

A LEVEL MATHS • EXTREME PAPERS

MATHS



MODEL ANSWERS

EXTREME PAPER: PURE MATHEMATICS

SCORE

/120

PERCENTAGE

%

STEMUP

Paper: Pure Mathematics (120 marks)

1 (11 marks) Edexcel | AQA | OCR A | OCR MEI

(a) Key idea: powers of 2 and 3 can never be equal, by parity.

Suppose $\log_2 3 = p/q$ with p, q positive integers (it is positive as $3 > 1$).

M1

Then $3 = 2^{p/q}$, so $3^q = 2^p$.

M1

LHS is odd but RHS is even ($p \geq 1$), a contradiction, so $\log_2 3$ is irrational.

E1

Answer: Proof.

3 marks

(b) Key idea: the expressions under the roots are perfect squares.

$x = t^2 + 1$, so $x + 2\sqrt{x-1} = t^2 + 2t + 1 = (t+1)^2$.

M1

and $x - 2\sqrt{x-1} = t^2 - 2t + 1 = (t-1)^2$.

M1

Since $t \geq 0$ and the root is non-negative: $\sqrt{(t+1)^2} = t+1$ and $\sqrt{(t-1)^2} = |t-1|$ (not $t-1$).

A1

So $f(x) = 1 + t + |t-1|$.

Answer: Shown.

3 marks

(c) If $t \leq 1$: $f = 1 + t + 1 - t = 2$ for every such t .

M1

If $t > 1$: $f = 2t > 2$, no solution. So the solution set is $0 \leq t \leq 1$, i.e. $1 \leq x \leq 2$.

A1

Answer: $1 \leq x \leq 2$

2 marks

(d) Case $t \leq 1$ ($1 \leq x \leq 2$): $f = 2$, so $x = 2$, and $x = 2$ does satisfy $t = 1 \leq 1$.

M1 A1

Case $t > 1$: $f = 2t$, so $2\sqrt{x-1} = x$, giving $4(x-1) = x^2$, $(x-2)^2 = 0$, $x = 2$, but then $t = 1$, not > 1 : rejected.

B1

So the only solution is $x = 2$.

Answer: $x = 2$ only

3 marks

2 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: the substitution $x \rightarrow 1 - 1/x$ has order 3, so three equations close up.

Let $y = 1 - \frac{1}{x} = \frac{(x-1)}{x}$. Then $1 - \frac{1}{y} = \frac{1}{(1-x)} = z$, and $1 - \frac{1}{z} = x$.

M1 A1

$f(x) + f(y) = 1 + x$, $f(y) + f(z) = 1 + y$, $f(z) + f(x) = 1 + z$.

M1

(first) - (second) + (third): $2f(x) = 1 + x - y + z$.

M1

$f(x) = \frac{1}{2}(1 + x - 1 + \frac{1}{x} + \frac{1}{(1-x)}) = \frac{1}{2}(x + \frac{1}{x} - \frac{1}{(x-1)})$.

A1

Answer: $f(x) = \frac{1}{2}(x + 1/x - 1/(x-1))$

5 marks

(b) Common denominator: $x + \frac{1}{x} - \frac{1}{(x-1)} = \frac{x^2(x-1) + (x-1) - x}{x(x-1)} = \frac{x^3 - x^2 - 1}{x(x-1)}$, so $f =$

M1

$\frac{x^3 - x^2 - 1}{2x(x-1)}$.

$f(2) = \frac{8 - 4 - 1}{4} = \frac{3}{4}$; $f(1/2) = \frac{1}{2}(\frac{1}{2} + 2 + 2) = \frac{9}{4}$, and $\frac{3}{4} + \frac{9}{4} = 3 = 1 + 2$ (note $1 - 1/2 = 1/2$).

A1

Answer: Shown; $f(2) = 3/4$, $f(1/2) = 9/4$

2 marks

(c) Use $f = \frac{1}{2}(x + 1/x - 1/(x-1))$ (partial fractions already done).

M1

$\int_2^3 f \, dx = \frac{1}{2} \left[\frac{x^2}{2} + \ln x - \ln(x-1) \right]_2^3$.

M1

$= \frac{1}{2} \left(\frac{9}{2} - 2 + \ln \frac{3}{2} - \ln 2 \right) = \frac{5}{4} + \frac{1}{2} \ln \frac{3}{4}$.

A1

Answer: $5/4 + (1/2) \ln(3/4)$

3 marks

3 (13 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: integrate a binomial expansion term by term.

Direct: $\int_0^1 (1-x)^n \, dx = \left[-\frac{(1-x)^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$.

M1

Expansion: $(1-x)^n = \sum_{k=0}^n C(n,k)(-x)^k$, so the integral is $\sum_{k=0}^n (-1)^k C(n,k) \frac{1}{k+1}$.

M1 A1

Answer: Shown.

3 marks

(b) Integrate $x(1-x)^n$ on $[0, 1]$; expansion gives $\sum_{k=0}^n (-1)^k C(n,k) \int_0^1 x^{k+1} dx = \sum_{k=0}^n (-1)^k \frac{C(n,k)}{k+2}$. M1 A1

Substitute $u = 1 - x$: $\int_0^1 (1-u)u^n du = \frac{1}{n+1} - \frac{1}{n+2}$. M1

$= \frac{1}{(n+1)(n+2)}$. A1

Answer: Shown. 4 marks

(c) Key idea: the same method with $x^2(1-x)^n$, which now needs $(1-u)^2$ after substituting $u = 1 - x$.

Expanding $(1-x)^n$ and integrating $x^2(1-x)^n$ over $[0, 1]$ gives $\sum_{k=0}^n (-1)^k \frac{C(n,k)}{k+3}$. M1

With $u = 1 - x$ the integral is $\int_0^1 (1-u)^2 u^n du = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$. M1

$= \frac{(n+2)(n+3) - 2(n+3) + 2}{(n+1)(n+2)(n+3)} = \frac{n^2 + 3n + 2}{(n+1)(n+2)(n+3)} = \frac{2}{(n+1)(n+2)(n+3)}$. A1

Answer: Shown. 3 marks

(d) Partial fractions: $\frac{1}{(k+1)(k+2)(k+3)} = \frac{1}{2} \frac{1}{k+1} - \frac{1}{k+2} + \frac{1}{2} \frac{1}{k+3}$. M1

Using (a), (b), (c): sum $= \frac{1}{2} \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)}$. M1

$= \frac{(n+2)(n+3) - 2(n+3) + 2}{2(n+1)(n+2)(n+3)} = \frac{1}{2(n+3)}$. A1

Answer: $1/(2(n+3))$ 3 marks

4 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta}$. M1

With $\tan 2\theta = \frac{2t}{1-t^2}$: numerator $= \frac{2t + t(1-t^2)}{1-t^2} = \frac{3t-t^3}{1-t^2}$, denominator $= \frac{1-t^2-2t^2}{1-t^2}$. M1 A1

Cancelling $1-t^2$ gives the result.

Answer: Shown. 3 marks

(b) Key idea: three distinct angles give three distinct roots of one cubic.

$3t - t^3 = \sqrt{3}(1-3t^2)$, i.e. $t^3 - 3\sqrt{3}t^2 - 3t + \sqrt{3} = 0$. M1 A1

$\tan 3\theta = \sqrt{3}$ for $\theta = 20^\circ, 80^\circ, 140^\circ$ ($3\theta = 60^\circ, 240^\circ, 420^\circ$), and $\tan 140^\circ = -\tan 40^\circ$, so the roots are $\tan 20^\circ, \tan 80^\circ, -\tan 40^\circ$, distinct. B1

Comparing coefficients with $(t-a)(t-b)(t-c)$: $a+b+c = 3\sqrt{3}$, $ab+bc+ca = -3$ and $abc = -\sqrt{3}$. M1

Sum of squares $= (3\sqrt{3})^2 - 2(-3) = 33$ (squares of $\tan 40^\circ$ and $-\tan 40^\circ$ agree). A1

Answer: 33 5 marks

(c) Key idea: $a^4 + b^4 + c^4 = (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2)$, and the last bracket is $(ab + bc + ca)^2 - 2abc(a+b+c)$.

$a^2b^2 + b^2c^2 + c^2a^2 = (-3)^2 - 2(-\sqrt{3})(3\sqrt{3}) = 9 + 18 = 27$. M1 A1

Fourth powers (even powers ignore the sign of c): $33^2 - 2 \times 27 = 1089 - 54 = 1035$. A1

Answer: 1035 3 marks

5 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $f'(x) = \frac{1 - \ln x}{x^2}$. M1

$f' = 0$ only at $x = e$; $f' > 0$ for $0 < x < e$ and $f' < 0$ for $x > e$. A1

So a maximum at $x = e$ and f strictly decreasing for $x > e$ (and increasing on $(0, e)$). A1

Answer: Shown. 3 marks

(b) $\Pi > e$ so $f(\Pi) < f(e)$, i.e. $\frac{\ln \Pi}{\Pi} < \frac{1}{e}$. M1

Hence $e \ln \Pi < \Pi$, so $\Pi^e < e^\Pi$. A1

Answer: Shown. 2 marks

(c) Key idea: $a^b = b^a$ means $f(a) = f(b)$, and f takes each value at most once on each side of e .

Taking logs: $b \ln a = a \ln b$, so $f(a) = f(b)$.

M1

$f(1) = 0 < f(x)$ for $x > 1$, so $a \neq 1$. Since f is strictly monotonic on $(0, e]$ and on $[e, \infty)$, equal values with $a < b$ force $a < e < b$; as a is an integer, $a = 2$.

M1 A1

$f(2) = \frac{\ln 2}{2} = \frac{\ln 4}{4} = f(4)$ and f is strictly decreasing for $x > e$, so $b = 4$ uniquely.

A1

Answer: $a = 2, b = 4$

4 marks

(d) $n^{1/n} = e^{f(n)}$, so we maximise $f(n)$.

M1

$f(2) = \frac{\ln 2}{2} = \frac{\ln 8}{6} < \frac{\ln 9}{6} = f(3)$, since $8 < 9$.

M1

$f(3) > f(n)$ for all $n \geq 4$ as f is strictly decreasing for $x > e$, and $f(1) = 0$. So $n = 3$.

A1

Answer: $n = 3$

3 marks

6 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $\tan\left(\frac{\pi}{4} - u\right) = \frac{1 - \tan u}{1 + \tan u}$.

M1

$1 + \frac{1 - \tan u}{1 + \tan u} = \frac{2}{1 + \tan u}$.

A1

Answer: Shown.

2 marks

(b) Key idea: the substitution maps the integral to a copy of itself, so I appears on both sides.

$dx = -du$; limits $x = 0 \rightarrow u = \pi/4$, $x = \pi/4 \rightarrow u = 0$.

M1

$I = \int_0^{\pi/4} \ln(1 + \tan(\frac{\pi}{4} - u)) du$.

A1

$= \int_0^{\pi/4} [\ln 2 - \ln(1 + \tan u)] du = \frac{\pi}{4} \ln 2 - I$.

M1 A1

So $2I = \frac{\pi}{4} \ln 2$, $I = \frac{\pi}{8} \ln 2$.

A1

Answer: Shown.

5 marks

(c) Key idea: $\sec^2 \theta / (1 + \tan \theta)$ is the derivative of $\ln(1 + \tan \theta)$, so integrate by parts and the integral from the start reappears.

$u = \theta$, $dv = \frac{\sec^2 \theta}{1 + \tan \theta} d\theta$, $v = \ln(1 + \tan \theta)$: integral $= [\theta \ln(1 + \tan \theta)]_0^{\pi/4} - \int_0^{\pi/4} \ln(1 + \tan \theta) d\theta$.

M1 A1

$= \frac{\pi}{4} \ln 2 - \frac{\pi}{8} \ln 2 = \frac{\pi}{8} \ln 2$.

A1

Answer: $(\pi/8) \ln 2$

3 marks

7 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Let $g(x) = x - \ln(1 + x)$; $g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$, negative for $-1 < x < 0$ and positive for $x > 0$.

M1

So g has its only minimum at $x = 0$, where $g(0) = 0$, hence $g(x) > 0$ for every other $x > -1$.

A1

Answer: Shown.

2 marks

(b) Key idea: put $x = 1/k$ and $x = -1/(k+1)$ in (a).

$x = 1/k$: $\ln(1 + \frac{1}{k}) < \frac{1}{k}$, so $k \ln(1 + \frac{1}{k}) < 1$, giving $(1 + \frac{1}{k})^k < e$.

M1 A1

$x = -\frac{1}{k+1}$: $\ln \frac{k}{k+1} < -\frac{1}{k+1}$, so $\ln \frac{k+1}{k} > \frac{1}{k+1}$, i.e. $(k+1) \ln(1 + \frac{1}{k}) > 1$ and $(1 + \frac{1}{k})^{k+1} > e$.

A1

Answer: Shown.

3 marks

(c) Key idea: the products telescope.

$\prod (1 + \frac{1}{k})^k = \prod \frac{(k+1)^k}{k^k} = \frac{n^{n-1}}{(n-1)!} = \frac{n^n}{n!}$ (every j with $2 \leq j \leq n-1$ contributes $j^{j-1}/j^j = 1/j$).

M1 A1

So $\frac{n^n}{n!} < e^{n-1}$, giving $n! > n^n e^{1-n} = e(n/e)^n$.

A1

Similarly $\prod (1 + \frac{1}{k})^{k+1} = \frac{n^{n+1}}{(n-1)!} = \frac{n^{n+1}}{n!} > e^{n-1}$, so $n! < n^{n+1} e^{1-n} = e n(n/e)^n$.

A1

Answer: Shown.

4 marks

(d) Take n -th roots in (c) and divide by n : $\frac{1}{e} e^{1/n} < u_n < \frac{1}{e} (e n)^{1/n}$.

M1 A1

$e^{1/n} \rightarrow 1$ and $(e/n)^{1/n} = e^{1/n} n^{1/n} \rightarrow 1$, so by the sandwich $u_n \rightarrow \frac{1}{e}$.

A1

Answer: $u_n \rightarrow 1/e$

3 marks

8 (13 marks) Edexcel | AQA | OCR A | OCR MEI

(a) Key idea: $2 \pm \sqrt{3}$ are the roots of $r^2 - 4r + 1 = 0$.

Each root satisfies $r^2 = 4r - 1$, so multiplying by r^n : $r^{n+2} = 4r^{n+1} - r^n$; add the two cases.

M1 A1

$S_0 = 2$ and $S_1 = 4$ are even integers and the recurrence preserves this, so all S_n are even integers (by induction).

B1

Answer: Shown.

3 marks

(b) $0 < 2 - \sqrt{3} < 1$, so $0 < (2 - \sqrt{3})^n < 1$ for $n \geq 1$.

M1

Hence $(2 + \sqrt{3})^n = S_n - (2 - \sqrt{3})^n$ lies strictly between $S_n - 1$ and S_n .

M1

So its integer part is $S_n - 1$, which is odd as S_n is even.

A1

Answer: Integer part is $S_n - 1$, odd.

3 marks

(c) Key idea: the last digits of S_n repeat, and the recurrence fixes the next term from the previous two.

$S_0, S_1, S_2, S_3, S_4 = 2, 4, 14, 52, 194$, with last digits 2, 4, 4, 2, 4.

M1

Since $S_{n+2} \equiv 4S_{n+1} - S_n \pmod{10}$, the pair of last digits (2, 4) of (S_3, S_4) equals that of (S_0, S_1) , so the last digits repeat with period 3: 2, 4, 4, 2, 4, 4, ...

M1 A1

$2027 = 3 \times 675 + 2$, so S_{2027} ends in 4 and the integer part, $S_{2027} - 1$, ends in 3.

A1

Answer: 3

4 marks

(d) Conjugating, $(2 - \sqrt{3})^n = a_n - b_n\sqrt{3}$; multiplying, $a_n^2 - 3b_n^2 = (2 + \sqrt{3})^n(2 - \sqrt{3})^n = 1$.

M1

So $\frac{a_n}{b_n} - \sqrt{3} = \frac{a_n - b_n\sqrt{3}}{b_n} = \frac{1}{b_n(a_n + b_n\sqrt{3})} > 0$, and $a_n > b_n\sqrt{3}$ gives $a_n + b_n\sqrt{3} > 2\sqrt{3} b_n$.

M1

$n = 6$, $b = 780$: error $< \frac{1}{2\sqrt{3} \times 780^2} \approx 4.74 \times 10^{-7} < 5 \times 10^{-7}$.

A1

Answer: Shown.

3 marks

9 (14 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Intercept form: $\frac{x}{\cos \theta} + \frac{y}{\sin \theta} = 1$.

M1

Multiply through by $\sin \theta \cos \theta$.

A1

Answer: Shown.

2 marks

(b) $\frac{dy}{dx} = \frac{3 \sin^2 \theta \cos \theta}{-3 \cos^2 \theta \sin \theta} = -\tan \theta$.

M1

Tangent: $y - \sin^3 \theta = -\tan \theta (x - \cos^3 \theta)$; multiply by $\cos \theta$: $y \cos \theta + x \sin \theta = \sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta$.

M1

$= \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) = \sin \theta \cos \theta$.

A1

Answer: Shown.

3 marks

(c) Key idea: the equation is symmetric in $\sin \theta$ and $\cos \theta$, so it is a quadratic in $s = \sin \theta + \cos \theta$.

$k(\sin \theta + \cos \theta) = \sin \theta \cos \theta$, and $\sin \theta \cos \theta = \frac{s^2 - 1}{2}$, so $s^2 - 2ks - 1 = 0$.

M1 A1

$s > 0$ forces $s = k + \sqrt{k^2 + 1}$ (the other root is negative).

A1

On $(0, \pi/2)$, $s = \sqrt{2} \sin(\theta + \pi/4)$ takes each value in $(1, \sqrt{2})$ twice and the value $\sqrt{2}$ once ($\theta = \pi/4$); it never exceeds $\sqrt{2}$.

M1

$k + \sqrt{k^2 + 1} < \sqrt{2}$ iff $0 < k < \frac{\sqrt{2}}{4}$, equal iff $k = \sqrt{2}/4$. So: 2 values if $0 < k < \sqrt{2}/4$; 1 value if $k =$

A1

$\sqrt{2}/4$; none if $k > \sqrt{2}/4$.

Answer: 2 if $0 < k < \sqrt{2}/4$; 1 if $k = \sqrt{2}/4$; 0 if $k > \sqrt{2}/4$

5 marks

(d) Area $= \int_0^1 y \, dx$, with $x = \cos^3 \theta$: x from 0 to 1 corresponds to θ from $\pi/2$ to 0, $dx = -3 \cos^2 \theta \sin \theta \, d\theta$.

M1

$$\text{Area} = \int_0^{\pi/2} 3 \sin^4 \theta \cos^2 \theta \, d\theta.$$

A1

$$\sin^4 \theta \cos^2 \theta = \frac{1}{4} \sin^2 2\theta \times \sin^2 \theta = \frac{1}{8} \sin^2 2\theta (1 - \cos 2\theta); \text{ integrate over } [0, \pi/2]: \text{ the } \sin^2 2\theta \text{ term}$$

M1

gives $\pi/4$ and the $\sin^2 2\theta \cos 2\theta$ term gives 0 (it is $\frac{1}{6} \sin^3 2\theta$).

$$\text{So } \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta = \frac{1}{8} \times \frac{\pi}{4} = \frac{\pi}{32} \text{ and Area} = \frac{3\pi}{32}.$$

A1

Answer: $3\pi/32$

4 marks

10 (14 marks) Edexcel | AQA | OCR A | OCR MEI

(a) Substitute $y = tx$: $x^3(1 + t^3) = 3tx^2$.

M1

$$x \neq 0, \text{ so } x = \frac{3t}{1+t^3} \text{ and } y = tx = \frac{3t^2}{1+t^3}.$$

A1

Answer: Shown.

2 marks

(b) $\frac{dx}{dt} = \frac{3(1+t^3) - 9t^2}{(1+t^3)^2} = \frac{3(1-2t^3)}{(1+t^3)^2}.$

M1 A1

$$\frac{dy}{dt} = \frac{6t(1+t^3) - 9t^4}{(1+t^3)^2} = \frac{3t(2-t^3)}{(1+t^3)^2}, \text{ and divide.}$$

A1

Answer: Shown.

3 marks

(c) (i) $dy/dt = 0$ with $t > 0$: $t^3 = 2$, $t = 2^{1/3}$; $x = \frac{3t}{3} = 2^{1/3}$, $y = \frac{3t^2}{3} = 2^{2/3}$.

M1 A1

(ii) $dx/dt = 0$: $t^3 = 1/2$, $t = 2^{-1/3}$; $1+t^3 = 3/2$ so $x = 2t = 2^{2/3}$, $y = 2t^2 = 2^{1/3}$.

A1

Answer: (i) $(2^{1/3}, 2^{2/3})$; (ii) $(2^{2/3}, 2^{1/3})$

3 marks

(d) Key idea: as t goes from 0 to ∞ the point traces the loop once, so the area is $-\int y \, dx$ taken round it, and the substitution $u = t^3$ makes everything rational.

Near $t = 0$ the point leaves the origin along the x -axis ($y = tx$, t small) and returns along the y -axis

B1

as $t \rightarrow \infty$, so the loop is traced anticlockwise and $\text{Area} = -\int_0^\infty y \frac{dx}{dt} dt$.

$$y \frac{dx}{dt} = \frac{9t^2(1-2t^3)}{(1+t^3)^3}.$$

M1

With $u = t^3$, $du = 3t^2 dt$: $\text{Area} = 3 \int_0^\infty \frac{2u-1}{(1+u)^3} du$.

M1

$2u-1 = 2(1+u) - 3$, so the integrand is $2(1+u)^{-2} - 3(1+u)^{-3}$, with antiderivative $-2(1+u)^{-1} + \frac{3}{2}(1+u)^{-2}$.

M1

This tends to 0 as $u \rightarrow \infty$ and equals $-\frac{1}{2}$ at $u = 0$, so the integral is $\frac{1}{2}$.

A1

$$\text{Area} = 3 \times \frac{1}{2} = \frac{3}{2}.$$

A1

Answer: $3/2$

6 marks