

A LEVEL MATHS • EXTREME PAPERS

MATHS



EXAM PAPER

EXTREME PAPER: PURE MATHEMATICS

SCORE

/120

PERCENTAGE

%

STEMUP

Paper: Pure Mathematics (120 marks)

1 Edexcel | AQA | OCR A | OCR MEI

In this question, a square root sign denotes the non-negative square root.

- (a) Prove by contradiction that $\log_2 3$ is irrational. [3]
- (b) For real $x \geq 1$ let $f(x) = \sqrt{x + 2\sqrt{x - 1}} + \sqrt{x - 2\sqrt{x - 1}}$.
Show that $f(x) = 1 + t + |t - 1|$, where $t = \sqrt{x - 1}$. [3]
- (c) Solve $f(x) = 2$. [2]
- (d) Solve $f(x) = x$. [3]

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The function f is defined for all real x other than 0 and 1, and satisfies, for all such x ,

$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x.$$

- (a) Find $f(x)$. (Hint: consider the values f takes at x , $1 - 1/x$ and $1/(1 - x)$.) [5]
- (b) Show that $f(x) = \frac{x^3 - x^2 - 1}{2x(x - 1)}$, and verify that your f really does satisfy the original equation at $x = 2$. [2]
- (c) Find the exact value of $\int_2^3 f(x) \, dx$. [3]

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Throughout, n is a positive integer.

- (a) By integrating $(1 - x)^n$ over $[0, 1]$ in two ways, show that

$$\sum_{k=0}^n (-1)^k \frac{C(n, k)}{k + 1} = \frac{1}{n + 1}, \text{ where } C(n, k) = \frac{n!}{k!(n - k)!}.$$
 [3]
- (b) Show that $\sum_{k=0}^n (-1)^k \frac{C(n, k)}{k + 2} = \frac{1}{(n + 1)(n + 2)}.$ [4]
- (c) Show that $\sum_{k=0}^n (-1)^k \frac{C(n, k)}{k + 3} = \frac{2}{(n + 1)(n + 2)(n + 3)}.$ [3]
- (d) Hence find, in terms of n , the value of $\sum_{k=0}^n (-1)^k \frac{C(n, k)}{(k + 1)(k + 2)(k + 3)}.$ [3]

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Throughout, $t = \tan \theta$.

- (a) Show that $\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}.$ [3]
- (b) By considering the equation $\tan 3\theta = \sqrt{3}$, find the exact value of $\tan^2 20^\circ + \tan^2 40^\circ + \tan^2 80^\circ.$ [5]
- (c) Hence find the exact value of $\tan^4 20^\circ + \tan^4 40^\circ + \tan^4 80^\circ.$ [3]

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Let $f(x) = \frac{\ln x}{x}$ for $x > 0$.

- (a) Show that f has a single stationary point, at $x = e$, that it is a maximum, and that f is strictly decreasing for $x > e$. [3]
- (b) Without a calculator, show that $e^\pi > \pi^e$. [2]
- (c) Positive integers a and b satisfy $a < b$ and $a^b = b^a$. Show that $a = 2$ and $b = 4$. [4]
- (d) Hence determine, with justification, the positive integer n for which $n^{1/n}$ is greatest. [3]

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Let $I = \int_0^{\pi/4} \ln(1 + \tan x) \, dx$.

(a) Show that $1 + \tan\left(\frac{\pi}{4} - u\right) = \frac{2}{1 + \tan u}$. [2]

(b) Use the substitution $x = \frac{\pi}{4} - u$ to show that $I = \frac{\pi}{8} \ln 2$. [5]

(c) Hence find the exact value of $\int_0^{\pi/4} \frac{\theta \sec^2 \theta}{1 + \tan \theta} \, d\theta$. [3]

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Throughout, n is an integer with $n \geq 2$, and $e = 2.718\dots$ is the base of natural logarithms.

(a) Show that $\ln(1 + x) < x$ for all $x > -1$ with $x \neq 0$. [2]

(b) Use (a) twice to show that, for every positive integer k ,
 $(1 + \frac{1}{k})^k < e < (1 + \frac{1}{k})^{k+1}$. [3]

(c) Multiply the inequalities in (b) over $k = 1, 2, \dots, n - 1$ and simplify the products to deduce that
 $e(n/e)^n < n! < e n(n/e)^n$. [4]

(d) Hence show that the sequence $u_n = \frac{(n!)^{1/n}}{n}$ satisfies $u_n \rightarrow \frac{1}{e}$ as $n \rightarrow \infty$. You may use the fact that $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$. [3]

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For integers $n \geq 0$ let $S_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n$. Here $\sqrt{3}$ denotes the positive square root.

(a) Show that $S_{n+2} = 4S_{n+1} - S_n$ for $n \geq 0$, and deduce that every S_n is an even integer. [3]

(b) Show that for $n \geq 1$ the integer part of $(2 + \sqrt{3})^n$ is odd. [3]

(c) Find the last digit of the integer part of $(2 + \sqrt{3})^{2027}$. [4]

(d) Write $(2 + \sqrt{3})^n = a_n + b_n\sqrt{3}$ with a_n, b_n positive integers. By considering $(2 - \sqrt{3})^n$, show that $a_n^2 - 3b_n^2 = 1$, and deduce that

$$0 < \frac{a_n}{b_n} - \sqrt{3} < \frac{1}{2\sqrt{3} b_n^2}.$$

Given that $(2 + \sqrt{3})^6 = 1351 + 780\sqrt{3}$, show that $\frac{1351}{780}$ approximates $\sqrt{3}$ with error less than 5×10^{-7} . [3]

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A ladder of length 1 m, modelled as a line segment, has its ends on the x -axis and the y -axis. In position L_θ ($0 < \theta < \pi/2$) its lower end is at $(\cos \theta, 0)$ and its upper end at $(0, \sin \theta)$.

(a) Show that the line through the ladder in position L_θ has equation $x \sin \theta + y \cos \theta = \sin \theta \cos \theta$. [2]

(b) The curve C has parametric equations $x = \cos^3 \theta$, $y = \sin^3 \theta$ for $0 < \theta < \pi/2$. Show that the tangent to C at the point with parameter θ is the line in (a). [3]

(c) Let $k > 0$. Find, in terms of k , the number of values of θ in $(0, \pi/2)$ for which the ladder line in (a) passes through the point (k, k) . [5]

(d) Find the exact area of the region in the first quadrant bounded by the axes and the curve C . [4]

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The curve C has equation $x^3 + y^3 = 3xy$. It has a closed loop that starts and ends at the origin.

- (a) For $t > 0$, the line $y = tx$ meets C again (other than at the origin) at the point P. Show that P has coordinates

$$x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}.$$

(These are parametric equations of the loop, for $0 \leq t < \infty$.)

[2]

- (b) Show that $\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3}$.

[3]

- (c) Find the exact coordinates of the points on the loop where the tangent is (i) horizontal, (ii) vertical.

[3]

- (d) Find the exact area enclosed by the loop.

[6]