

A LEVEL MATHS • EXTREME PAPERS

MATHS



MODEL ANSWERS

EXTREME PAPER: STATISTICS

SCORE

/90

PERCENTAGE

%

STEMUP

Paper: Statistics (90 marks)

1 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: only the last two tosses matter, so use states and set up equations for Alia's winning probability.

Let s , h , t be the probability that Alia wins from the states: no useful toss yet (or last toss T); last toss H only; last two tosses H T.

From the state H H, further H's keep the state H H and the first T gives H H T, so Ben certainly wins: Alia's probability is 0. B1

Tree from each state: $s = \frac{1}{2}h + \frac{1}{2}s$, $h = \frac{1}{2}(0) + \frac{1}{2}t$, $t = \frac{1}{2}(1) + \frac{1}{2}s$. M1 A1

After H T, a T gives H T T, which has no useful ending, so the state returns to s .

First equation gives $s = h$, so $s = \frac{1}{2}t = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}s) = \frac{1}{4} + \frac{1}{4}s$. M1

$$\frac{3}{4}s = \frac{1}{4}, \text{ so } s = \frac{1}{3}.$$

A1

Answer: 1/3 5 marks

(b) $s = ph + qs$, $h = qt$, $t = p + qs$. M1

$ps = ph$ so $s = h$; $s = q(p + qs)$, so $s(1 - q^2) = pq$, and $1 - q^2 = (1 - q)(1 + q) = p(1 + q)$, giving A1

$$s = \frac{q}{1 + q}.$$

$\frac{q}{1 + q} < \frac{1}{2}$ because $2q < 1 + q$ when $q < 1$, so Alia's probability is below $1/2$ for every p . B1

Answer: $q/(1+q) < 1/2$ 3 marks

(c) Alia wins on toss 3 only by H T H: probability $\frac{1}{8}$. On toss 4 only by T H T H (H H T H is M1

impossible since H H T has already ended the game at toss 3): probability $\frac{1}{16}$.

$$\text{Required probability} = \frac{3}{16} \div \frac{1}{3} = \frac{9}{16}.$$

A1

Answer: 9/16 2 marks

2 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: $M < 60$ means both scores are below 60; $L > 40$ means both scores are above 40.

$P(M < 60) = [P(X_1 < 60)]^2 = 0.64$, so $P(X_1 < 60) = 0.8$. M1

$$\text{Hence } \frac{60 - \mu}{\sigma} = 0.8416.$$

A1

$P(L > 40) = [P(X_1 > 40)]^2 = 0.81$, so $P(X_1 > 40) = 0.9$ and $P(X_1 < 40) = 0.1$. M1

$$\text{Hence } \frac{40 - \mu}{\sigma} = -1.2816.$$

A1

Subtracting: $20 = 2.1232\sigma$, so $\sigma = 9.42$ and $\mu = 60 - 0.8416\sigma = 52.1$. A1

Answer: $\sigma = 9.42$, $\mu = 52.1$ 5 marks

(b) $M_n < \mu$ only if every score is below μ , and each is below μ with probability $\frac{1}{2}$. M1

$$P(M_n > \mu) = 1 - \left(\frac{1}{2}\right)^n = 1 - 2^{-n}.$$

A1

Answer: shown 2 marks

(c) Key idea: use the complement of $M_n < \mu + 2\sigma$ or $L_n > \mu - 2\sigma$ and inclusion-exclusion.

$P(M_n < \mu + 2\sigma) = [\Phi(2)]^n$, and by symmetry $P(L_n > \mu - 2\sigma) = [\Phi(2)]^n$. M1

Both together: all scores lie between $\mu - 2\sigma$ and $\mu + 2\sigma$, probability $[\Phi(2) - \Phi(-2)]^n = [2\Phi(2) - 1]^n$. M1

$P(\text{neither}) = 1 - P(\text{first}) - P(\text{second}) + P(\text{both}) = 1 - 2[\Phi(2)]^n + [2\Phi(2) - 1]^n$. A1

$$n = 20: 1 - 2(0.9772)^{20} + (0.9544)^{20} = 0.132.$$

A1

Answer: 0.132 4 marks

3 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

$$(a) \bar{x} \sim N(\mu, \frac{144}{n}). P(E1) = P(\bar{x} > c \mid \mu = 80) = 1 - \Phi(\frac{(c - 80)\sqrt{n}}{12}) = \Phi(-\frac{(c - 80)\sqrt{n}}{12}). \quad \text{M1}$$

$$P(E2) = P(\bar{x} < c \mid \mu = 84) = \Phi(\frac{(c - 84)\sqrt{n}}{12}). \quad \text{M1}$$

Φ is one-to-one, so $-(c - 80) = c - 84$, giving $c = 82$ independent of n . A1

Answer: $c = 82$ 3 marks

$$(b) \text{ Each error probability is } \Phi(-\frac{2\sqrt{n}}{12}) = \Phi(-\frac{\sqrt{n}}{6}). \quad \text{M1}$$

$$\text{Need } \frac{\sqrt{n}}{6} \geq 2.3263, \text{ so } \sqrt{n} \geq 13.958. \quad \text{A1}$$

$$n \geq 194.8, \text{ so the least } n \text{ is } 195. \quad \text{A1}$$

Answer: 195 3 marks

$$(c) (i) \bar{x} \text{ has standard deviation } 12/6 = 2, \text{ so } P(E1) = P(Z > 1) = 0.1587 \text{ and } P(E2) = P(Z < -1) = 0.1587. \quad \text{B1}$$

$$(ii) P(\text{decide } 84 \mid \mu = 80) = 0.1587 \text{ and } P(\text{decide } 84 \mid \mu = 84) = 1 - 0.1587 = 0.8413. \quad \text{B1}$$

$$P(\mu = 80 \mid \text{decide } 84) = \frac{0.7 \times 0.1587}{0.7 \times 0.1587 + 0.3 \times 0.8413}. \quad \text{M1 M1}$$

$$= \frac{0.1111}{0.1111 + 0.2524} = 0.306 \text{ (3 s.f.)}. \quad \text{A1}$$

Answer: (i) 0.1587 each; (ii) 0.306 5 marks

4 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: condition on the position j of the best candidate, which is equally likely to be 1, 2, 3 or 4.

$$\text{Each position has probability } \frac{1}{4}. \quad \text{B1}$$

$j = 1$: the best candidate is rejected, so the probability of success is 0. M1

$j = 2$: candidate 2 beats candidate 1 and is accepted: probability 1. $j = 3$: success needs candidate 2 M1

not to beat candidate 1, i.e. candidate 1 is the best of the first two: probability $\frac{1}{2}$. $j = 4$: candidate

1 must be the best of the first three: probability $\frac{1}{3}$.

$$P = \frac{1}{4}(0 + 1 + \frac{1}{2} + \frac{1}{3}) = \frac{1}{4} \times \frac{11}{6} = \frac{11}{24}. \quad \text{A1}$$

Answer: 11/24 4 marks

(b) Key idea: the best candidate, in position j , is accepted exactly when no candidate before j (after the first k) was accepted.

$$P(\text{best is in position } j) = \frac{1}{n}, \text{ and for } j \leq k \text{ the best candidate is rejected.} \quad \text{B1}$$

For $j > k$, a candidate in positions $k + 1$ to $j - 1$ is accepted early only if they beat all earlier ones; so the best is accepted exactly when the best of the first $j - 1$ candidates lies in the first k positions. M1

By symmetry the best of the first $j - 1$ candidates is equally likely to be in any of those positions, M1

so this has probability $\frac{k}{j - 1}$.

$$P = \sum_{j=k+1}^n \frac{1}{n} \times \frac{k}{j - 1} = \frac{k}{n} \sum_{j=k+1}^n \frac{1}{j - 1}. \quad \text{A1}$$

Answer: shown 4 marks

$$(c) k = 1: \frac{1}{5}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{5}{12}; k = 2: \frac{2}{5}(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{13}{30}. \quad \text{M1}$$

$$k = 3: \frac{3}{5}(\frac{1}{3} + \frac{1}{4}) = \frac{7}{20}; k = 4: \frac{4}{5} \times \frac{1}{4} = \frac{1}{5}. \quad \text{M1}$$

The greatest is at $k = 2$, with probability $\frac{13}{30}$. A1

Answer: $k = 2, 13/30$ 3 marks

5 (11 marks) CIE

(a) $\frac{10!}{3! 3! 2!}$

M1

$= 50\,400.$

A1

Answer: 50 400

2 marks

(b) Arrange the other seven letters T, T, T, A, I, I, C first: $\frac{7!}{3! 2!} = 420$ ways.

M1

The three S go in three different gaps of the eight gaps (including the ends): ${}^8C_3 = 56$ ways.

M1

$420 \times 56 = 23\,520.$

A1

Answer: 23 520

3 marks

(c) Key idea: the letters A, I, I, C split the row into 5 gaps; the six letters S, T then fill the gaps, and the only restriction is that each gap holds an alternating string of S and T.

A, I, I, C can be arranged in $\frac{4!}{2!} = 12$ ways, giving 5 gaps (including the ends).

B1

Letters in different gaps are separated by A, I or C, so the gaps are independent and each must be an alternating string.

B1

A gap holds: nothing (1 way); length $2j$, $j \geq 1$: STST... or TSTS... (2 ways, j S and j T); length $2j + 1$, $j \geq 0$: starts and ends with S ($j + 1$ S, j T) or with T (1 way each).

M1

Weight an S by x and a T by y : $G = 1 + \frac{2xy}{1 - xy} + \frac{x + y}{1 - xy} = \frac{(1 + x)(1 + y)}{1 - xy}.$

M1

Number $= 12 \times [x^3y^3] G^5$, and $G^5 = (1 + x)^5(1 + y)^5(1 - xy)^{-5}$, with $(1 - xy)^{-5} = \sum j^{+4} C_4(xy)^j.$

M1

$[x^3y^3] = \sum_{j=0}^3 j^{+4} C_4 ({}^5C_{3-j})^2 = 100 + 500 + 375 + 35 = 1010$, and $12 \times 1010 = 12\,120.$

A1

Answer: 12 120

6 marks

6 (12 marks) Edexcel | AQA | OCR A | OCR MEI

(a) Key idea: split each deviation from the combined mean $\bar{x}$ into a deviation from the group mean plus a shift.

For group 1, $\sum(x - \bar{x})^2 = S_{xx}^{(1)} + m(\bar{x}_1 - \bar{x})^2$, because the cross term $\sum(x - \bar{x}_1)$ is zero. Similarly for group 2.

M1 A1

$\bar{x} = \frac{m\bar{x}_1 + n\bar{x}_2}{m + n}$, so $\bar{x}_1 - \bar{x} = \frac{n(\bar{x}_1 - \bar{x}_2)}{m + n}$ and $\bar{x}_2 - \bar{x} = -\frac{m(\bar{x}_1 - \bar{x}_2)}{m + n}.$

M1

The extra terms total $\frac{mn^2 + nm^2}{(m + n)^2}(\bar{x}_1 - \bar{x}_2)^2 = \frac{mn}{m + n}(\bar{x}_1 - \bar{x}_2)^2.$

A1

Answer: shown

4 marks

(b) Group P: $r = \frac{18}{\sqrt{12 \times 30}} = 0.949$. Group Q: $r = \frac{14}{\sqrt{20 \times 16}} = 0.783.$

B1

$\frac{mn}{m + n} = \frac{96}{20} = \frac{24}{5}$. Combined: $S_{xx} = 12 + 20 + \frac{24}{5} \times 16 = \frac{544}{5}$, $S_{xy} = 18 + 14 + \frac{24}{5} \times 4 \times (-2) =$

M1

$-\frac{32}{5}$, $S_{yy} = 30 + 16 + \frac{24}{5} \times 4 = \frac{326}{5}.$

$r = -\frac{32}{5} / \sqrt{\frac{544}{5} \times \frac{326}{5}} = -0.0760.$

A1

Answer: 0.949, 0.783, -0.0760

3 marks

(c) Gradient $= \frac{-32}{5} / \frac{544}{5} = -\frac{1}{17}.$

M1

Combined mean: $\bar{x} = \frac{8 \times 2 + 12 \times 6}{20} = 4.4$, $\bar{y} = \frac{8 \times 5 + 12 \times 3}{20} = 3.8.$

M1

$y - 3.8 = -\frac{1}{17}(x - 4.4)$, so $y = \frac{69}{17} - \frac{x}{17} = \frac{69 - x}{17}.$

A1

Answer: $y = (69 - x)/17$

3 marks

(d) Only S_{xy} depends on d (the y -mean difference is d): combined $S_{xy} = 18 + 14 + \frac{24}{5} \times 4 \times d =$

M1

$32 + \frac{96}{5}d.$

$r = 0$ when $S_{xy} = 0$, so $d = -\frac{5}{3}.$

A1

Answer: $d = -5/3$ **2 marks****7 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE****(a)** $(q + p)^n = \sum {}^nC_r p^r q^{n-r} = 1$ and $(q - p)^n = \sum (-1)^r {}^nC_r p^r q^{n-r}$.**M1**Adding, the odd r cancel and the even r double: $1 + (q - p)^n = 2 \sum_{r \text{ even}} {}^nC_r p^r q^{n-r}$.**M1**

$$\text{So } P(X \text{ even}) = \frac{1 + (q - p)^n}{2}.$$

A1**Answer: shown****3 marks**

$$\text{(b)} \frac{1 + (1 - 2p)^9}{2} = 0.55, \text{ so } (1 - 2p)^9 = 0.1.$$

M1

$$1 - 2p = 0.1^{1/9} = 0.7743, \text{ so } p = 0.113.$$

A1**Answer: 0.113****2 marks****(c)** Key idea: absorb the factor $1/(r + 1)$ into the binomial coefficient.

$$\frac{{}^nC_r}{r + 1} = \frac{n!}{(r + 1)r!(n - r)!} = \frac{(n + 1)!}{(n + 1)(r + 1)!(n - r)!} = \frac{{}^{n+1}C_{r+1}}{n + 1}.$$

M1

$$E\left(\frac{1}{X + 1}\right) = \sum_{r=0}^n \frac{{}^{n+1}C_{r+1}}{n + 1} p^r q^{n-r} = \frac{1}{(n + 1)p} \sum_{r=0}^n {}^{n+1}C_{r+1} p^{r+1} q^{(n+1)-(r+1)}.$$

M1

$$\text{With } s = r + 1, \text{ the sum is } \sum_{s=1}^{n+1} {}^{n+1}C_s p^s q^{n+1-s} = (q + p)^{n+1} - q^{n+1} = 1 - q^{n+1}.$$

M1

$$\text{So } E\left(\frac{1}{X + 1}\right) = \frac{1 - q^{n+1}}{(n + 1)p}.$$

A1**Answer: shown****4 marks**

$$\text{(d)} \frac{{}^nC_r}{(r + 1)(r + 2)} = \frac{{}^{n+2}C_{r+2}}{(n + 1)(n + 2)}.$$

M1

$$E = \frac{1}{(n + 1)(n + 2)p^2} \sum_{s=2}^{n+2} {}^{n+2}C_s p^s q^{n+2-s}, \text{ with } s = r + 2.$$

M1

$$\text{The sum is } (q + p)^{n+2} - q^{n+2} - (n + 2)p q^{n+1} = 1 - q^{n+2} - (n + 2)p q^{n+1}, \text{ giving the result.}$$

A1**Answer: shown****3 marks****8 (12 marks) CIE | Edexcel IAL****(a)** Key idea: write F_r as 1 minus a finite sum, differentiate each term and watch the sum telescope.

$$F_r(\lambda) = 1 - \sum_{k=0}^{r-1} \frac{e^{-\lambda} \lambda^k}{k!}.$$

M1

$$\text{For } k \geq 1: \frac{d}{d\lambda} \left(\frac{e^{-\lambda} \lambda^k}{k!} \right) = \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} - \frac{e^{-\lambda} \lambda^k}{k!}; \text{ for } k = 0 \text{ it is } -e^{-\lambda}.$$

M1

$$\text{Writing } u_k = \frac{e^{-\lambda} \lambda^k}{k!}, \text{ the sum of the derivatives is } -u_0 + (u_0 - u_1) + (u_1 - u_2) + \dots + (u_{r-2} - u_{r-1}) \\ = -u_{r-1}.$$

M1

$$\text{Hence } \frac{dF_r}{d\lambda} = u_{r-1} = \frac{e^{-\lambda} \lambda^{r-1}}{(r-1)!}.$$

A1**Answer: shown****4 marks**

$$\text{(b)} F_3(0) = P(\text{Po}(0) \geq 3) = 0, \text{ so by (a) with } r = 3, F_3(\lambda) = \int_0^\lambda \frac{t^2 e^{-t}}{2} dt.$$

B1 M1

$$F_3(\lambda) = 1 - e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2} \right), \text{ so the integral is } 2F_3(\lambda) = 2 - e^{-\lambda} (\lambda^2 + 2\lambda + 2).$$

A1**Answer: shown****3 marks**

$$\text{(c)} P(N = n) = \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} e^{-\lambda} d\lambda = \frac{1}{n!} \int_0^\infty \lambda^n e^{-2\lambda} d\lambda.$$

M1 M1

$$= \frac{1}{n!} \times \frac{n!}{2^{n+1}} = \left(\frac{1}{2} \right)^{n+1}.$$

A1**Answer: shown****3 marks**

$$\text{(d)} P(N \text{ even}) = \frac{1}{2} + \frac{1}{2^3} + \dots = \frac{1}{2 \div 1 - 1/4} = \frac{2}{3}, \text{ and } P(N = 0) = \frac{1}{2}, \text{ so } P(N \text{ even and } N \geq 1) = \frac{2}{3}$$

M1

$$- \frac{1}{2} = \frac{1}{6}.$$

$P(N \geq 1) = \frac{1}{2}$, so the required probability is $\frac{1}{6} \div \frac{1}{2} = \frac{1}{3}$.

Answer: 1/3

A1

2 marks