2 - \ln(x + 1), \text{ so } \ln(x + 1) = 2 - y \quad \text{M1}$$

$$x + 1 = e^{2-y}, \text{ so } f^{-1}(x) = e^{2-x} - 1 \quad \text{A1}$$

$$f \text{ takes every real value, so the domain of } f^{-1} \text{ is all real } x \quad \text{B1}$$

$$e^{2-x} > 0, \text{ so the range of } f^{-1} \text{ is } f^{-1}(x) > -1 \quad \text{B1}$$

$$\text{Answer: } f^{-1}(x) = e^{2-x} - 1; \text{ domain all real } x; \text{ range } f^{-1}(x) > -1 \quad \text{4 marks}$$

$$(b) f^{-1}(2) = e^0 - 1 = 0 \quad \text{B1}$$

$$2 - \ln(x + 1) = 1, \text{ so } \ln(x + 1) = 1 \quad \text{M1}$$

$$x = e - 1 \quad \text{A1}$$

$$\text{Answer: } f^{-1}(2) = 0; x = e - 1 \quad \text{3 marks}$$

20 (9 marks)

$$(a) f(x) = (x - 2)^2 + k - 4 \quad \text{M1}$$

$$\text{Range: } f(x) \geq k - 4 \quad \text{A1}$$

$$\text{Answer: } f(x) \geq k - 4 \quad \text{2 marks}$$

$$(b) g(2) = 3 \quad \text{B1}$$

$$f(3) = 9 - 12 + k = k - 3 \quad \text{M1}$$

$$k - 3 = 13, \text{ so } k = 16 \quad \text{A1}$$

$$\text{Answer: } k = 16 \quad \text{3 marks}$$

$$(c) f(x) = (x - 2)^2 + 12 \text{ has its vertex at } x = 2, \text{ so } a = 2 \quad \text{M1 A1}$$

$$y = (x - 2)^2 + 12 \text{ gives } x = 2 + \sqrt{y - 12}, \text{ taking } x \geq 2 \quad \text{M1}$$

$$f^{-1}(x) = 2 + \sqrt{x - 12} \quad \text{A1}$$

$$\text{Answer: } a = 2; f^{-1}(x) = 2 + \sqrt{x - 12} \quad \text{4 marks}$$

21 (6 marks)

(a) Let $u = 3x - 2$, so $x = \frac{u+2}{3}$

M1

$g(u) = (u+2)^2 - 4(u+2) + 5$

M1

$g(x) = x^2 + 1$

A1

Answer: $g(x) = x^2 + 1$

3 marks

(b) $f^{-1}(x) = \frac{x+2}{3}$

B1

$x^2 + 1 = \frac{x+2}{3}$, so $3x^2 - x + 1 = 0$

M1

Discriminant $1 - 12 = -11 < 0$, so no real roots

E1

Answer: no real solutions

3 marks

22 (9 marks)

(a) $y = \ln(2x - 1)$, so $e^y = 2x - 1$

M1

$f^{-1}(x) = \frac{e^x + 1}{2}$

A1

Answer: $f^{-1}(x) = \frac{e^x + 1}{2}$

2 marks

(b) Domain of f^{-1} = range of f : all real x

B1

Range of f^{-1} = domain of f : $f^{-1}(x) > \frac{1}{2}$

B1

Answer: Domain: all real x ; range: $f^{-1}(x) > \frac{1}{2}$

2 marks

(c) $fg(x) = \ln(2e^{2x} - 1)$

M1

$2e^{2x} - 1 = 7$, so $e^{2x} = 4$

M1

$x = \ln 2$

A1

Answer: $x = \ln 2$

3 marks

(d) Need $2e^{2x} - 1 > 0$, so $e^{2x} > \frac{1}{2}$

M1

$2x > -\ln 2$, so $x > -\frac{1}{2}\ln 2$

A1

Answer: $x > -\frac{1}{2}\ln 2$

2 marks

23 (10 marks)

(a) $e^x > 0$, so $f(x) > -1$

B1

Answer: $f(x) > -1$

1 mark

(b) $y + 1 = 2e^x$, so $x = \ln\left(\frac{y+1}{2}\right)$

M1

$f^{-1}(x) = \ln\left(\frac{x+1}{2}\right)$

A1

Domain $x > -1$

B1

Answer: $f^{-1}(x) = \ln\left(\frac{x+1}{2}\right)$, $x > -1$

3 marks

(c) $fg(x) = 2e^{\ln(x+4)} - 1$

M1

$= 2(x+4) - 1 = 2x + 7$

A1

Domain $x > -4$, so range $fg(x) > -1$

B1 B1

Answer: $fg(x) = 2x + 7$, $x > -4$; range > -1

4 marks

(d) $2e^x = 8$, $e^x = 4$

M1

$x = \ln 4$

A1

Answer: $x = \ln 4$

2 marks

24 (9 marks)

(a) $2(1) + 1 = 3$ and $1^2 + 2 = 3$, the same value

B1

First piece gives $f(x) \leq 3$, second gives $f(x) > 3$, so range is all of $\mathbb{R}$

M1 A1

Answer: continuous; the range is all real numbers

3 marks

(b) $x \leq 3$: $y = 2x + 1$ gives $x = \frac{y-1}{2}$ M1 A1

$x > 3$: $y = x^2 + 2$ gives $x = \sqrt{y-2}$ (positive) M1 A1

Answer: $\frac{x-1}{2}$ for $x \leq 3$; $\sqrt{x-2}$ for $x > 3$ 4 marks

(c) $\frac{x-1}{2} = 2$ gives $x = 5$, but this is not ≤ 3 , rejected M1

$\sqrt{x-2} = 2$ gives $x = 6$, valid A1

Answer: $x = 6$ 2 marks

25 (7 marks)

(a) $f(x) = (x-1)^2 - 4$, minimum at $x = 1$: $f(x) \geq -4$ M1 A1

Answer: $f(x) \geq -4$ 2 marks

(b) $fg(x) = (2x-1)^2 - 2(2x-1) - 3$ M1

$= 4x^2 - 4x + 1 - 4x + 2 - 3 = 4x^2 - 8x$ A1

$g(x)$ must lie in the domain of f : $2x-1 \geq 1$, so $x \geq 1$ B1

Answer: domain $x \geq 1$ 3 marks

(c) $4x(x-2) = 0$, so $x = 0$ or $x = 2$ M1

$x = 0$ is outside the domain, so $x = 2$ A1

Answer: $x = 2$ 2 marks

26 (8 marks)

(a) $x^2 \geq 0$, so $f(x) \geq -6$ B1

$\ln x$ takes every real value B1

Answer: $f(x) \geq -6$; g takes every real value 2 marks

(b) $f(1) = -5$ M1

-5 is not in the domain of g ($x > 0$), so $g(-5)$ is undefined E1

Answer: $f(1) = -5$ is not in the domain of g 2 marks

(c) $gf(x) = \ln(x^2 - 6)$ needs $x^2 - 6 > 0$ M1

$x < -\sqrt{6}$ or $x > \sqrt{6}$ A1

Answer: $x < -\sqrt{6}$ or $x > \sqrt{6}$ 2 marks

(d) $\ln(x^2 - 6) = \ln 10$, so $x^2 - 6 = 10$ M1

$x = \pm 4$, and both values satisfy $|x| > \sqrt{6}$ A1

Answer: $x = \pm 4$ 2 marks

27 (10 marks)

(a) $f(x) = (x-3)^2 + 1$ M1

$x \geq 3$ so $f(x) \geq 1$ A1

Answer: $f(x) \geq 1$ 2 marks

(b) $(x-3)^2 = y-1$, and $x \geq 3$ so $x = 3 + \sqrt{y-1}$ M1

$f^{-1}(x) = 3 + \sqrt{x-1}$ A1

Domain $x \geq 1$ B1

Answer: $f^{-1}(x) = 3 + \sqrt{x-1}$, $x \geq 1$ 3 marks

(c) $gf(x) = \sqrt{x^2 - 6x + 10} - 1 = \sqrt{(x-3)^2}$ M1

$= |x-3| = x-3$ as $x \geq 3$, with domain $x \geq 3$ A1

Answer: shown; domain $x \geq 3$ 2 marks

(d) $g(x)$ must lie in the domain of f , so $\sqrt{x-1} \geq 3$ M1

Hence $x \geq 10$; for $x \in [1, 10)$ the value $g(x) < 3$ is not allowed A1

$fg(x) = (\sqrt{x-1} - 3)^2 + 1$ with $\sqrt{x-1} \geq 3$ gives range $fg(x) \geq 1$ A1

Answer: Domain $x \geq 10$; range $fg(x) \geq 1$ 3 marks

28 (9 marks)

(a) f is many-to-one on $\mathbb{R}$ (for example $f(0) = f(6) = 5$) so f has no inverse function E1

$3 + \sqrt{x+4}$ is the inverse of f restricted to $x \geq 3$ (positive root, range of the inverse is at least 3) E1

Answer: f is many-to-one; valid for the domain $x \geq 3$ 2 marks

(b) $g(x) = (x - 3)^2 - 4$ M1

$(x - 3)^2 = y + 4$, and $x \leq 3$ so $x = 3 - \sqrt{y + 4}$ M1

$g^{-1}(x) = 3 - \sqrt{x + 4}$ A1

Domain $x \geq -4$, range $g^{-1}(x) \leq 3$ A1

Answer: $g^{-1}(x) = 3 - \sqrt{x + 4}$; domain $x \geq -4$; range ≤ 3 4 marks

(c) At A , $g(x) = x$: $x^2 - 7x + 5 = 0$ M1

$x = \frac{7 \pm \sqrt{29}}{2}$, and $x \leq 3$ so $x = \frac{7 - \sqrt{29}}{2}$ M1

A is $(\frac{7 - \sqrt{29}}{2}, \frac{7 - \sqrt{29}}{2})$ A1

Answer: $A = (\frac{7 - \sqrt{29}}{2}, \frac{7 - \sqrt{29}}{2})$ 3 marks

29 (9 marks)

(a) $ff(x) = \frac{2(2x + 5) + 5(kx - 2)}{k(2x + 5) - 2(kx - 2)}$ M1

Numerator $= 4x + 10 + 5kx - 10 = (5k + 4)x$ A1

Denominator $= 2kx + 5k - 2kx + 4 = 5k + 4$, so $ff(x) = x$ A1

Answer: shown 3 marks

(b) f is self-inverse, since $ff(x) = x$, so $f^{-1}(x) = f(x)$ B1 E1

Answer: $f^{-1}(x) = \frac{2x + 5}{kx - 2}$ 2 marks

(c) $2x + 5 = x(x - 2)$, so $x^2 - 4x - 5 = 0$ M1

$(x - 5)(x + 1) = 0$, so $x = 5$ or $x = -1$ A1

Answer: $x = 5$ or $x = -1$ 2 marks

(d) Range: $f(x) \neq 2$, which is the domain of $f^{-1} = f$ B1

Reflecting $y = f(x)$ in $y = x$ gives $y = f^{-1}(x)$, which is the same graph E1

Answer: $f(x) \neq 2$; self-inverse 2 marks

30 (12 marks)

(a) $f(x) = (x - 1)^2 + 1$, so $x = 1 + \sqrt{y - 1}$ M1 A1

$f^{-1}(x) = 1 + \sqrt{x - 1}$, $x \geq 1$ A1

Answer: $f^{-1}(x) = 1 + \sqrt{x - 1}$, $x \geq 1$ 3 marks

(b) $x^2 - 3x + 2 = 0$ M1

$x = 1$ or $x = 2$ A1

Answer: $x = 1$ or 2 2 marks

(c) $f(a) = b$ and $f^{-1}(a) = b$, so $f(b) = a$, and (b, a) lies on $y = f(x)$ too M1 E1

If $a < b$ then $f(a) < f(b)$, i.e. $b < a$, a contradiction (similarly $a > b$) M1 E1

Answer: all intersections lie on $y = x$ 4 marks

(d) Points of intersection: $(1, 1)$ and $(2, 2)$ B1

$ff(x) = (x - 1)^4 + 1 = x$ gives $(x - 1)((x - 1)^3 - 1) = 0$ M1

$x = 1$ or $x = 2$, as before A1

Answer: $(1, 1)$ and $(2, 2)$ 3 marks

Graph Transformations

1 (3 marks)

(a) Replacing x by $(x - 3)$: translation by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ B1

Answer: translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ 1 mark

(b) Adding 5 to every y value: translation by the vector $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$ B1

Answer: translation $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$ 1 mark

(c) Every y value changes sign: reflection in the x -axis B1

Answer: reflection in the x -axis 1 mark

2 (2 marks)

Translation $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$: minimum at $(3, 2)$ B1

Translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: minimum at $(4, -2)$ B1

Answer: (a) $(3, 2)$ (b) $(4, -2)$ 2 marks

3 (3 marks)

Translation by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ B1

Translation by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ B1

Stretch parallel to the y -axis, scale factor 2 B1

Answer: (a) translation $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ **(b) translation** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ **(c) vertical stretch sf 2** 3 marks

4 (3 marks)

$(1, 5)$ B1

$(2, 7)$ B1

$(2, 15)$ B1

Answer: (a) $(1, 5)$ (b) $(2, 7)$ (c) $(2, 15)$ 3 marks

5 (2 marks)

x is replaced by $x - 3$ M1

$y = \sqrt{x - 3}$ A1

Answer: $y = \sqrt{x - 3}$ 2 marks

6 (2 marks)

$y = -2^x$ B1

$y = 2^{-x}$ B1

Answer: (a) $y = -2^x$ (b) $y = 2^{-x}$ 2 marks

7 (4 marks)

(a) Stretch parallel to the y -axis, scale factor 3: $(4, 18)$ B1

Answer: $(4, 18)$ 1 mark

(b) Stretch parallel to the x -axis, scale factor $\frac{1}{2}$: $(2, 6)$ B1

Answer: $(2, 6)$ 1 mark

(c) Reflection in the x -axis: $(4, -6)$ B1

Answer: $(4, -6)$ 1 mark

(d) Reflection in the y -axis: $(-4, 6)$ B1

Answer: $(-4, 6)$ **1 mark****8 (5 marks)****(a)** Translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$: maximum at $(1, 2)$ **B1**Minimum at $(3, -2)$ **B1****Answer: $(1, 2)$ and $(3, -2)$** **2 marks****(b)** Stretch scale factor 2 parallel to the y-axis: maximum $(-1, 4)$ **B1**Minimum $(1, -4)$ **B1****Answer: $(-1, 4)$ and $(1, -4)$** **2 marks****(c)** Reflection in the y-axis: the old minimum $(1, -2)$ becomes $(-1, -2)$ and the old maximum $(-1, 2)$ becomes $(1, 2)$, so the maximum is $(1, 2)$ **B1****Answer: $(1, 2)$** **1 mark****9 (2 marks)**

$$x^2 + 4x + 7 = (x + 2)^2 + 3$$

M1Translation by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **A1****Answer: translation $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$** **2 marks****10 (3 marks)**Stretch parallel to the x-axis, scale factor $\frac{1}{2}$ **B1**

Stretch parallel to the y-axis, scale factor 3

B1Translation by $\begin{pmatrix} -\frac{\pi}{3} \\ 0 \end{pmatrix}$ **B1****Answer: (a) horizontal stretch sf $\frac{1}{2}$ (b) vertical stretch sf 3 (c) translation $\begin{pmatrix} -\frac{\pi}{3} \\ 0 \end{pmatrix}$** **3 marks****11 (3 marks)**Translation gives $y = e^{x-2}$ **M1**

$$e^{x-2} = e^{-2} e^x$$

M1This is a stretch parallel to the y-axis with scale factor e^{-2} **A1****Answer: vertical stretch, sf e^{-2}** **3 marks****12 (6 marks)**

(a) $x^2 + 6x + 5 = (x + 3)^2 - 4$

M1So $y = (x + 3)^2 - 4$ is $y = x^2$ shifted 3 to the left**A1**and 4 down: translation $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$ **A1****Answer: translation $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$** **3 marks****(b)** $y = \sin x$ to $y = \sin 3x$: stretch parallel to the x-axis, scale factor $\frac{1}{3}$ **B1** $y = \sin 3x$ to $y = 2 \sin 3x$: stretch parallel to the y-axis, scale factor 2**B1**Then translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to give $y = 2 \sin 3x + 1$ **B1****Answer: stretch x factor $\frac{1}{3}$; stretch y factor 2; translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$** **3 marks**

13 (6 marks)

(a) Translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$: add 2 to each x

M1

A(-1, 0), B(1, 3), C(3, -1), D(5, 2)

A1

Answer: A(-1, 0), B(1, 3), C(3, -1), D(5, 2)

2 marks

(b) Reflection in the x-axis: change the sign of each y

M1

A(-3, 0), B(-1, -3), C(1, 1), D(3, -2)

A1

Answer: A(-3, 0), B(-1, -3), C(1, 1), D(3, -2)

2 marks

(c) Stretch parallel to the x-axis, scale factor $\frac{1}{2}$: halve each x

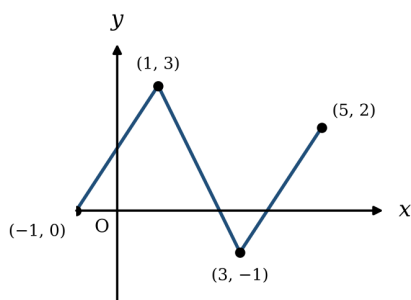
M1

A(-1.5, 0), B(-0.5, 3), C(0.5, -1), D(1.5, 2)

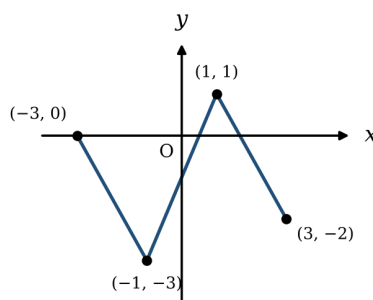
A1

Answer: A(-1.5, 0), B(-0.5, 3), C(0.5, -1), D(1.5, 2)

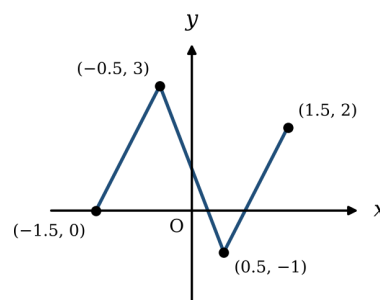
2 marks



$$y = f(x - 2)$$



$$y = -f(x)$$



$$y = f(2x)$$

14 (6 marks)

(a) Translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$: turning points (-1, 0) and (1, -4)

B1 B1

Meets x-axis at (-1, 0) (touching) and (2, 0)

B1

y-intercept $f(1) = -2$, so (0, -2)

B1

Answer: turning points (-1, 0) and (1, -4); meets axes at (-1, 0), (2, 0), (0, -2)

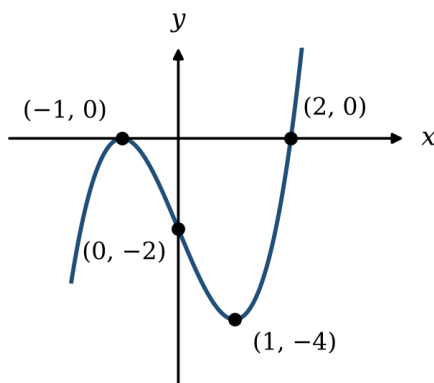
4 marks

(b) Reflection in the x-axis: (0, 0) and (2, 4)

B1 B1

Answer: (0, 0) and (2, 4)

2 marks



15 (5 marks)

(a) Stretch parallel to the x-axis, scale factor $\frac{1}{3}$

B1

Stretch parallel to the y-axis, scale factor 2

B1

Translation by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

B1

Answer: stretch x by $\frac{1}{3}$; stretch y by 2; translate $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

3 marks

(b) Max $2(1) - 1 = 1$, min $2(-1) - 1 = -3$

B1

Period $\frac{2\pi}{3}$

B1

Answer: max 1, min -3, period $\frac{2\pi}{3}$

2 marks

16 (4 marks)

(a) After translation: $y = (x + 4)^2$

M1

Stretch sf $\frac{1}{2}$ in x replaces x by $2x$: $y = (2x + 4)^2$

M1 A1

Answer: shown

3 marks

(b) $(2x + 4)^2 = 0$ when $x = -2$, so $(-2, 0)$

B1

Answer: $(-2, 0)$

1 mark

17 (5 marks)

(a) After reflection: $y = -x^2$

M1

After translation: $y = -(x - 3)^2 + 5$

A1

Answer: $y = 5 - (x - 3)^2$

2 marks

(b) Maximum at $(3, 5)$

B1

$(x - 3)^2 = 5$

M1

$x = 3 \pm \sqrt{5}$

A1

Answer: $(3, 5)$; $x = 3 \pm \sqrt{5}$

3 marks

18 (4 marks)

x -coordinates decrease by 2: $x - 2$

M1

y becomes $3 - y$

M1

$(-4, 2), (-2, -1), (1, 5)$

A1 A1

Answer: $(-4, 2), (-2, -1), (1, 5)$

4 marks

19 (10 marks)

(a) $2x^2 - 12x + 13 = 2(x^2 - 6x) + 13$

M1

$= 2(x - 3)^2 - 18 + 13 = 2(x - 3)^2 - 5$

M1 A1

Answer: $y = 2(x - 3)^2 - 5$

3 marks

(b) Translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ gives $y = (x - 3)^2$

B1

Stretch parallel to the y -axis, scale factor 2, gives $y = 2(x - 3)^2$

B1

Translation by $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ gives $y = 2(x - 3)^2 - 5$

B1

Minimum point $(3, -5)$

B1

Answer: translate $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$; stretch y by 2; translate $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$; minimum $(3, -5)$

4 marks

(c) $2(x - 3)^2 = 5$, so $(x - 3)^2 = \frac{5}{2}$

M1

$x - 3 = \pm \frac{\sqrt{10}}{2}$

M1

$x = 3 \pm \frac{\sqrt{10}}{2}$

A1

Answer: $x = 3 \pm \frac{\sqrt{10}}{2}$

3 marks

20 (8 marks)

(a) Maximum 5 and minimum -3 : $a = \frac{5 + (-3)}{2} = 1$, $b = \frac{5 - (-3)}{2} = 4$

M1 A1

Period is π , so $c = \frac{2\pi}{\pi} = 2$

B1

Answer: $a = 1$, $b = 4$, $c = 2$

3 marks

(b) Stretch parallel to the x -axis, scale factor $\frac{1}{2}$

B1

Stretch parallel to the y -axis, scale factor 4

B1

Translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

B1

Answer: stretch x factor $\frac{1}{2}$; stretch y factor 4; translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3 marks

(c) The line $y = 3$ lies strictly between the minimum -3 and the maximum 5
so it meets each of the two complete cycles twice: 4 solutions

M1

A1

Answer: 4 solutions

2 marks

21 (9 marks)

(a) Reflection: $y = -(x^2 - 4x + 1)$

M1

Translation: $y = -[(x - 2)^2 - 4(x - 2) + 1] + 3$

M1

$= -(x^2 - 8x + 13) + 3 = -x^2 + 8x - 10$

A1

Answer: shown

3 marks

(b) $y = -(x - 4)^2 + 6$, maximum $(4, 6)$

M1 A1

$(x - 4)^2 = 6$, $x = 4 \pm \sqrt{6}$

A1

Answer: $(4, 6)$; $x = 4 \pm \sqrt{6}$

3 marks

(c) Translation: $y = x^2 - 8x + 13 + 3 = x^2 - 8x + 16$, reflect: $y = -(x - 4)^2$

M1 A1

A different curve (it touches the x-axis at $(4, 0)$): the order matters

E1

Answer: $y = -(x - 4)^2$; the order matters

3 marks

22 (6 marks)

(a) Translation: $y = 2^{x+3}$

M1

$2^{x+3} = 2^3 \times 2^x = 8 \times 2^x$, a stretch sf 8 in the y direction

A1

Answer: shown

2 marks

(b) $y = 2^{-x} - 4$

B1

Asymptote $y = -4$

B1

y-intercept: $1 - 4 = -3$, so $(0, -3)$

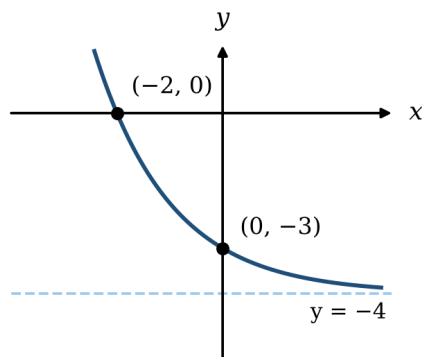
B1

$2^{-x} = 4$ gives $x = -2$, so $(-2, 0)$

B1

Answer: $y = 2^{-x} - 4$, asymptote $y = -4$, crossing at $(0, -3)$ and $(-2, 0)$

4 marks



23 (6 marks)

(a) $y = f(x - 3) - 2 = (x - 3)^2 + b(x - 3) + c - 2$

M1

$= x^2 - 6x + 9 + bx - 3b + c - 2$

M1

$= x^2 + (b - 6)x + c - 3b + 7$

A1

Answer: shown

3 marks

(b) $b - 6 = -10$, so $b = -4$

M1 A1

$c + 12 + 7 = 27$, so $c = 8$

A1

Answer: $b = -4$, $c = 8$

3 marks

24 (6 marks)

(a) $f(4 - x) = (4 - x)^2 - 4(4 - x) + 3$

M1

$= 16 - 8x + x^2 - 16 + 4x + 3 = x^2 - 4x + 3 = f(x)$

A1

Answer: shown

2 marks

(b) $y = f(4 - x)$ is $f(-x)$ translated 4 right, i.e. $y = f(-(x - 4))$: reflection in the y-axis then translation 4 to the right, which equals reflection in $x = 2$

M1 E1

The graph is unchanged by this reflection, so $x = 2$ is a line of symmetry

A1

Answer: reflection in $x = 2$; line of symmetry $x = 2$ **3 marks****(c) Translate 1 to the left: $x = 1$** **B1****Answer: $x = 1$** **1 mark****25 (8 marks)****(a) Passing through O means $f(0 + \alpha) = 0$, so $f(\alpha) = 0$** **M1** **$\alpha = 1$ or $\alpha = 5$** **A1****Answer: $\alpha = 1$ or $\alpha = 5$** **2 marks****(b) $f(10k) = 0$** **M1** **$10k = 1$ or $10k = 5$, so $k = \frac{1}{10}$ or $k = \frac{1}{2}$** **M1 A1****Answer: $k = \frac{1}{10}$ or $k = \frac{1}{2}$** **3 marks****(c) Translation by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: minimum at $(4, -4)$** **M1****Stretch factor 2 in y and translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$: $(4, 2 \times (-4) + 3) = (4, -5)$** **M1 A1****Answer: $(4, -5)$** **3 marks****26 (10 marks)****(a) Stretch: $y = f(2x)$** **M1****Then translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: replace x by $x - 3$, giving $y = f(2(x - 3)) = f(2x - 6)$** **A1****Answer: $y = f(2x - 6)$** **2 marks****(b) Translation first: $y = f(x - 3)$** **M1****Then stretch: replace x by $2x$, giving $y = f(2x - 3)$** **A1****Answer: $y = f(2x - 3)$** **2 marks****(c) C_1 : minimum when $2x - 6 = 2$, so $x = 4$: $(4, -1)$** **M1** **C_2 : minimum when $2x - 3 = 2$, so $x = \frac{5}{2}$: $(\frac{5}{2}, -1)$** **A1****The minimum points are different, so the two curves are different****E1****Answer: C_1 : $(4, -1)$; C_2 : $(\frac{5}{2}, -1)$; the curves differ****3 marks****(d) $(2x - 6)^2 - 4(2x - 6) + 3 = 3$, so $(2x - 6)(2x - 10) = 0$** **M1 A1** **$x = 3$ or $x = 5$** **A1****Answer: $x = 3$ or $x = 5$** **3 marks****27 (9 marks)****(a) Student A: replacing x by $(x - 4)$ moves the graph 4 units to the right (when $x = 4$ the output is 0)****E1 E1****Student B: multiplying the output by 2 is a stretch parallel to the y-axis, scale factor 2; a stretch****E1 E1****parallel to the x-axis would give $y = (x/2)^2 = \frac{x^2}{4}$** **Answer: A: translation is to the right; B: stretch is parallel to the y-axis****4 marks****(b) Stretch parallel to the x-axis with scale factor $\frac{1}{k}$ gives $y = (kx)^2 = k^2x^2$** **M1****Need $k^2 = 2$, so $k = \sqrt{2}$** **A1****Scale factor $\frac{1}{\sqrt{2}}$** **A1****Answer: scale factor $\frac{1}{\sqrt{2}}$** **3 marks****(c) $3e^x = e^{\ln 3}e^x = e^{x + \ln 3}$** **M1****Translation $\begin{pmatrix} -\ln 3 \\ 0 \end{pmatrix}$** **A1****Answer: translation $\begin{pmatrix} -\ln 3 \\ 0 \end{pmatrix}$** **2 marks**

28 (10 marks)

(a) Roots increase by 2: replace x by $(x - 2)$ (translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$) M1

$$(x - 2)^3 - 4(x - 2)^2 + (x - 2) + 6 = 0$$
M1

$$x^3 - 6x^2 + 12x - 8 - 4x^2 + 16x - 16 + x - 2 + 6 = 0$$
M1

$$x^3 - 10x^2 + 29x - 20 = 0$$
A1

Answer: $x^3 - 10x^2 + 29x - 20 = 0$ 4 marks

(b) Roots doubled: replace x by $\frac{x}{2}$ (stretch parallel to the x -axis, scale factor 2) M1

$$\frac{x^3}{8} - x^2 + \frac{x}{2} + 6 = 0$$
M1

$$\text{Multiply by 8: } x^3 - 8x^2 + 4x + 48 = 0$$
A1

Answer: $x^3 - 8x^2 + 4x + 48 = 0$ 3 marks

(c) Reflection in the y -axis: replace x by $-x$ M1

$$-x^3 - 4x^2 - x + 6 = 0$$
M1

$$x^3 + 4x^2 + x - 6 = 0$$
A1

Answer: $x^3 + 4x^2 + x - 6 = 0$ 3 marks

29 (12 marks)

(a) A: after translation $y = (x - 2)^2$, then x replaced by $\frac{x}{3}$: $y = (\frac{x}{3} - 2)^2$ M1 A1

B: after stretch $y = (\frac{x}{3})^2$, then translation: $y = (\frac{x - 2}{3})^2$ M1 A1

Answer: A: $y = (\frac{x}{3} - 2)^2$; B: $y = (\frac{x - 2}{3})^2$ 4 marks

(b) A has vertex (6, 0); B has vertex (2, 0) B1 B1

Curve A is the one with vertex (6, 0) (the navy curve), curve B the red curve E1

Answer: A (6, 0); B (2, 0) 3 marks

(c) In A the stretch also scales the translation distance: 2 becomes 6 E1

Translation by $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ after the stretch gives A B1

Answer: $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ 2 marks

(d) $(\frac{x}{3} - 2)^2 = (\frac{x - 2}{3})^2$, so $|\frac{x - 6}{3}| = |\frac{x - 2}{3}|$ M1

$$x - 6 = -(x - 2) \text{ (the other case gives } -6 = -2)$$
M1

$$2x = 8, x = 4$$
A1

Answer: $x = 4$ 3 marks

30 (12 marks)

(a) Roots of $f(x) + k = 0$ are the intersections of $y = f(x)$ with $y = -k$ M1

Three intersections when the horizontal line lies strictly between the minimum -2 and maximum M1 E1

$$2: -2 < -k < 2$$

That is $-2 < k < 2$ A1

Answer: shown 4 marks

(b) $f(-a) = 0$, so $-a^3 + 3a = 0$ M1

$$a(3 - a^2) = 0, \text{ so } a = 0 \text{ or } a = \pm\sqrt{3}$$
A1 A1

Answer: $a = 0, \pm\sqrt{3}$ 3 marks

(c) Touching occurs at a turning point lying on the x -axis M1

Turning points are (0, $2 + k$) and (2, $-2 + k$) M1 A1

$k = -2$: touches at $x = 0$; $k = 2$: touches at $x = 2$ A1 A1

Answer: $k = -2$ (touching at $x = 0$) or $k = 2$ (touching at $x = 2$) 5 marks

Polynomial Graphs

1 (2 marks)

(a) Roots of each bracket: $x = 1$, $x = -2$, $x = 4$

Answer: (1, 0), (-2, 0), (4, 0)

B1

1 mark

(b) $x = 0$: $y = (-1)(2)(-4) = 8$

Answer: (0, 8)

B1

1 mark

2 (3 marks)

Positive cubic: starts bottom left and ends top right

B1

Crosses the x-axis at $x = -2$, $x = 0$ and $x = 3$

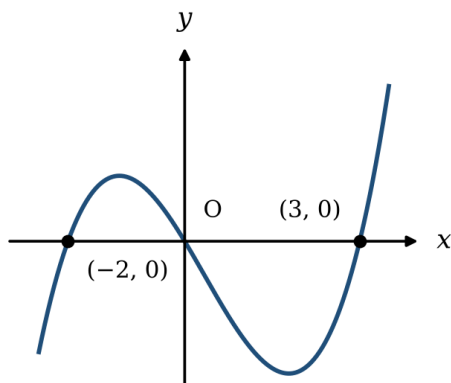
B1

Passes through the origin, so (0, 0) is the y-intercept; each root is a simple crossing

B1

Answer: cubic through (-2, 0), (0, 0) and (3, 0), positive cubic shape

3 marks



3 (2 marks)

Each factor in turn is zero

M1

$x = -1$, $x = \frac{3}{2}$, $x = 4$

A1

Answer: $x = -1, \frac{3}{2}, 4$

2 marks

4 (3 marks)

Touches the x-axis at the origin (repeated root $x = 0$)

B1

Crosses the x-axis at (4, 0), positive cubic shape

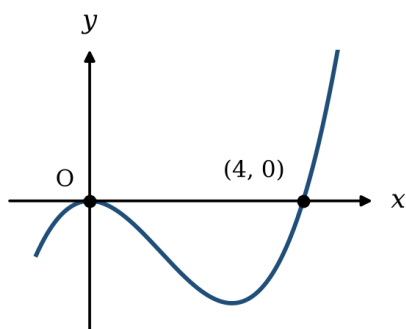
B1

Passes through the origin, shape: rises from bottom left, touches at O, dips, then rises through (4, 0)

B1

Answer: touches at the origin, crosses at (4, 0)

3 marks



5 (3 marks)

Roots at $x = 1$, $x = -2$, $x = -4$

B1

y-intercept: $(1)(2)(4) = 8$

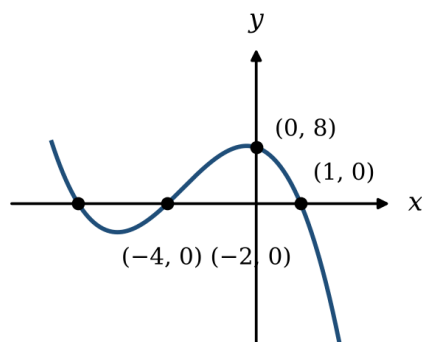
B1

Coefficient of x^3 is negative, so the curve falls from top left to bottom right

B1

Answer: negative cubic through (-4, 0), (-2, 0), (1, 0), y-intercept (0, 8)

3 marks



6 (4 marks)

Positive cubic shape

Repeated root at $x = 2$: the curve touches the x -axis at $(2, 0)$

Single root at $x = -1$: crosses the x -axis at $(-1, 0)$

$x = 0$: $y = (-2)^2(1) = 4$, so the y -intercept is $(0, 4)$

Answer: cubic touching at $(2, 0)$, crossing at $(-1, 0)$, y -intercept $(0, 4)$

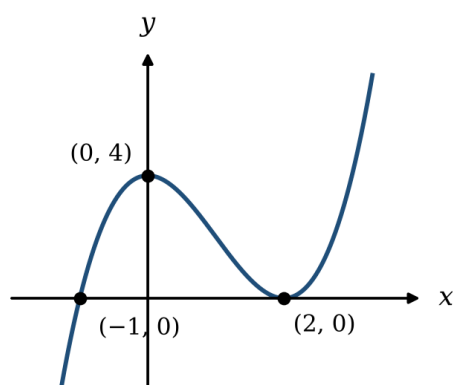
B1

B1

B1

B1

4 marks



7 (5 marks)

(a) $f(x) = k(x + 3)(x - 1)(x - 4)$

$f(0) = k(3)(-1)(-4) = 12k = 24$, so $k = 2$

$f(x) = 2(x + 3)(x - 1)(x - 4)$

Answer: $f(x) = 2(x + 3)(x - 1)(x - 4)$

M1

M1

A1

3 marks

(b) $(x + 3)(x - 1) = x^2 + 2x - 3$; $(x^2 + 2x - 3)(x - 4) = x^3 - 2x^2 - 11x + 12$

$f(x) = 2x^3 - 4x^2 - 22x + 24$

Answer: $2x^3 - 4x^2 - 22x + 24$

M1

A1

2 marks

8 (4 marks)

Roots $x = -2$ and $x = 2$ (crossing), $x = 0$ (touching)

Positive quartic, W-shape with the middle touching the x -axis at O

Passes through the origin

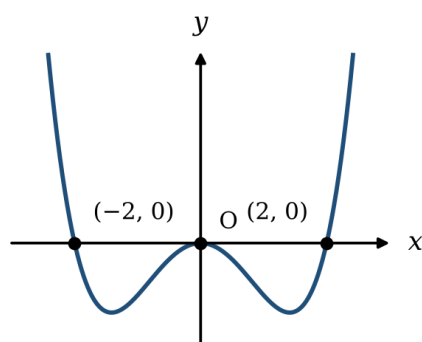
Answer: quartic touching at O, crossing at $(-2, 0)$ and $(2, 0)$

B1 B1

B1

B1

4 marks



9 (3 marks)

Triple root: crosses the x-axis at (1, 0) with a point of inflection there

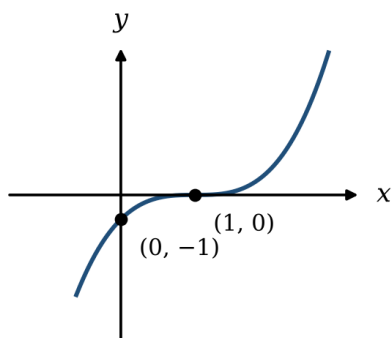
B1 B1

y-intercept: $(-1)^3 = -1$, so (0, -1)

B1

Answer: cubic with inflection at (1, 0), y-intercept (0, -1)

3 marks

**10 (4 marks)**

(a) $f(x) = k(x + 2)(x - 1)(x - 3)$

B1

$f(0) = k(2)(-1)(-3) = 6k = 12$

M1

$k = 2$

A1

Answer: $k = 2$

3 marks

(b) $f(4) = 2(6)(3)(1) = 36$

B1

Answer: 36

1 mark

11 (3 marks)

Positive cubic, roots -1, 2, 4

M1

Curve above the x-axis between -1 and 2, and for $x > 4$

M1

$-1 < x < 2$ or $x > 4$

A1

Answer: $-1 < x < 2$ or $x > 4$

3 marks

12 (6 marks)

(a) Leading term x^3 is positive: the curve rises on the right and falls on the left

M1

$y \rightarrow \infty$ as $x \rightarrow \infty$, and $y \rightarrow -\infty$ as $x \rightarrow -\infty$

A1

Answer: $y \rightarrow \infty$ as $x \rightarrow \infty$; $y \rightarrow -\infty$ as $x \rightarrow -\infty$

2 marks

(b) Leading term is $-x^3$

M1

$y \rightarrow -\infty$ as $x \rightarrow \infty$, and $y \rightarrow \infty$ as $x \rightarrow -\infty$

A1

Answer: $y \rightarrow -\infty$ as $x \rightarrow \infty$; $y \rightarrow \infty$ as $x \rightarrow -\infty$

2 marks

(c) Leading term is x^4 , an even power with positive coefficient

M1

$y \rightarrow \infty$ as $x \rightarrow \infty$, and $y \rightarrow \infty$ as $x \rightarrow -\infty$

A1

Answer: $y \rightarrow \infty$ in both directions

2 marks

13 (6 marks)

(a) $f(3) = 27 - 27 - 12 + 12 = 0$, so $(x - 3)$ is a factor

B1

$x^3 - 3x^2 - 4x + 12 = x^2(x - 3) - 4(x - 3) = (x - 3)(x^2 - 4)$

M1

$= (x - 3)(x - 2)(x + 2)$

A1

Answer: $(x - 3)(x - 2)(x + 2)$

3 marks

(b) Positive cubic shape

B1

Crosses the x-axis at $(-2, 0)$, $(2, 0)$ and $(3, 0)$

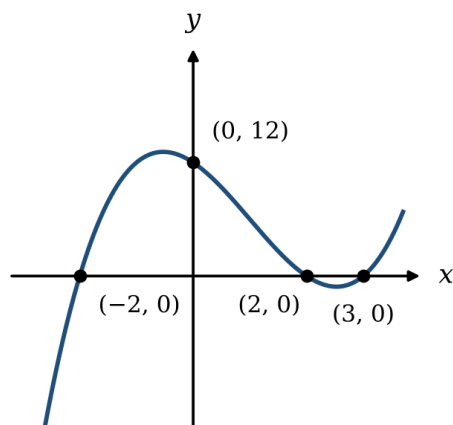
B1

y-intercept at $(0, 12)$

B1

Answer: cubic crossing at $(-2, 0)$, $(2, 0)$, $(3, 0)$, y-intercept $(0, 12)$

3 marks



14 (7 marks)

(a) $x^3 - 4x = x(x - 2)(x + 2)$: positive cubic crossing the x-axis at $-2, 0, 2$

B1 B1

The line $y = x$ passes through the origin with gradient 1, and meets the curve three times

B1

Answer: cubic through $(-2, 0)$, $(0, 0)$, $(2, 0)$ with the line $y = x$ through the origin

3 marks

(b) $x^3 - 4x = x$, so $x^3 - 5x = 0$

M1

$$x(x^2 - 5) = 0$$

A1

$$x = 0 \text{ or } x = \pm\sqrt{5}$$

A1

Answer: $x = 0$, $x = \pm\sqrt{5}$

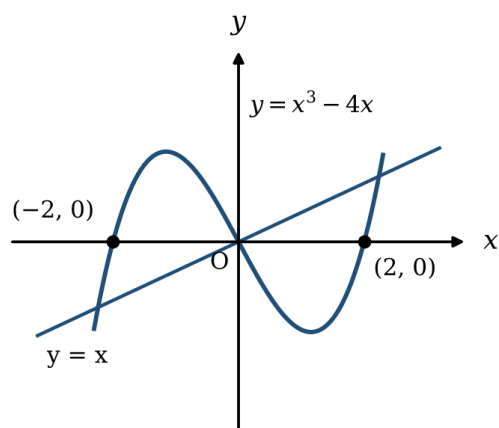
3 marks

(c) The line meets the curve at three places, one at the origin and one each side of it

E1

Answer: three intersections

1 mark



15 (6 marks)

(a) Roots $x = -2$, $x = 3$ (crossing) and $x = 0$ (touching)

B1 B1

Coefficient of x^4 is -1 : both ends go down, hump on each side of the touching point

B1 B1

Answer: quartic touching at O, crossing at $(-2, 0)$ and $(3, 0)$, both ends down

4 marks

(b) The curve is above the x-axis between -2 and 3 (excluding the touching point at $x = 0$)

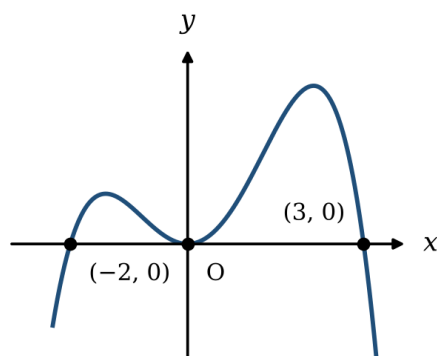
M1

$$-2 < x < 3, x \neq 0$$

A1

Answer: $-2 < x < 3, x \neq 0$

2 marks



16 (6 marks)

(a) $f(2) = 8 + 4a + 2b - 6 = 0$, so $2a + b = -1$

M1 A1

$f(-1) = -1 + a - b - 6 = -12$, so $a - b = -5$

M1

$a = -2, b = 3$

A1

Answer: $a = -2, b = 3$

4 marks

(b) $f(x) = x^3 - 2x^2 + 3x - 6 = (x - 2)(x^2 + 3)$

M1

$x^2 + 3 = 0$ has no real solutions, so the only real root is $x = 2$

E1

Answer: only $x = 2$

2 marks

17 (6 marks)

(a) $x(x^2 - 6x + 9)$

M1

$= x(x - 3)^2$

A1

Answer: $x(x - 3)^2$

2 marks

(b) $1 - 6 + 9 - 4 = 0$

B1

$x^3 - 6x^2 + 9x - 4 = (x - 1)(x^2 - 5x + 4)$

M1

$= (x - 1)^2(x - 4)$, so $x = 1$ (repeated) or 4

A1

Answer: $x = 1$ (twice), $x = 4$

3 marks

(c) The line $y = 4$ is a tangent to the curve at $x = 1$ (a turning point), and cuts it again at $x = 4$

E1

Answer: $y = 4$ touches C at $x = 1$

1 mark

18 (5 marks)

(a) $x(x^2 - 1)$

M1

$= x(x - 1)(x + 1)$

A1

Answer: $x(x - 1)(x + 1)$

2 marks

(b) $f(x + 2) = (x + 2)(x + 1)(x + 3)$

B1

Roots at $-3, -2, -1$ and y-intercept 6

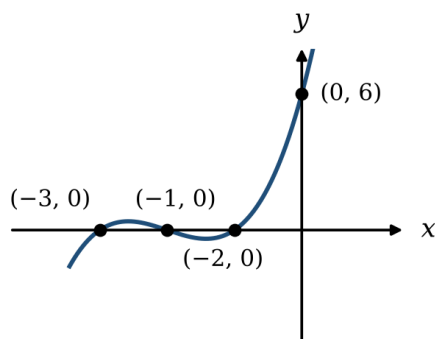
B1

Positive cubic shape through these points

B1

Answer: $y = (x + 1)(x + 2)(x + 3)$, crossing at $(-3, 0), (-2, 0), (-1, 0), (0, 6)$

3 marks

**19 (7 marks)**

(a) $x^2(x - 3) = x(3 - x)$

M1

$x^3 - 3x^2 = 3x - x^2$, so $x^3 - 2x^2 - 3x = 0$, i.e. $x(x - 3)(x + 1) = 0$

M1 A1

$(0, 0), (3, 0)$ and $(-1, -4)$

A1

Answer: $(0, 0), (3, 0), (-1, -4)$

4 marks

(b) C_1 touches the x-axis at O and crosses it at $(3, 0)$: positive cubic

B1

C_2 is an inverted parabola through O and $(3, 0)$

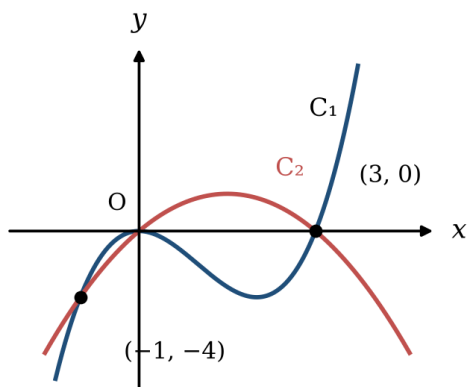
B1

The intersections $(0, 0), (3, 0)$ and $(-1, -4)$ are marked

B1

Answer: cubic touching at O, crossing at $(3, 0)$; parabola through O and $(3, 0)$

3 marks



20 (5 marks)

(a) $y = \frac{1}{10}(-1)(-4)(-7) = -2.8$

M1

The road is 2.8 km below sea level at $x = 0$

A1

Answer: -2.8 km

2 marks

(b) The curve is below the x-axis for $x < 1$ and for $4 < x < 7$

M1

$0 \leq x < 1$ and $4 < x < 7$

A1

Answer: $0 \leq x < 1$ and $4 < x < 7$

2 marks

(c) For example, a road section 2.8 km below sea level is implausible; or the cubic gives a very smooth profile

E1

Answer: e.g. height of -2.8 km is not plausible

1 mark

21 (8 marks)

(a) $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4)$

M1

$= (x - 1)(x + 1)(x - 2)(x + 2)$

A1

Answer: $(x - 1)(x + 1)(x - 2)(x + 2)$

2 marks

(b) Positive quartic: W shape

B1

Crosses the x-axis at $(-2, 0)$, $(-1, 0)$, $(1, 0)$, $(2, 0)$

B1

y-intercept $(0, 4)$

B1

Answer: W-shaped quartic crossing at $(-2, 0)$, $(-1, 0)$, $(1, 0)$, $(2, 0)$, y-intercept $(0, 4)$

3 marks

(c) Let $u = x^2$: $u^2 - 5u + 4 - k = 0$ needs two distinct positive roots

M1

Discriminant $25 - 4(4 - k) > 0$, so $k > -\frac{9}{4}$

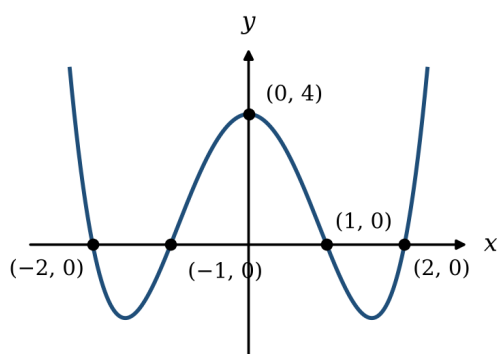
A1

Product of roots $4 - k > 0$ (and sum $5 > 0$), so $k < 4$: $-\frac{9}{4} < k < 4$

A1

Answer: $-\frac{9}{4} < k < 4$

3 marks



22 (8 marks)

(a) $f(x) = k(x - 1)(x - 3)^2$, since 3 is a repeated root

M1

$f(0) = k(-1)(9) = -9k = -9$, so $k = 1$

M1

$f(x) = (x - 1)(x - 3)^2$

A1

Answer: shown

3 marks

(b) $y = (x + 2 - 1)(x + 2 - 3)^2 = (x + 1)(x - 1)^2$

B1

Meets the x-axis at $(-1, 0)$ and $(1, 0)$ (touching at $(1, 0)$)

B1

y-intercept: $(1)(-1)^2 = 1$, so $(0, 1)$

B1

Answer: $y = (x + 1)(x - 1)^2$; $(-1, 0)$, $(1, 0)$, $(0, 1)$

3 marks

(c) $f(a) = 0$

M1

$a = 1$ or $a = 3$

A1

Answer: $a = 1$ or $a = 3$

2 marks

23 (10 marks)

(a) $y = k(x + 2)^2(x - 3)$

B1

$-12 = k(4)(-3)$, so $k = 1$

M1 A1

Answer: $k = 1$

3 marks

(b) Touching at $(-2, 0)$ and crossing at $(3, 0)$

B1

Correct shape with $(0, -12)$ marked

B1

Answer: touches at $(-2, 0)$, crosses at $(3, 0)$, y-intercept $(0, -12)$

2 marks

(c) $x^3 + x^2 - 8x - 12 = -12$, so $x(x^2 + x - 8) = 0$

M1 A1

$x = 0$ or $x = \frac{-1 \pm \sqrt{33}}{2}$

A1

Answer: $x = 0, \frac{-1 \pm \sqrt{33}}{2}$

3 marks

(d) The curve is on or above the x-axis for $x \geq 3$ and at the touching point

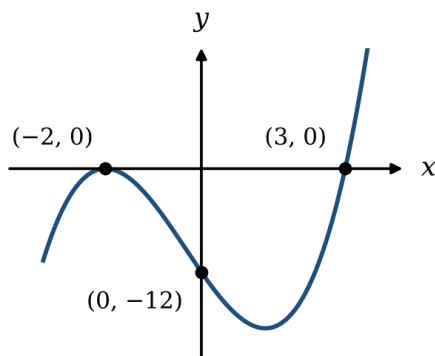
M1

$x \geq 3$ or $x = -2$

A1

Answer: $x \geq 3$ or $x = -2$

2 marks



24 (9 marks)

(a) $f(3) = 54 - 27 - 33 + 6 = 0$

M1 A1

$f(x) = (x - 3)(2x^2 + 3x - 2)$

M1

$= (x - 3)(2x - 1)(x + 2)$

A1

Answer: $(x - 3)(2x - 1)(x + 2)$

4 marks

(b) Roots $-2, 0.5, 3$

B1

y-intercept $(0, 6)$

B1

Positive cubic shape

B1

Answer: cubic crossing at $(-2, 0)$, $(1/2, 0)$, $(3, 0)$, y-intercept $(0, 6)$

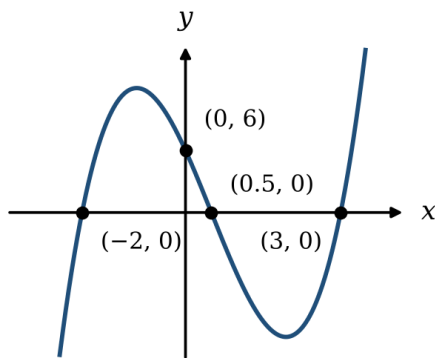
3 marks

(c) The curve is below the x-axis for $x < -2$ and for $\frac{1}{2} < x < 3$

M1 A1

Answer: $x < -2$ or $\frac{1}{2} < x < 3$

2 marks

**25 (10 marks)**

(a) $f(2) = 8 + 4p + 2q + 12 = 0$, so $2p + q = -10$

$f(-3) = -27 + 9p - 3q + 12 = 0$, so $3p - q = 5$

$p = -1, q = -8$

Answer: $p = -1, q = -8$

(b) Write $f(x) = (x - 2)(x + 3)(x + c)$; constant terms: $(-2)(3)c = 12$, so $c = -2$

So $f(x) = (x - 2)(x + 3)(x - 2) = (x - 2)^2(x + 3)$

Answer: shown

(c) Touches the x-axis at $(2, 0)$, crosses at $(-3, 0)$, y-intercept $(0, 12)$

$f(x) > 0$ for $x > -3$ except at $x = 2$

Answer: $x > -3, x \neq 2$

M1 A1

M1

A1

4 marks

M1

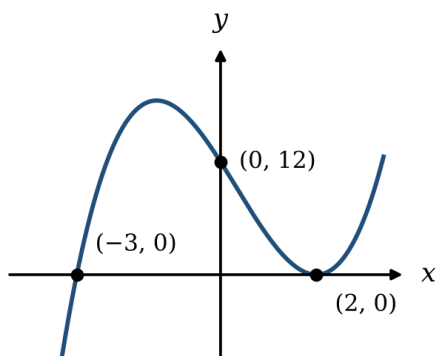
A1

2 marks

B1 B1

M1 A1

4 marks

**26 (9 marks)**

(a) C crosses the x-axis at $(-p, 0)$ and touches it at $(q, 0)$

Positive cubic: starts bottom left, touches the axis at $(q, 0)$, then rises

y-intercept: $(p)(q^2)$ so $(0, pq^2)$

Answer: crosses at $(-p, 0)$, touches at $(q, 0)$, y-intercept $(0, pq^2)$

(b) Replace x by $x + q$: $y = (x + q + p)((x + q) - q)^2$

$y = x^2(x + p + q)$

x^2 is a repeated factor, so the image touches the x-axis at the origin

Answer: $y = x^2(x + p + q)$; touches the x-axis at O

(c) $(x - q)^2 \geq 0$ for all x , with equality at $x = q$

So the product is non-negative exactly when $x + p \geq 0$ or $x = q$: $x \geq -p$

Answer: $x \geq -p$

B1 B1

B1

B1

4 marks

M1

A1

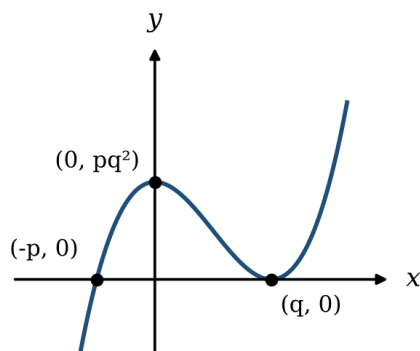
E1

3 marks

M1

A1

2 marks

**27 (11 marks)**

(a) $f(x) = k(x - 2)^2(x - b)$

M1

$f(0) = -4kb = -12$, so $kb = 3$

M1

$f(5) = 9k(5 - b) = 18$, so $5k - kb = 2$, giving $5k - 3 = 2$, $k = 1$

M1

$b = 3$, so $f(x) = (x - 2)^2(x - 3)$

A1

Answer: $f(x) = (x - 2)^2(x - 3)$

4 marks

(b) $(x - 2)^2(x - 3) = 4$ gives $x^3 - 7x^2 + 16x - 16 = 0$

M1

$x = 4$ is a root: $(x - 4)(x^2 - 3x + 4) = 0$

A1

$x^2 - 3x + 4$ has discriminant $9 - 16 < 0$, so the only point is $(4, 4)$

A1

Answer: $(4, 4)$

3 marks

(c) $f'(x) = 3x^2 - 14x + 16 = (3x - 8)(x - 2) = 0$

M1

Minimum at $x = \frac{8}{3}$

A1

$f(\frac{8}{3}) = (\frac{2}{3})^2(-\frac{1}{3}) = -\frac{4}{27}$

A1

The local maximum is $(2, 0)$, so three solutions when $-\frac{4}{27} < k < 0$

A1

Answer: min point $(\frac{8}{3}, -\frac{4}{27})$; $-\frac{4}{27} < k < 0$

4 marks

28 (9 marks)

(a) Base is $(30 - 2x)$ by $(20 - 2x)$ and height is x

M1

$V = x(30 - 2x)(20 - 2x) = 2(15 - x) \times 2(10 - x) \times x = 4x(10 - x)(15 - x)$

A1

Answer: shown

2 marks

(b) Need $x > 0$ and $20 - 2x > 0$, so $0 < x < 10$

B1

Positive cubic crossing the x -axis at 0, 10 and 15

B1

Line $V = 1000$ meets the curve at the two points found in part (c)

B1

Answer: $0 < x < 10$; cubic crossing at 0, 10 and 15, meeting $V = 1000$ twice

3 marks

(c) $4x(10 - x)(15 - x) = 1000$, so $x^3 - 25x^2 + 150x - 250 = 0$

M1 A1

$x = 5$ is a root: $(x - 5)(x^2 - 20x + 50) = 0$

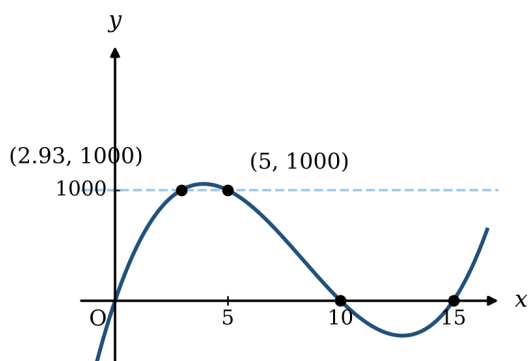
M1

$x = 10 \pm 5\sqrt{2}$; only $x = 10 - 5\sqrt{2}$ lies in the range, so $x = 5$ or $x = 10 - 5\sqrt{2}$

A1

Answer: $x = 5$ or $x = 10 - 5\sqrt{2} (\approx 2.93)$

4 marks



29 (14 marks)

(a) $(x - a)^2(x + 2a) = (x^2 - 2ax + a^2)(x + 2a)$ M1

$= x^3 + 2ax^2 - 2ax^2 - 4a^2x + a^2x + 2a^3$ M1

$= x^3 - 3a^2x + 2a^3$ A1

Answer: shown 3 marks

(b) Touches the x-axis at $(a, 0)$ B1

Crosses the x-axis at $(-2a, 0)$ B1

y-intercept $(0, 2a^3)$ B1

Answer: touches at $(a, 0)$, crosses at $(-2a, 0)$, $(0, 2a^3)$ 3 marks

(c) $2a^3 = 16$, so $a = 2$ M1 A1

Touch at $(2, 0)$, cross at $(-4, 0)$, through $(0, 16)$ B1

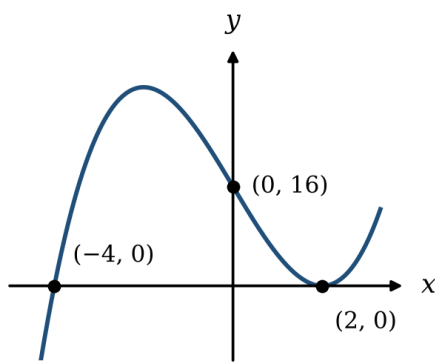
Answer: $a = 2$; touches at $(2, 0)$, crosses at $(-4, 0)$, y-intercept $(0, 16)$ 3 marks

(d) $y' = 3x^2 - 12 = 0$, so $x = \pm 2$ M1 A1

Turning points $(2, 0)$ and $(-2, 32)$ A1

A horizontal line meets the curve three times between the local minimum and maximum: $0 < k < 32$ M1 A1

Answer: $(2, 0)$, $(-2, 32)$; $0 < k < 32$ 5 marks

**30 (14 marks)**

(a) $f(3) = 27 - 81 + 3k - 24 = 0$ M1

$3k = 78$, so $k = 26$ A1

Answer: shown 2 marks

(b) $f(x) = (x - 3)(x^2 - 6x + 8)$ M1 A1

$= (x - 3)(x - 2)(x - 4)$ A1

Answer: $(x - 2)(x - 3)(x - 4)$ 3 marks

(c) Roots 2, 3, 4 B1

y-intercept $(0, -24)$ B1

Positive cubic shape B1

Answer: cubic crossing at $(2, 0)$, $(3, 0)$, $(4, 0)$, y-intercept $(0, -24)$ 3 marks

(d) Common difference 1 B1

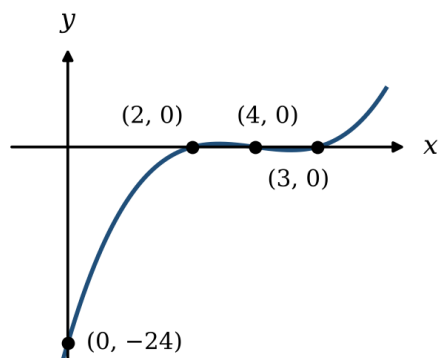
$y = (x - 1)(x - 2)(x - 3)$ M1 A1

Roots 1, 2, 3 still arithmetic: a translation does not change the spacing E1

Answer: difference 1; $y = (x - 1)(x - 2)(x - 3)$; yes 4 marks

(e) Below the x-axis for $x < 2$ and for $3 < x < 4$ M1 A1

Answer: $x < 2$ or $3 < x < 4$ 2 marks



Reciprocal and Rational Graphs

1 (2 marks)

(a) The denominator is zero when $x = 3$, and y approaches 0 for large x : $x = 3$ and $y = 0$

B1

Answer: $x = 3$ and $y = 0$

1 mark

(b) Vertical asymptote $x = 0$; the curve is $y = \frac{1}{x}$ shifted up 2: $y = 2$

B1

Answer: $x = 0$ and $y = 2$

1 mark

2 (3 marks)

Two branches, in the first and third quadrants

B1

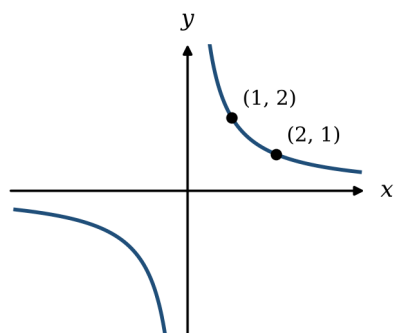
Asymptotes are the axes: $x = 0$ and $y = 0$

B1

(1, 2) and (2, 1) lie on the curve

B1

Answer: two branches in quadrants 1 and 3, asymptotes $x = 0$ and $y = 0$, through (1, 2) and (2, 1) **3 marks**



3 (3 marks)

Asymptotes are the axes: $x = 0$ and $y = 0$

B1

Reflection of $y = \frac{1}{x}$ in the x -axis: branches in quadrants 2 and 4

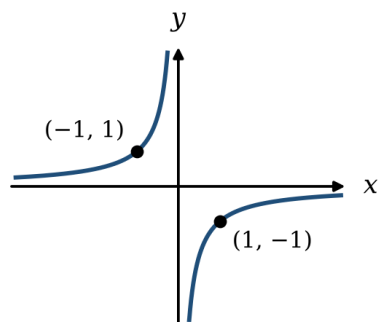
B1

Points such as (1, -1) and (-1, 1)

B1

Answer: branches in quadrants 2 and 4, asymptotes $x = 0$ and $y = 0$

3 marks



4 (3 marks)Asymptotes $x = 0$ and $y = 0$

B1

Both branches lie above the x -axis and the graph is symmetric about the y -axis

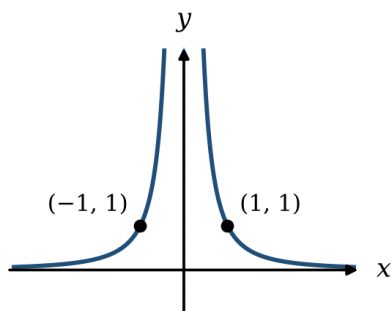
B1

Points such as $(1, 1)$ and $(-1, 1)$

B1

Answer: both branches above the x -axis, asymptotes $x = 0$ and $y = 0$

3 marks

**5 (2 marks)**

$$1 = 3(x - 2)$$

M1

$$x = \frac{7}{3}$$

A1

Answer: $x = \frac{7}{3}$

2 marks

6 (4 marks)Translation of $y = \frac{1}{x}$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$: two branches

B1

Asymptotes $x = -2$ and $y = 0$

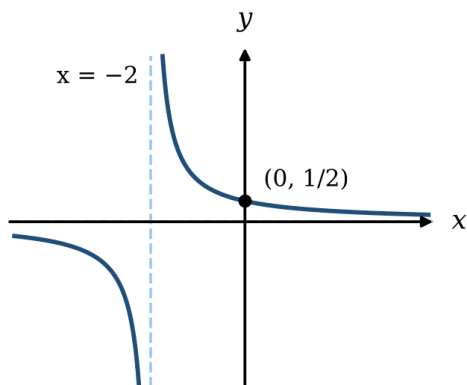
B1 B1

 $x = 0$: $y = \frac{1}{2}$, so the curve crosses the y -axis at $(0, \frac{1}{2})$ and never crosses the x -axis

B1

Answer: asymptotes $x = -2$ and $y = 0$, y -intercept $(0, 1/2)$, no x -intercept

4 marks

**7 (5 marks)**Translation of $y = \frac{3}{x}$: two branches, one in the top right and one in the bottom left of the asymptotes

B1

Asymptotes $x = 1$ and $y = 2$

B1 B1

$$y = 0: \frac{3}{x-1} = -2, \text{ so } x = -\frac{1}{2}: \text{ crosses the } x\text{-axis at } (-\frac{1}{2}, 0)$$

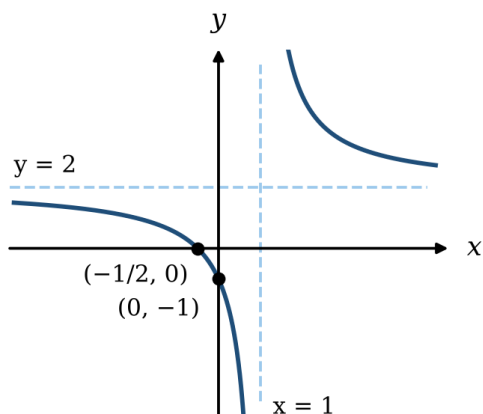
B1

$$x = 0: y = -3 + 2 = -1: \text{ crosses the } y\text{-axis at } (0, -1)$$

B1

Answer: asymptotes $x = 1$ and $y = 2$, crossing the axes at $(-1/2, 0)$ and $(0, -1)$

5 marks



8 (3 marks)

Asymptotes $x = 2$ and $y = 0$

B1

$$y\text{-intercept: } \frac{1}{(-2)^2} = \frac{1}{4}$$

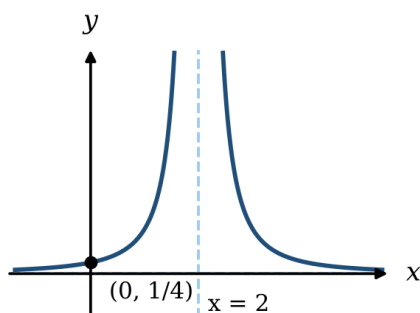
B1

Both branches above the x -axis, approaching $x = 2$ upwards

B1

Answer: asymptotes $x = 2$ and $y = 0$, y -intercept $(0, 1/4)$, both branches above the x -axis

3 marks



9 (5 marks)

$$(a) 1 + \frac{3}{x-2} = \frac{x-2+3}{x-2}$$

M1

$$= \frac{x+1}{x-2}$$

A1

Answer: shown

2 marks

(b) Asymptotes $x = 2$, $y = 1$

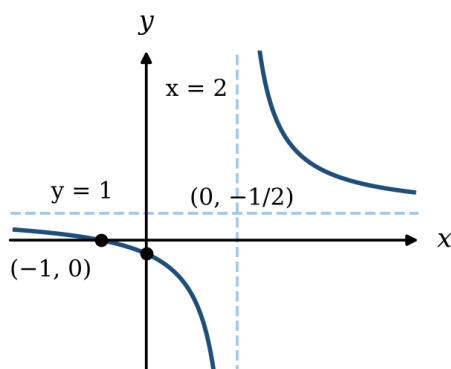
B1

x -intercept $(-1, 0)$, y -intercept $(0, -\frac{1}{2})$

B1 B1

Answer: asymptotes $x = 2$ and $y = 1$, crossing at $(-1, 0)$ and $(0, -1/2)$

3 marks



10 (3 marks)

$$1 = (x-1)(x+1) = x^2 - 1$$

M1

$$x^2 = 2$$

M1

$$x = \pm\sqrt{2}$$

A1

Answer: $x = \pm\sqrt{2}$

3 marks

11 (4 marks)

$$\frac{4}{x} = x - 3, \text{ so } 4 = x^2 - 3x$$

M1

$$x^2 - 3x - 4 = 0, (x - 4)(x + 1) = 0$$

M1

$$x = 4, y = 1 \text{ and } x = -1, y = -4: (4, 1) \text{ and } (-1, -4)$$

A1 A1

Answer: (4, 1) and (-1, -4)

4 marks

12 (3 marks)**(a)** The horizontal asymptote is $y = b = -1$

B1

Answer: $b = -1$

1 mark

$$\text{(b)} 1 = \frac{a}{-2} - 1$$

M1

$$a = -4$$

A1

Answer: $a = -4$

2 marks

13 (4 marks)**(a)** Translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

B1

Answer: translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

1 mark

(b) Stretch parallel to the y-axis, scale factor 5

B1

Answer: stretch parallel to the y-axis, scale factor 5

1 mark

(c) Translation by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$

B1

Answer: translation by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$

1 mark

(d) Reflection in the x-axis

B1

Answer: reflection in the x-axis

1 mark

14 (6 marks)**(a)** Asymptotes give $b = 1$ and $c = 3$

B1 B1

$$f(0) = \frac{a}{1} + 3 = 5, \text{ so } a = 2$$

B1

Answer: $a = 2, b = 1, c = 3$

3 marks

$$\text{(b)} \frac{2}{x+1} + 3 = 0, \text{ so } \frac{2}{x+1} = -3$$

M1

$$x + 1 = -\frac{2}{3}, \text{ so } x = -\frac{5}{3}: (-\frac{5}{3}, 0)$$

A1

Answer: $(-\frac{5}{3}, 0)$

2 marks

$$\text{(c)} \frac{2}{x+1} > 0, \text{ so } x > -1$$

B1

Answer: $x > -1$

1 mark

15 (6 marks)

$$\text{(a)} -1 + \frac{4}{x+1} = \frac{-(x+1) + 4}{x+1}$$

M1

$$= \frac{3-x}{x+1}$$

A1

Answer: shown

2 marks

(b) Asymptotes $x = -1$ and $y = -1$

B1

x-intercept: $3 - x = 0$ gives $(3, 0)$; y-intercept $(0, 3)$

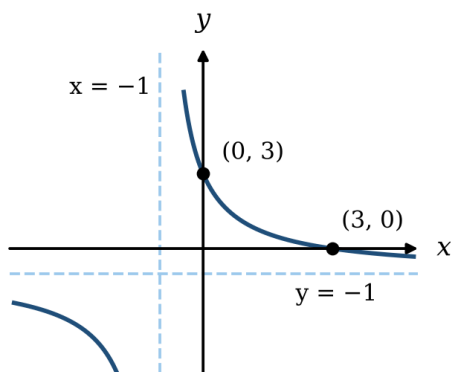
B1 B1

Both branches drawn correctly (decreasing)

B1

Answer: asymptotes $x = -1$ and $y = -1$, crossing at $(3, 0)$ and $(0, 3)$

4 marks



16 (6 marks)

(a) Stretch: $y = \frac{3}{x}$; translation: $y = \frac{3}{x-2} - 1$

M1 A1

Asymptotes $x = 2$ and $y = -1$

B1

Answer: $y = \frac{3}{x-2} - 1$; $x = 2$, $y = -1$

3 marks

(b) $y = 0$: $\frac{3}{x-2} = 1$, $x = 5$; $(5, 0)$

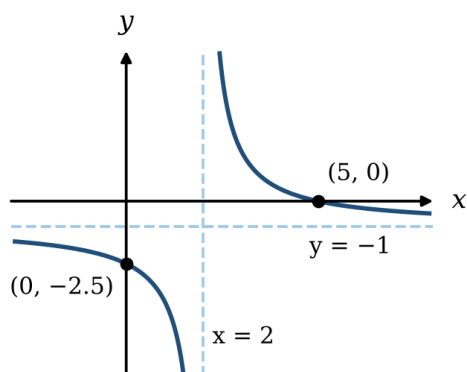
M1 A1

$x = 0$: $y = -\frac{3}{2} - 1 = -\frac{5}{2}$; $(0, -\frac{5}{2})$

B1

Answer: $(5, 0)$ and $(0, -\frac{5}{2})$

3 marks



17 (7 marks)

(a) $2 + \frac{7}{x-3} = \frac{2(x-3)+7}{x-3}$

M1

$= \frac{2x+1}{x-3}$

A1

Answer: shown

2 marks

(b) Asymptotes $x = 3$ and $y = 2$

B1

$y = 0$: $2x + 1 = 0$, so $(-\frac{1}{2}, 0)$

B1

$x = 0$: $y = \frac{1}{-3}$, so $(0, -\frac{1}{3})$

B1

Answer: $x = 3$, $y = 2$; $(-\frac{1}{2}, 0)$, $(0, -\frac{1}{3})$

3 marks

(c) $2x + 1 = x(x - 3)$, so $x^2 - 5x - 1 = 0$

M1

$x = \frac{5 \pm \sqrt{29}}{2}$

A1

Answer: $x = \frac{5 \pm \sqrt{29}}{2}$

2 marks

18 (5 marks)

(a) $x^2 - 4 = 0$ gives $x = 2$ and $x = -2$

B1

$y = 0$

B1

Answer: $x = 2$, $x = -2$, $y = 0$

2 marks

(b) y-intercept: $\frac{1}{-4}$, so $(0, -\frac{1}{4})$

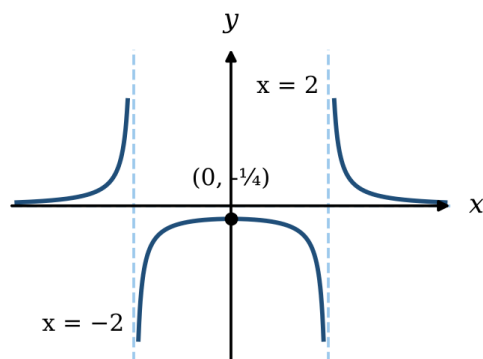
B1

Outer branches above the x-axis, middle branch below, symmetric about the y-axis

B1 B1

Answer: y-intercept $(0, -1/4)$, outer branches above the x-axis, middle branch below

3 marks



19 (5 marks)

(a) Horizontal line $y = 2$ meeting the right-hand branch

B1

$$\frac{1}{x-1} = 2, \text{ so } x-1 = \frac{1}{2}, x = \frac{3}{2}$$

M1 A1

Answer: $x = \frac{3}{2}$

3 marks

(b) The curve lies below the line $y = 2$ on the left branch (all $x < 1$) and to the right of $x = \frac{3}{2}$

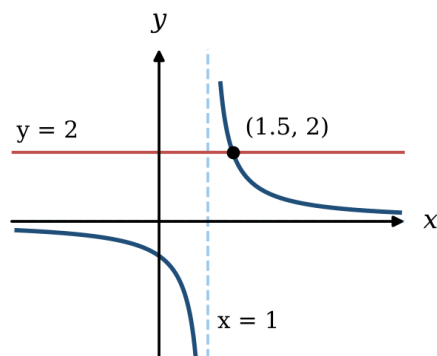
M1

$$x < 1 \text{ or } x > \frac{3}{2}$$

A1

Answer: $x < 1$ or $x > \frac{3}{2}$

2 marks



20 (8 marks)

(a) Both branches lie above the horizontal asymptote (shape of $y = \frac{1}{x^2}$ translated)

B1

Asymptotes $x = -1$ and $y = -4$

B1 B1

$$y = 0: (x+1)^2 = \frac{1}{4}, \text{ so } x = -\frac{1}{2} \text{ or } x = -\frac{3}{2}: \text{ crosses the x-axis at } (-\frac{3}{2}, 0) \text{ and } (-\frac{1}{2}, 0)$$

B1

$$x = 0: y = 1 - 4 = -3: (0, -3)$$

B1

Answer: asymptotes $x = -1$ and $y = -4$, crossing at $(-3/2, 0)$, $(-1/2, 0)$ and $(0, -3)$

5 marks

(b) Need $\frac{1}{(x+1)^2} - 4 > 0$: the part of C above the x-axis

M1

This is between the two roots found, either side of $x = -1$

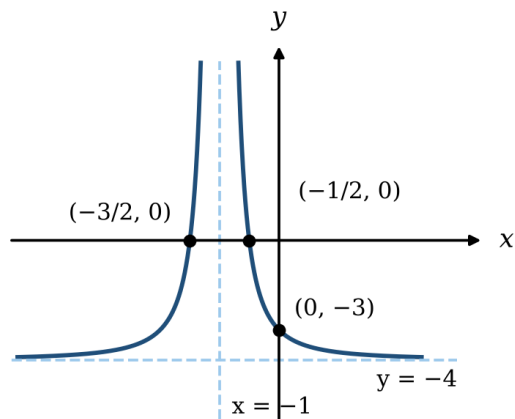
A1

$$-\frac{3}{2} < x < -\frac{1}{2}, x \neq -1$$

A1

Answer: $-\frac{3}{2} < x < -\frac{1}{2}, x \neq -1$

3 marks



21 (10 marks)

(a) $f(-x) = \frac{-x}{x^2 - 1} = -f(x)$

M1 A1

The graph has rotational symmetry of order 2 about the origin

E1

Answer: odd; rotational symmetry about O

3 marks

(b) $x = 1, x = -1$

B1

$y = 0$ (as $x \rightarrow \pm\infty, f(x) \rightarrow 0$)

B1

Answer: $x = \pm 1, y = 0$

2 marks

(c) Passes through the origin, $f(x) = 0$ only at $x = 0$

B1

Sign of f in each region: $x < -1$ negative, $-1 < x < 0$ positive, $0 < x < 1$ negative, $x > 1$ positive

M1

Three branches approaching the asymptotes correctly

A1

Answer: odd function through the origin with asymptotes $x = 1, x = -1$ and $y = 0$

3 marks

(d) From the sketch, above the x -axis for $-1 < x < 0$

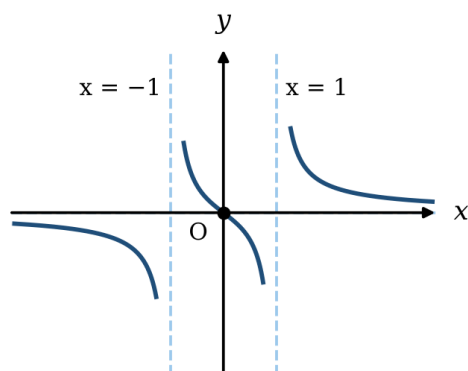
M1

and for $x > 1$

A1

Answer: $-1 < x < 0$ or $x > 1$

2 marks



22 (10 marks)

(a) $\frac{1}{v} = \frac{1}{12} - \frac{1}{u} = \frac{u - 12}{12u}$

M1 A1

$v = \frac{12u}{u - 12}$

A1

Answer: shown

3 marks

(b) $u = 12$: as the object approaches the focal point the image distance becomes arbitrarily large

B1 E1

$v = 12$: for very distant objects the image forms close to the focal length 12 cm

B1 E1

Answer: $u = 12, v = 12$

4 marks

(c) $\frac{12u}{u - 12} = 2u$

M1

$12 = 2(u - 12)$ (as $u \neq 0$)

M1

$u = 18$ cm

A1

Answer: $u = 18$ cm

3 marks

23 (9 marks)

$$(a) x + \frac{1}{x} - 2 = \frac{x^2 - 2x + 1}{x}$$

$$= \frac{(x-1)^2}{x} \geq 0 \text{ for } x > 0, \text{ so } f(x) \geq 2$$

M1

A1 E1

Answer: shown

3 marks

(b) $x = 0$ and $y = x$ (oblique)

B1 B1

Minimum at $(1, 2)$, odd function so maximum at $(-1, -2)$

B1

Correct shape

B1

Answer: $x = 0$ and $y = x$; minimum $(1, 2)$, maximum $(-1, -2)$

4 marks

(c) $2x^2 + 2 = 5x$, so $2x^2 - 5x + 2 = 0$

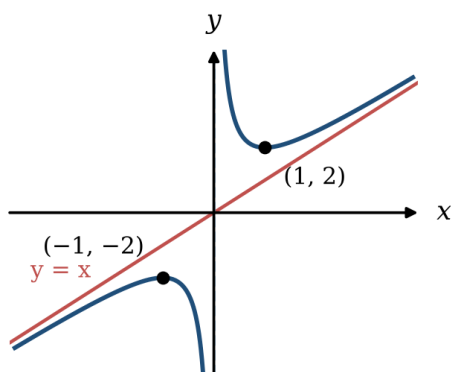
M1

$(2x - 1)(x - 2) = 0$, so $x = \frac{1}{2}$ or $x = 2$

A1

Answer: $x = \frac{1}{2}$ or 2

2 marks



24 (8 marks)

(a) $x = 0$: $y = -1$; $y = 0$: $3x = 2$

M1

$(0, -1)$ and $(\frac{2}{3}, 0)$

A1

Answer: $(0, -1)$, $(\frac{2}{3}, 0)$

2 marks

(b) $\frac{3x-2}{x+2} = x$ gives $3x - 2 = x^2 + 2x$

M1

$x^2 - x + 2 = 0$

A1

Discriminant $1 - 8 = -7 < 0$, so no real solutions

E1

Answer: no intersection

3 marks

(c) $3 - \frac{8}{x+2} = \frac{3x+6-8}{x+2}$

B1

$f(x) < 0$ between the root and the asymptote: $-2 < x < \frac{2}{3}$

M1 A1

Answer: $-2 < x < \frac{2}{3}$

3 marks

25 (9 marks)

(a) $2 - \frac{5}{x+3} = \frac{2(x+3)-5}{x+3}$

M1

$= \frac{2x+1}{x+3}$

A1

Answer: shown

2 marks

(b) Stretch parallel to the y-axis, scale factor 5: $y = \frac{5}{x}$

B1

Reflection in the x-axis: $y = -\frac{5}{x}$

B1

Translation by $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$: $y = 2 - \frac{5}{x+3}$

B1

Answer: stretch y by 5; reflect in the x-axis; translate $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

3 marks

(c) Asymptotes $x = -3$ and $y = 2$

B1 B1

$$y = 0: 2x + 1 = 0, \text{ so } (-\frac{1}{2}, 0)$$

M1

$$x = 0: y = \frac{1}{3}, \text{ so } (0, \frac{1}{3})$$

A1

Answer: $x = -3, y = 2; (-\frac{1}{2}, 0), (0, \frac{1}{3})$

4 marks

26 (10 marks)

(a) Asymptotes $x = 2$ and $y = 3$, shape of $\frac{7}{x-2} + 3$

B1

$$x\text{-intercept } (-\frac{1}{3}, 0)$$

B1

$$y\text{-intercept } (0, -\frac{1}{2})$$

B1

Answer: asymptotes $x = 2$ and $y = 3$, crossing at $(-1/3, 0)$ and $(0, -1/2)$

3 marks

(b) $3x + 1 = (x + k)(x - 2)$

M1

$$x^2 + (k - 5)x - 2k - 1 = 0$$

A1

$$\text{Discriminant } (k - 5)^2 + 4(2k + 1) = k^2 - 2k + 29$$

M1

$$= (k - 1)^2 + 28 > 0 \text{ for all } k$$

A1

Substituting $x = 2$ gives $-7 \neq 0$, so no root is at the asymptote: always two points

E1

Answer: shown

5 marks

(c) $\alpha + \beta = 5 - k = 3$, so $k = 2$

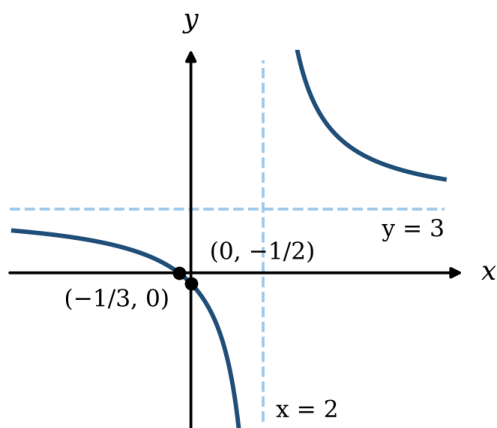
M1

$$\alpha\beta = -2k - 1 = -5$$

A1

Answer: $k = 2, \alpha\beta = -5$

2 marks



27 (9 marks)

(a) Asymptotes $x = 1, x = 3$ (denominator zero) and $y = 0$

B1 B1

$$x = 0: y = \frac{1}{(-1)(-3)} = \frac{1}{3}, \text{ so } A(0, \frac{1}{3})$$

B1

Answer: $x = 1, x = 3, y = 0; A(0, \frac{1}{3})$

3 marks

(b) $(x - 1)(x - 3) = (x - 2)^2 - 1 \geq -1$, and $\neq 0$

M1

So $u = (x - 1)(x - 3)$ takes values $u \geq -1, u \neq 0$

A1

$$y = \frac{1}{u} \leq -1 \text{ or } y > 0$$

A1

Answer: $y \leq -1$ or $y > 0$

3 marks

(c) $y > 0$ needs $(x - 1)(x - 3) > 0$, so $x < 1$ or $x > 3$; then $\frac{1}{u} > \frac{1}{3}$ means $u < 3$

M1

$$x^2 - 4x + 3 < 3, \text{ so } x(x - 4) < 0, \text{ giving } 0 < x < 4$$

A1

Combining: $0 < x < 1$ or $3 < x < 4$

A1

Answer: $0 < x < 1$ or $3 < x < 4$

3 marks

28 (12 marks)

(a) $\frac{4}{x} = k - x$

M1

$4 = kx - x^2$, so $x^2 - kx + 4 = 0$

A1

Answer: shown

2 marks

(b) Discriminant $k^2 - 16 = 0$

M1

$k = 4$ ($k > 0$)

A1

$x^2 - 4x + 4 = 0$, $x = 2$, $y = 2$; point (2, 2)

M1 A1

Answer: $k = 4$; (2, 2)

4 marks

(c) $k = 5$: $25 - 16 > 0$, two intersections

B1

$k = 3$: $9 - 16 < 0$, none

B1

Answer: 2 and 0

2 marks

(d) $\frac{4}{x} - 5 + x > 0$ gives $\frac{x^2 - 5x + 4}{x} > 0$

M1

$\frac{(x-1)(x-4)}{x} > 0$

M1

Sign analysis: $0 < x < 1$ or $x > 4$

A1 A1

Answer: $0 < x < 1$ or $x > 4$

4 marks

29 (17 marks)

(a) $1 - \frac{3}{x^2 - 1} = \frac{x^2 - 1 - 3}{x^2 - 1}$

M1

$= \frac{x^2 - 4}{x^2 - 1}$

A1

Answer: shown

2 marks

(b) $x = 1$, $x = -1$, $y = 1$

B1 B1

Even function: symmetric about the y-axis

B1

Answer: $x = \pm 1$, $y = 1$; symmetric about y-axis

3 marks

(c) x-intercepts $(\pm 2, 0)$, y-intercept $(0, 4)$

B1 B1

Outer branches below $y = 1$ approaching it from below

B1

Middle branch above, rising to infinity at $x = \pm 1$

B1

Answer: crossing at $(-2, 0)$, $(2, 0)$, $(0, 4)$; asymptotes $x = 1$, $x = -1$ and $y = 1$

4 marks

(d) From the sketch, on or above the x-axis for $x \leq -2$, for $-1 < x < 1$ and for $x \geq 2$

M1 A1 A1

Answer: $x \leq -2$, $-1 < x < 1$, $x \geq 2$

3 marks

(e) For $|x| > 1$, $x^2 - 1 > 0$ so $f(x) = 1 - \frac{3}{x^2 - 1} < 1$, taking all values below 1

M1 A1

For $|x| < 1$, $0 \leq x^2 < 1$ so $-1 \leq x^2 - 1 < 0$, giving $\frac{3}{x^2 - 1} \leq -3$ and $f(x) \geq 4$

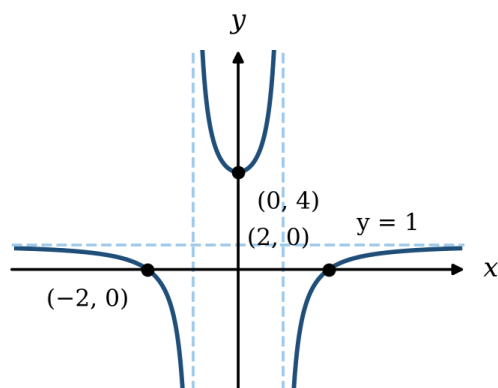
M1 A1

Range: $f(x) < 1$ or $f(x) \geq 4$

A1

Answer: $f(x) < 1$ or $f(x) \geq 4$

5 marks

**30 (11 marks)**

(a) $x = a$ and $y = b$

B1 B1

Answer: $x = a$, $y = b$

2 marks

$$(b) x = 0: y = \frac{k}{-a} + b = b - \frac{k}{a}$$

M1 A1

$$y = 0: \frac{k}{x - a} = -b, \text{ so } x - a = -\frac{k}{b}$$

M1

$$x = a - \frac{k}{b}$$

A1

Answer: shown

4 marks

$$(c) ab > k \text{ gives } b > \frac{k}{a}, \text{ so } b - \frac{k}{a} > 0$$

M1 E1

$$ab > k \text{ gives } a > \frac{k}{b}, \text{ so } a - \frac{k}{b} > 0$$

E1

Answer: $ab > k$ makes both $b - \frac{k}{a}$ and $a - \frac{k}{b}$ positive

3 marks

(d) Shape of $y = \frac{k}{x}$ centred on the point (a, b) where the asymptotes meet

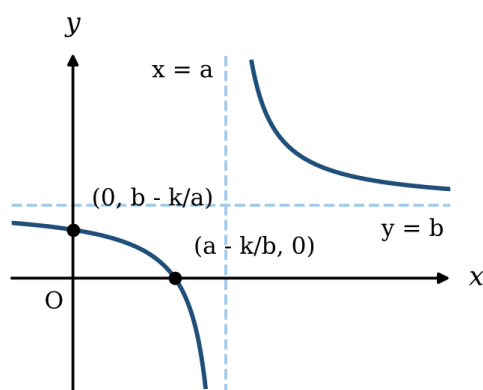
B1

Both branches drawn, with the lower left branch crossing the axes at the intercepts from part (b)

B1

Answer: reciprocal shape centred on (a, b) crossing the axes at $(a - \frac{k}{b}, 0)$ and $(0, b - \frac{k}{a})$

2 marks



Modulus Graphs

1 (2 marks)

$$|-4| + |3| = 4 + 3 = 7$$

B1

$$|5 - 12| = |-7| = 7$$

B1

Answer: (a) 7 (b) 7

2 marks

2 (3 marks)

V-shaped graph, entirely on or above the x-axis

B1

Vertex at $(3, 0)$

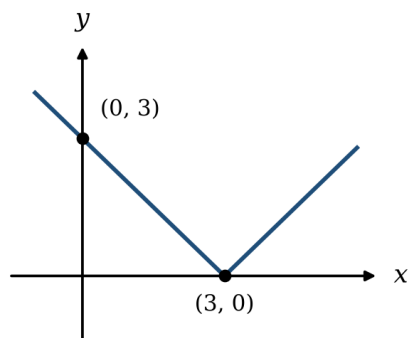
B1

y-intercept: $|0 - 3| = 3$, so $(0, 3)$

B1

Answer: V-shape with vertex $(3, 0)$ and y-intercept $(0, 3)$

3 marks



3 (2 marks)

$$x - 3 = 5 \text{ or } x - 3 = -5$$

M1

$$x = 8 \text{ or } x = -2$$

A1

Answer: $x = 8$ or $x = -2$

2 marks

4 (2 marks)

$$2x + 1 = 7 \text{ or } 2x + 1 = -7$$

M1

$$x = 3 \text{ or } x = -4$$

A1

Answer: $x = 3$ or $x = -4$

2 marks

5 (2 marks)V-shape with vertex at $(-2, 0)$

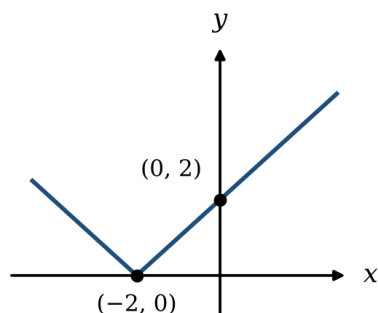
B1

y-intercept: $|2| = 2$, so $(0, 2)$

B1

Answer: V with vertex $(-2, 0)$ and y-intercept $(0, 2)$

2 marks

**6 (2 marks)**

$$f(-1) = |3 + 2| = 5$$

B1

$$f(4) = |3 - 8| = 5$$

B1

Answer: $f(-1) = 5$, $f(4) = 5$

2 marks

7 (4 marks)(a) V-shape with vertex on the x-axis at $(2, 0)$

B1

y-intercept $(0, 4)$

B1

Answer: V-shape with vertex $(2, 0)$ and y-intercept $(0, 4)$

2 marks

(b) $2x - 4 = 6$ gives $x = 5$

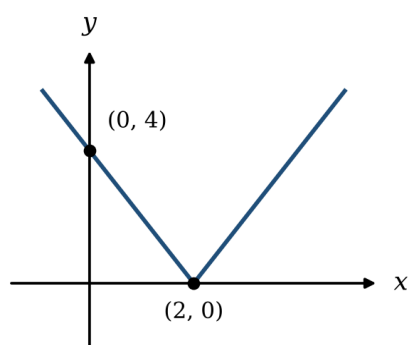
M1

$$2x - 4 = -6 \text{ gives } x = -1$$

A1

Answer: $x = 5$ or $x = -1$

2 marks

**8 (5 marks)**(a) Start with the parabola $y = x^2 - 4$, crossing the x-axis at $x = \pm 2$

B1

The part below the x-axis is reflected above it, so the minima are at $(-2, 0)$ and $(2, 0)$

B1

y-intercept is $(0, 4)$, a local maximum

B1

Answer: W-shape with minima $(-2, 0)$, $(2, 0)$ and maximum $(0, 4)$

3 marks

(b) $x^2 - 4 = 5$ gives $x = \pm 3$

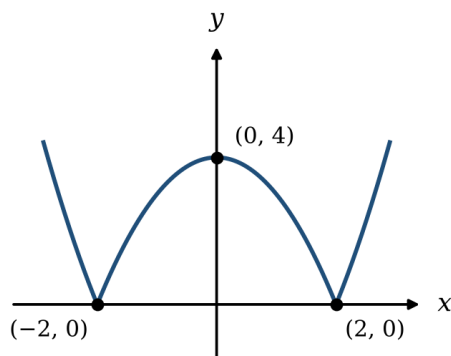
M1

$$x^2 - 4 = -5 \text{ gives } x^2 = -1, \text{ no real solutions}$$

A1

Answer: $x = \pm 3$

2 marks



9 (4 marks)

(a) $-3 < x - 4 < 3$

M1

$1 < x < 7$

A1

Answer: $1 < x < 7$

2 marks

(b) $2x + 1 \geq 5$ or $2x + 1 \leq -5$

M1

$x \geq 2$ or $x \leq -3$

A1

Answer: $x \geq 2$ or $x \leq -3$

2 marks

10 (3 marks)

Inverted V with maximum at (0, 4), the y-intercept

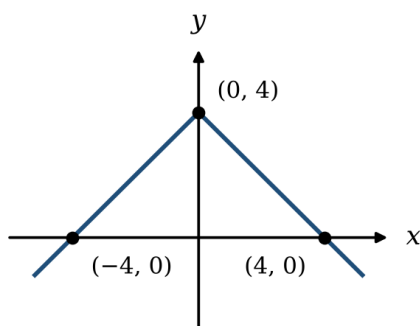
B1 B1

Meets the x-axis where $|x| = 4$: (-4, 0) and (4, 0)

B1

Answer: inverted V with maximum (0, 4), crossing at (-4, 0) and (4, 0)

3 marks



11 (4 marks)

$2x - 3 = x$ gives $x = 3$

M1 A1

$-(2x - 3) = x$ gives $3 = 3x$, $x = 1$

A1

Check both: $|3| = 3$ and $|-1| = 1$, so both valid

B1

Answer: $x = 1$ or $x = 3$

4 marks

12 (3 marks)

(a) $-3 \leq m - 500 \leq 3$

M1

$497 \leq m \leq 503$

A1

Answer: $497 \leq m \leq 503$

2 marks

(b) 503.2 and 496.9 are outside the interval

B1

Answer: 503.2 g and 496.9 g

1 mark

13 (7 marks)

(a) Either $2x - 1 = x + 4$

M1

giving $x = 5$, or $2x - 1 = -(x + 4)$

A1

giving $3x = -3$, $x = -1$

A1

Answer: $x = 5$ or $x = -1$

3 marks

(b) Between the two intersection points the graph of $y = |2x - 1|$ lies below that of $y = |x + 4|$

M1

(check: $x = 0$ gives $1 < 4$)

$-1 < x < 5$

A1

Answer: $-1 < x < 5$

2 marks

(c) Two V-shaped graphs with vertices $(\frac{1}{2}, 0)$ and $(-4, 0)$

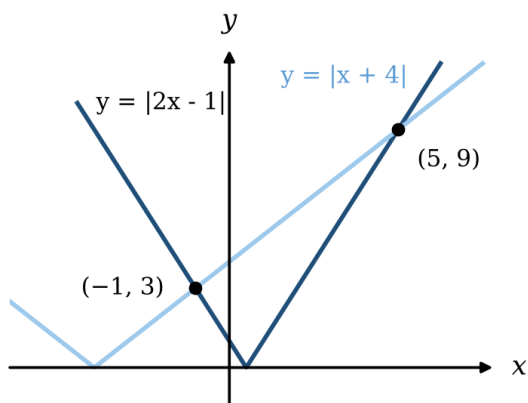
B1

Meeting at $(-1, 3)$ and $(5, 9)$

B1

Answer: V graphs with vertices $(\frac{1}{2}, 0)$ and $(-4, 0)$, meeting at $(-1, 3)$ and $(5, 9)$

2 marks



14 (4 marks)

$3x - 2 = x$ gives $x = 1$

M1 A1

$3x - 2 = -x$ gives $4x = 2$, $x = \frac{1}{2}$

M1 A1

Answer: $x = 1$ or $x = \frac{1}{2}$

4 marks

15 (6 marks)

(a) Reflect the part of the curve below the x -axis in the x -axis

M1

x -intercepts $(-1, 0)$ and $(3, 0)$, y -intercept $(0, 3)$

A1

Local maximum at $(1, 4)$

A1

Answer: x -intercepts $(-1, 0)$, $(3, 0)$, y -intercept $(0, 3)$, local maximum $(1, 4)$

3 marks

(b) For $x \geq 0$ keep the curve; for $x < 0$ reflect the $x \geq 0$ part in the y -axis

M1

x -intercepts $(-3, 0)$ and $(3, 0)$

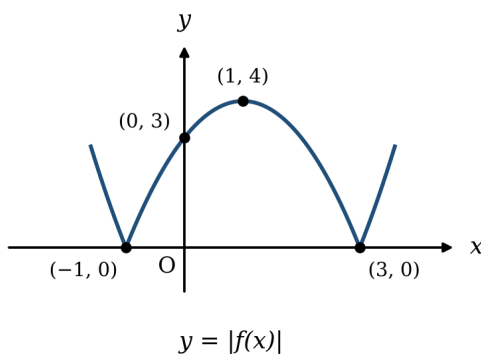
A1

y -intercept $(0, -3)$

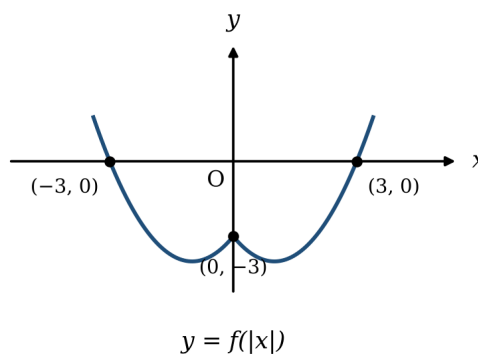
A1

Answer: W-shape through $(-3, 0)$, $(0, -3)$, $(3, 0)$

3 marks



$$y = |f(x)|$$



$$y = f(|x|)$$

16 (4 marks)

(a) $(x - 2)^2 < (x + 1)^2$

M1

$x^2 - 4x + 4 < x^2 + 2x + 1$

M1

$3 < 6x$, so $x > \frac{1}{2}$

A1

Answer: $x > \frac{1}{2}$

3 marks

(b) Both sides are non-negative, so $a < b \Leftrightarrow a^2 < b^2$ for non-negative a, b

E1

Answer: both sides non-negative

1 mark

17 (5 marks)**(a)** V-shape with vertex (2, 0)

B1

y-intercept (0, 2)

B1

Answer: V with vertex (2, 0) and y-intercept (0, 2)

2 marks

(b) $y = |x| - 2$: V-shape, vertex (0, -2), x-intercepts (± 2 , 0)

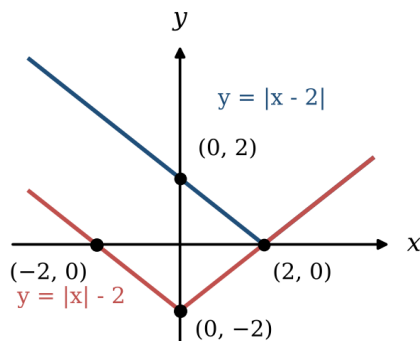
B1 B1

One has its vertex on the x-axis (reflects the negative part in the x-axis), the other is symmetric in the y-axis

E1

Answer: V with vertex (0, -2) and x-intercepts (-2, 0), (2, 0)

3 marks

**18 (3 marks)** $|x| - 3 = 2$ or $|x| - 3 = -2$

M1

 $|x| = 5$ or $|x| = 1$

A1

 $x = \pm 5$ or $x = \pm 1$

A1

Answer: $x = \pm 1$ or $x = \pm 5$

3 marks

19 (6 marks)**(a)** Asymptotes $x = 2$ and $y = 0$

B1

y-intercept: $|\frac{1}{-2}| = \frac{1}{2}$, so $(0, \frac{1}{2})$; there is no x-intercept

B1

Both branches lie above the x-axis, since the lower branch of $y = f(x)$ is reflected

B1

Answer: asymptotes $x = 2$ and $y = 0$; y-intercept $(0, \frac{1}{2})$

3 marks

(b) $|x - 2| < 1$, with $x \neq 2$

M1

 $1 < x < 3$

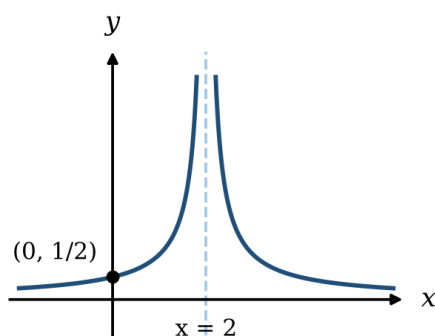
A1

Excluding $x = 2$: $1 < x < 2$ or $2 < x < 3$

A1

Answer: $1 < x < 2$ or $2 < x < 3$

3 marks

**20 (7 marks)****(a)** For $0 < x < 1$ we have $\ln x < 0$, so that part of $y = \ln x$ is reflected in the x-axis

B1

The graph meets the x-axis only at (1, 0)

B1

The y-axis ($x = 0$) is a vertical asymptote, with $y \rightarrow \infty$ as $x \rightarrow 0$

B1

Answer: meets the x-axis at (1, 0); asymptote $x = 0$

3 marks

(b) $\ln x = 2$ or $\ln x = -2$

M1

 $x = e^2$ or $x = e^{-2}$

A1

Answer: $x = e^2$ or $x = e^{-2}$

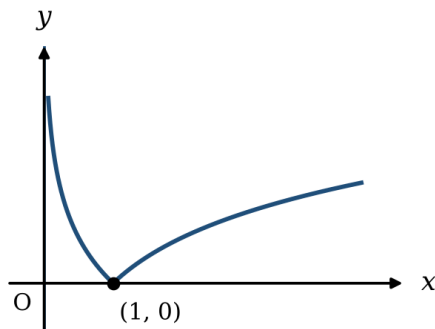
2 marks

(c) $-1 < \ln x < 1$

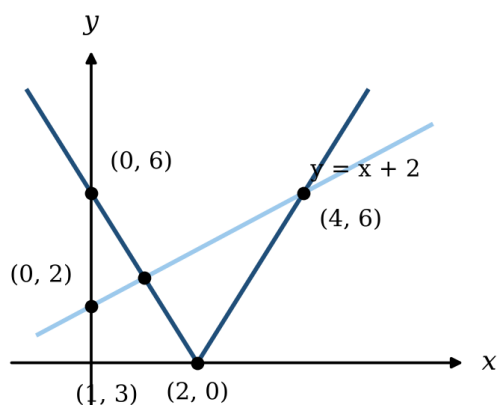
M1

 $e^{-1} < x < e$

A1

Answer: $e^{-1} < x < e$ **2 marks****21 (8 marks)****(a)** V-shape with vertex (2, 0) and y-intercept (0, 6)**B1 B1**

Line with intercepts (0, 2) and (-2, 0), meeting the V twice

B1**Answer: V with vertex (2, 0), y-intercept (0, 6); line through (0, 2), (-2, 0); two intersections****3 marks****(b)** $3x - 6 = x + 2$ gives $x = 4$ **M1 A1** $-(3x - 6) = x + 2$ gives $6 - 3x = x + 2$, so $x = 1$ **A1****Answer: $x = 1$ or $x = 4$** **3 marks****(c)** The V is above the line outside the two intersections**M1** $x < 1$ or $x > 4$ **A1****Answer: $x < 1$ or $x > 4$** **2 marks****22 (8 marks)****(a)** $fg(x) = |2(x^2 - 3) - 5| = |2x^2 - 11|$ **B1** $gf(x) = |2x - 5|^2 - 3 = (2x - 5)^2 - 3$ **M1** $= 4x^2 - 20x + 25 - 3 = 4x^2 - 20x + 22$ **A1****Answer: $fg(x) = |2x^2 - 11|$** **3 marks****(b)** $2x^2 - 11 = 3$ gives $x^2 = 7$ **M1** $2x^2 - 11 = -3$ gives $x^2 = 4$ **M1** $x = \pm\sqrt{7}$ or $x = \pm 2$ **A1****Answer: $x = \pm 2, \pm\sqrt{7}$** **3 marks****(c)** $fg(x) \geq 0$, and 0 is attained at $x^2 = \frac{11}{2}$, so the range is $fg(x) \geq 0$ **M1 A1****Answer: $fg(x) \geq 0$** **2 marks**

23 (8 marks)

(a) $x^2 - 3x = 2$ or $x^2 - 3x = -2$ M1

$x^2 - 3x + 2 = 0$ gives $x = 1$ or $x = 2$ A1

$x^2 - 3x - 2 = 0$ gives $x = \frac{3 \pm \sqrt{17}}{2}$ M1

So $x = \frac{3 - \sqrt{17}}{2}, 1, 2, \frac{3 + \sqrt{17}}{2}$ A1

Answer: $x = \frac{3 - \sqrt{17}}{2}, 1, 2, \frac{3 + \sqrt{17}}{2}$ 4 marks

(b) The graph lies below the line between the 1st and 2nd intersections, and between the 3rd and 4th M1

$\frac{3 - \sqrt{17}}{2} < x < 1$ A1

$2 < x < \frac{3 + \sqrt{17}}{2}$ A1

Answer: $\frac{3 - \sqrt{17}}{2} < x < 1$ or $2 < x < \frac{3 + \sqrt{17}}{2}$ 3 marks

(c) The central hump has maximum value $\frac{9}{4} < 3$, so $y = 3$ only meets the two outer branches: 2 solutions E1

Answer: 2 solutions 1 mark

24 (7 marks)

(a) The modulus is never negative, so $2x - 3 \geq 0$ M1

$x \geq \frac{3}{2}$ A1

Answer: shown 2 marks

(b) $x^2 - 4x = 2x - 3$ gives $x^2 - 6x + 3 = 0$, $x = 3 \pm \sqrt{6}$ M1 A1

$-(x^2 - 4x) = 2x - 3$ gives $x^2 - 2x - 3 = 0$, $x = 3$ or -1 M1 A1

Reject $3 - \sqrt{6} \approx 0.55$ and -1 (both $< \frac{3}{2}$): $x = 3$ or $x = 3 + \sqrt{6}$ A1

Answer: $x = 3$ or $x = 3 + \sqrt{6}$ 5 marks

25 (8 marks)

(a) $x < -2$: $-(x + 2) + (x - 1) = -3$ B1

$-2 \leq x \leq 1$: $(x + 2) + (x - 1) = 2x + 1$ B1

$x > 1$: $(x + 2) - (x - 1) = 3$ B1

Answer: -3 ; $2x + 1$; 3 3 marks

(b) Horizontal at -3 , rising line through $(-2, -3)$ and $(1, 3)$, horizontal at 3 B1 B1

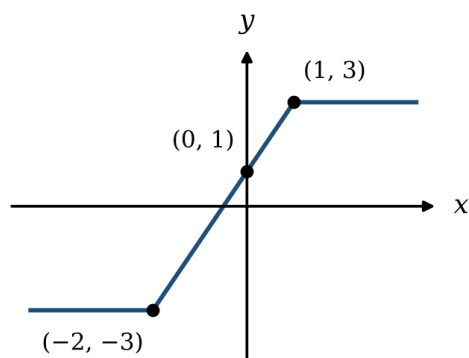
Range $-3 \leq f(x) \leq 3$ B1

Answer: rising line through $(-2, -3)$ and $(1, 3)$ between horizontal ends; range $-3 \leq f(x) \leq 3$ 3 marks

(c) $2x + 1 = 1$ (the only region where f takes the value 1) M1

$x = 0$ A1

Answer: $x = 0$ 2 marks



26 (7 marks)

(a) Substitute $x = 4$: $|8 - k| = 7$

M1

$8 - k = 7$ or $8 - k = -7$

M1

$k = 1$ or $k = 15$

A1

Answer: shown

3 marks

(b) $k = 1$: $1 - 2x = x + 3$ gives $x = -\frac{2}{3}$, valid since $x + 3 > 0$

M1 A1

$k = 15$: $2x - 15 = x + 3$ gives $x = 18$, valid since $x + 3 > 0$

M1 A1

Answer: $k = 1$: $x = -\frac{2}{3}$; $k = 15$: $x = 18$

4 marks

27 (9 marks)

(a) $x = -3$: $|2(-3) - 1| = 7$ but $3(-3) + 2 = -7$, so it is not a solution

E1

A modulus is never negative, so the right-hand side must satisfy $3x + 2 \geq 0$, which the student did not check

E1

Answer: $x = -3$ is not a solution

2 marks

(b) Case 1: $2x - 1 = 3x + 2$ gives $x = -3$

M1

Case 2: $1 - 2x = 3x + 2$ gives $x = -\frac{1}{5}$

A1

Check: $x = -3$ rejected, $x = -\frac{1}{5}$ gives $\frac{7}{5} = \frac{7}{5}$, so $x = -\frac{1}{5}$

A1

Answer: $x = -\frac{1}{5}$

3 marks

(c) The V-shape lies above the line only to the left of the intersection, since for $x = 0$ we get $1 > 2$ false

M1

$x < -\frac{1}{5}$

A1

Answer: $x < -\frac{1}{5}$

2 marks

(d) V-shape with vertex $(\frac{1}{2}, 0)$ and y-intercept $(0, 1)$

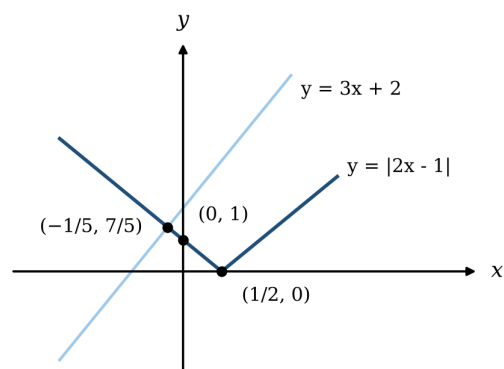
B1

Line with gradient 3 and y-intercept $(0, 2)$, meeting the V once at $x = -\frac{1}{5}$

B1

Answer: V with vertex $(1/2, 0)$, line of gradient 3 through $(0, 2)$; one intersection

2 marks



28 (10 marks)

(a) $|a + b|^2 = (a + b)^2 = a^2 + 2ab + b^2$

M1

$ab \leq |ab| = |a||b|$, so $|a + b|^2 \leq (|a| + |b|)^2$

M1 E1

Both sides are non-negative, so taking square roots, $|a + b| \leq |a| + |b|$

E1

Answer: proved

4 marks

(b) Take $a = x$ and $b = 3$: $|x + 3| \leq |x| + |3| = |x| + 3$

M1

Equality when $ab \geq 0$, i.e. $x \geq 0$

A1

Answer: equality for $x \geq 0$

2 marks

(c) $|x + 2| = |(x - 2) + 4| \leq |x - 2| + 4 < 5$

M1 E1

$|x^2 - 4| = |x - 2||x + 2| < 1 \times 5 = 5$

M1 A1

Answer: shown

4 marks

29 (10 marks)

(a) $x < 1$: $(1 - x) + (4 - x) = 5 - 2x$

B1

$1 \leq x \leq 4$: $(x - 1) + (4 - x) = 3$

B1

$x > 4$: $(x - 1) + (x - 4) = 2x - 5$

B1**Answer: $5 - 2x$; 3 ; $2x - 5$** **3 marks**

(b) $5 - 2x = 7$ gives $x = -1$

M1

$2x - 5 = 7$ gives $x = 6$ (the middle region gives $3 \neq 7$)

A1**Answer: $x = -1$ or $x = 6$** **2 marks****(c)** The graph is below the line $y = 7$ between its two intersections**M1**

$-1 < x < 6$

A1**Answer: $-1 < x < 6$** **2 marks****(d)** The minimum value of the graph is 3, attained for $1 \leq x \leq 4$ **M1**For $k > 3$ the line $y = k$ meets each sloping part once: $k > 3$ **A1****Answer: $k > 3$** **2 marks****(e)** $|x - 1| + |4 - x| \geq |(x - 1) + (4 - x)| = 3$ by the triangle inequality**E1****Answer: proved****1 mark****30 (15 marks)**

(a) $x^2 - 6x + 5 = (x - 3)^2 - 4$, minimum (3, -4)

M1 A1

The modulus reflects the part below the x-axis, turning the minimum into a maximum (3, 4)

E1**Answer: shown****3 marks**

(b) $k < 0$: none; $k = 0$: 2; $0 < k < 4$: 4

B1 B1 B1

$k = 4$: 3; $k > 4$: 2

B1 B1**Answer: 0, 2, 4, 3, 2****5 marks**

(c) $x^2 - 6x + 2 = 0$ gives $x = 3 \pm \sqrt{7}$

M1 A1

$x^2 - 6x + 8 = 0$ gives $x = 2$ or 4

M1 A1**Answer: $x = 2, 4, 3 \pm \sqrt{7}$** **4 marks****(d)** Graph below the line $y = 3$ between the first two and between the last two intersections**M1**

$3 - \sqrt{7} < x < 2$ or $4 < x < 3 + \sqrt{7}$

A1 A1**Answer: $3 - \sqrt{7} < x < 2$ or $4 < x < 3 + \sqrt{7}$** **3 marks**